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Math 8652 : Theory of Probability

Homework #6

(1) Let (Ω, F, P) be a probability space and $(F_n)_{n\geq 0}$ be an increasing filtration of σ-subfields of F. Let ξ_n be F_n-measurable with $E|\xi_n|<\infty$. Suppose that for any bounded stopping times $\tau \leq \sigma$, we have $E\xi_\tau \leq E\xi_\sigma$. We show that the sequence (ξ_n) is a submartingale with respect to (F_n).

Take any $m \in \{0,1,2,\ldots\}$. It suffices to show $E(\xi_{m+1}|F_m) \geq \xi_m$. First, we show the following lemma.

[Let X be a ~~vari~~ random variable on a probability space $(\Omega, \mathcal{G}, P)$ and $EXI_A \geq 0$ for every $A \in \mathcal{G}$. Then $X \geq 0$ a.s.]

Proof of the lemma

Put $B = \{\omega \in \Omega : X(\omega) < 0\}$. For each $n \in \mathbb{N}$, we put

$$B_n = \{\omega \in \Omega : X(\omega) < -\tfrac{1}{n}\}.$$

Then $B = \bigcup_{n=1}^{\infty} B_n$. By the hypothesis,

$$0 \leq EXI_{B_n} \leq \left(-\tfrac{1}{n}\right) EI_{B_n} = -\tfrac{1}{n} P(B_n).$$

Thus, $P(B_n) = 0$. Then $P(B) \leq \sum_{n=1}^{\infty} P(B_n) = 0$. This means $P(B) = 0$. ∎

Return to the problem. Put $\eta = E(\xi_{m+1}|F_n)$, $X = \eta - \xi_m$ and $\mathcal{G} = F_m$. Because η and X are F_m-measurable, they are random variables on the probability space $(\Omega, \mathcal{G}, P)$. Our goal is to show $X \geq 0$ a.s. By the above lemma, it suffices to show $EXI_A \geq 0$ for every $A \in \mathcal{G}$. Fix $A \in \mathcal{G}$. By the definition of conditional expectations,

$$E\eta I_A = E\xi_{m+1} I_A, \qquad (1)$$

which is a finite number because $E|\xi_{m+1}| < \infty$. Define two maps $\sigma, \tau : \Omega \to \{0,1,2,\ldots\}$,

$$\sigma(\omega) = \begin{cases} m+1 & \text{if } \omega \in A, \\ m & \text{if } \omega \in A^c = \Omega \setminus A, \end{cases}$$

$$\tau(\omega) = m \quad \forall \omega \in \Omega.$$

Then $\sigma \geq \tau$ in Ω. For each $n \in \{0,1,2,\ldots\}$,

$$\{\omega : \tau(\omega) > n\} = \begin{cases} \Omega & \text{if } n < m, \\ \phi & \text{if } n \geq m, \end{cases}$$

which belongs to $\mathcal{F}_n$. Thus, τ is a bounded stopping time.

$$\{\omega : \sigma(\omega) > n\} = \begin{cases} \Omega & \text{if } n < m, \\ A & \text{if } n = m, \\ \phi & \text{if } n > m, \end{cases}$$

which belongs to $\mathcal{F}_n$. Thus, σ is also a bounded stopping time. By the hypothesis, $E\xi_\sigma \geq E\xi_\tau$. Then

$$E\xi_m = E\xi_\tau \leq E\xi_\sigma = E\xi_\sigma I_A + E\xi_\sigma I_{A^c} = E\xi_{m+1} I_A + E\xi_m I_{A^c}$$

Then $E\xi_m I_A + E\xi_m I_{A^c} \leq E\xi_{m+1} I_A + E\xi_m I_{A^c}$. $\qquad (2)$

Because $E|\xi_m| < \infty$, $E\xi_m I_{A^c} \in \mathbb{R}$. Then (2) implies $E\xi_m I_A \leq E\xi_{m+1} I_A$.

We get

$$EXI_A = E\eta I_A - E\xi_m I_A \overset{(1)}{=\!=} E\xi_{m+1} I_A - E\xi_m I_A \geq 0.$$

(2) Let $X_1, X_2, X_3, \dots$ be an independent and identically distributed sequence of real-valued random variables. Suppose $E|X_1| < \infty$ and $EX_1 = 0$. Put $S_0 = 0$ and $S_n = X_1 + X_2 + \dots + X_n$ for $n \in \mathbb{N}$. For $\tau = \inf\{n \geq 0 : S_n > 0\}$, we show that $E\tau = \infty$. We need the following results.

Lemma 1: Let $\eta_1, \eta_2, \eta_3, \dots$ be an independent and identically distributed sequence of random variables with $E\eta_1 = 0$. Put

$$\mathcal{F}_0 = \{\phi, \Omega\}, \quad \mathcal{F}_n = \sigma(\eta_1, \eta_2, \dots, \eta_n) \quad \forall n \in \mathbb{N},$$

$$\xi_0 = 0, \quad \xi_n = \eta_1 + \eta_2 + \dots + \eta_n \quad \forall n \in \mathbb{N}.$$

Then $(\xi_n, \mathcal{F}_n)_{n \geq 0}$ is a martingale.

Lemma 2: Let $(\xi_n, \mathcal{F}_n)_{n \geq 0}$ be a martingale. Then $E\xi_n = E\xi_0$ for all $n \in \mathbb{N}$.

Doob's theorem (Problem 18, Fristedt-Gray page 470; Lecture on 02/23/2015 by Professor Krylov)

Let $(\xi_n, \mathcal{F}_n)_{n \geq 0}$ be a submartingale and let τ be a stopping time. Then $(\xi_{n \wedge \tau}, \mathcal{F}_n)_{n \geq 0}$ is also a submartingale.

Proof of Lemma 1

It is clear that ξ_n is $\mathcal{F}_n$-measurable and $(\mathcal{F}_n)_{n \geq 0}$ is an increasing sequence of σ-fields. For each $n \in \{0, 1, 2, \dots\}$ and $A \in \mathcal{F}_n$,

$$E\xi_{n+1}I_A = E(\xi_n+\eta_{n+1})I_A = E\xi_n I_A + E\eta_{n+1}I_A. \qquad (1)$$

Since $(\eta_1,\dots,\eta_n)$ and η_{n+1} are independent, I_A which is $\mathcal{F}_n$-measurable and η_{n+1} are independent. Then $E\eta_{n+1}I_A = E\eta_{n+1}\,EI_A = 0.\,P(A)=0$. Then (1) implies $E\xi_{n+1}I_A = E\xi_n I_A$. Therefore, $E(\xi_{n+1}|\mathcal{F}_n) = \xi_n$.

Proof of Lemma 2

Because $(\xi_n,\mathcal{F}_n)_{n\geq 0}$ is a martingale, $E(\xi_n|\mathcal{F}_0) = \xi_0$ for all $n\in\mathbb{N}$. By the definition of conditional expectation, $E\xi_n I_\Omega = E\xi_0 I_\Omega$. In other words, $E\xi_n = E\xi_0$.

Return to the problem. Put

$$\mathcal{F}_0 = \{\phi,\Omega\},\quad \mathcal{F}_n = \sigma(X_1,X_2,\dots,X_n) \quad \forall n\in\mathbb{N},$$

$$\bar{S}_0 = 0,\quad \bar{S}_n = |X_1|+|X_2|+\dots+|X_n| \quad \forall n\in\mathbb{N},$$

$$c = E|X_1| \in \mathbb{R}.$$

First, we show that τ is a stopping time with respect to the increasing filtration $(\mathcal{F}_n)_{n\geq 0}$. For each $n\in\{0,1,2,\dots\}$,

$$\begin{aligned}\{\omega : \tau(\omega) > n\} &= \{\omega : S_k(\omega)\leq 0 \ \forall k=0,1,\dots,n\}\\ &= \{\omega : \omega\in S_k^{-1}((-\infty,0]) \ \forall k=0,1,\dots,n\}\\ &= \bigcap_{k=0}^{n} S_k^{-1}((-\infty,0]).\end{aligned}$$

Each set $S_k^{-1}((-\infty,0])$ is $\mathcal{F}_k$-measurable, thus is $\mathcal{F}_n$-measurable. Then

$\{\omega : \tau(\omega) > n\} \in \mathcal{F}_n$.

Next, put $\zeta_n = |X_n| - c$ and $Y_0 = 0$, $Y_n = \zeta_1 + \zeta_2 + \dots + \zeta_n = \bar{S}_n - nc$. We show that $(Y_n, \mathcal{F}_n)_{n \geq 0}$ is a martingale based on the proof of Lemma 1. It is clear that each Y_n is $\mathcal{F}_n$-measurable. For each $n \in \{0, 1, 2, \dots\}$ and $A \in \mathcal{F}_n$,

$$E\, Y_{n+1} I_A = E(Y_n + \zeta_n) I_A = E Y_n I_A + E \zeta_n I_A. \qquad (2)$$

Because $(X_1, \dots, X_n)$ and X_{n+1} are independent, I_A and ζ_{n+1} are also independent. Then $E\zeta_n I_A = E\zeta_n\, E I_A = 0 . P(A) = 0$. Then (2) implies $E\, Y_{n+1} I_A = E\, Y_n I_A$. Therefore, $E(Y_{n+1} | \mathcal{F}_n) = Y_n$. We have showed that $(Y_n, \mathcal{F}_n)_{n \geq 0}$ is a martingale.

Then $(Y_n, \mathcal{F}_n)_{n \geq 0}$ and $(-Y_n, \mathcal{F}_n)_{n \geq 0}$ are submartingales. By Doob's theorem, $(Y_{n\wedge\tau}, \mathcal{F}_n)_{n \geq 0}$ and $(-Y_{n\wedge\tau}, \mathcal{F}_n)_{n \geq 0}$ are submartingales. Then $(Y_{n\wedge\tau}, \mathcal{F}_n)_{n \geq 0}$ is a martingale. By Lemma 2,

$$E\, Y_{n\wedge\tau} = E\, Y_{0\wedge\tau} = E\, Y_0 = E(0) = 0.$$

In other words, $E(\bar{S}_{n\wedge\tau} - (n\wedge\tau)c) = 0$. Then $E\bar{S}_{n\wedge\tau} = c\, E(n\wedge\tau)$.

Suppose by contradiction that $E\tau < \infty$. Because $0 \leq n\wedge\tau \leq \tau$, $E\tau < \infty$ and $\lim_{n\to\infty} (n\wedge\tau) = \tau$, by the Dominated Convergence theorem we get

$$\lim_{n\to\infty} E(n\wedge\tau) = E\tau.$$

Thus, $\lim_{n\to\infty} E\bar{S}_{n\wedge\tau} = cE\tau$. Because the sequence $(\bar{S}_{n\wedge\tau})_{n\geq 0}$ is nonnegative increasing and $\lim_{n\to\infty} \bar{S}_{n\wedge\tau} = \bar{S}_\tau$, by the Monotone Convergence Theorem we get

$$E\bar{S}_\tau = \lim_{n\to\infty} E\bar{S}_{n\wedge\tau} = cE\tau < \infty.$$

We have $|S_{n\wedge\tau}| \leq \bar{S}_{n\wedge\tau} \leq \bar{S}_\tau$, $E\bar{S}_\tau < \infty$ and $\lim_{n\to\infty} S_{n\wedge\tau} = S_\tau$. By the Dominated Convergence theorem,

$$ES_\tau = \lim_{n\to\infty} ES_{n\wedge\tau}. \qquad (3)$$

Applying Lemma 1 for $\eta_n = X_n$, $\xi_n = S_n$, we conclude that $(S_n, \mathcal{F}_n)_{n\geq 0}$ is a martingale. Then $(S_n, \mathcal{F}_n)_{n\geq 0}$ and $(-S_n, \mathcal{F}_n)_{n\geq 0}$ are submartingales. ~~Thus,~~ By Doob's theorem, $(S_{n\wedge\tau}, \mathcal{F}_n)_{n\geq 0}$ and $(-S_{n\wedge\tau}, \mathcal{F}_n)_{n\geq 0}$ are submartingales. Thus, $(S_{n\wedge\tau}, \mathcal{F}_n)_{n\geq 0}$ is a martingale. By Lemma 2, $ES_{n\wedge\tau} = ES_{0\wedge\tau} = ES_0 = 0$. Then (3) implies $ES_\tau = 0$.

By the definition of τ, $S_{\tau(\omega)}(\omega) > 0$ for every $\omega \in \Omega$. Then $P(S_\tau = 0) = 0$. On the other hand, the identity $ES_\tau = 0$ and the fact that $S_\tau \geq 0$ imply $S_\tau = 0$ a.s., which means $P(S_\tau = 0) = 1$. This is a contradiction.

3 Let $\lambda, \mu > 0$, $\lambda + \mu \leq 1$. Suppose by contradiction that there is a renewal sequence $X_0, X_1, X_2, \dots$ such that

$$X_0 = 1,$$

$$\mathbb{P}(X_1 = 1) = \lambda + \mu, \qquad (1)$$

$$\mathbb{P}(X_2 = 1) = \lambda^2 + \mu^2. \qquad (2)$$

Define a sequence of random variables $T_0, T_1, T_2, \dots$ as follows.

$$T_0 = 0, \quad T_m = \inf\{n > T_{m-1} : X_n = 1\}.$$

For each $n \in \mathbb{N}$, we put

$$A_n = \{\omega : T_1 - T_0 = 1, T_2 - T_1 = 1, \dots, T_n - T_{n-1} = 1\},$$

$$B_n = \{\omega : X_1(\omega) = 1, X_2(\omega) = 1, \dots, X_n(\omega) = 1\}.$$

We show that $A_n = B_n$. For each $\omega \in A_n$, $T_1(\omega) = 1$, $T_2(\omega) = 2, \dots, T_n(\omega) = n$. By the definition of T_m, we conclude that $X_1(\omega) = 1, X_2(\omega) = 1, \dots, X_n(\omega) = 1$. Thus $\omega \in B_n$. For each $\omega \in B_n$, the definition of T_m gives us $T_1(\omega) = 1$, $T_2(\omega) = 2, \dots, T_n(\omega) = n$. Then $\omega \in A_n$.

Now that $A_n = B_n$, $\mathbb{P}(A_n) = \mathbb{P}(B_n)$. In other words,

$$\begin{aligned}\mathbb{P}(X_1 = 1, X_2 = 1, \dots, X_n = 1) &= \mathbb{P}(T_1 - T_0 = 1, T_2 - T_1 = 1, \dots, T_n - T_{n-1} = 1) \\ &= \mathbb{P}(T_1 = 1)\,\mathbb{P}(T_1 = 1) \cdots \mathbb{P}(T_1 = 1) \quad \text{(because } (X_n) \text{ is a renewal sequence)} \\ &= \mathbb{P}(T_1 = 1)^n. \qquad (3)\end{aligned}$$

For $n = 1$, (3) gives $\mathbb{P}(X_1 = 1) = \mathbb{P}(T_1 = 1)$. For $n = 2$, (3) gives

$$\mathbb{P}(X_1 = 1, X_2 = 1) = \mathbb{P}(T_1 = 1)^2 = \mathbb{P}(X_1 = 1)^2.$$

Then $\mathbb{P}(X_1=1)^2 = \mathbb{P}(X_1=1, X_2=1) \le \mathbb{P}(X_2=1)$.

Substituting (1) and (2) into the above estimate, we get

$$(\lambda+\mu)^2 \le \lambda^2+\mu^2.$$

This is a contradiction because λ and μ are two positive numbers.

4 We determine all $q \in (0,1]$ such that the sequence $(1, 0, q, q, q, \dots)$ is a potential sequence. That is to determine $q \in (0,1]$ such that there exists a renewal sequence $X_0, X_1, X_2, \dots$ such that

$$\begin{cases} \mathbb{P}(X_0=1)=1, \\ \mathbb{P}(X_1=1)=0, \\ \mathbb{P}(X_n=1)=q \quad \forall n\ge 2. \end{cases} \qquad (1)$$

Suppose $q\in(0,1]$ is such a value. The waiting time T_1 was defined by

$$T_1 = \inf\{n\ge 1 : X_n = 1\}.$$

Define

$$\Psi(s) = \sum_{n=0}^{\infty} \mathbb{P}(X_n=1)s^n,$$

$$\varphi(s) = \sum_{n=0}^{\infty} \mathbb{P}(T_1=n)s^n \qquad \forall s\in[0,1).$$

By Theorem 4, Friestedt-Gray page 493, $\Psi(s)=\frac{1}{1-\varphi(s)}$. This implies

$$\varphi(s) = \frac{\Psi(s)-1}{\Psi(s)} \qquad \forall s\in[0,1). \qquad (2)$$

Replacing the data in (1) into the expression of $\Psi(s)$, we get

$$\Psi(s) = 1+\sum_{n=2}^{\infty} q s^n = 1+\frac{qs^2}{1-s} \qquad \forall s\in[0,1).$$

Then (2) becomes
$$\varphi(s) = \frac{qs^2}{qs^2 - s + 1} \quad \forall s \in [0,1).$$

Put $\mu = \frac{1}{q} \in [1, \infty)$. Then
$$\varphi(s) = \frac{s^2}{s^2 - \mu s + \mu} \quad \forall s \in [0,1).$$

Thus,
$$\frac{s^2}{s^2 - \mu s + \mu} = \sum_{n=0}^{\infty} P(T_1 = n) s^n \quad \forall s \in [0,1). \qquad (3)$$

Suppose $\mu \neq 4$. Then the polynomial $s^2 - \mu s + \mu$ has ~~no~~ two distinct roots α_1, α_2. They are nonzero, either real or complex. Then

$$\mathrm{LHS}(3) = \frac{s^2}{(s-\alpha_1)(s-\alpha_2)} = \frac{s^2}{\alpha_1 - \alpha_2} \Bigg(\underbrace{\frac{\frac{1}{\alpha_2}}{1 - \frac{s}{\alpha_2}}}_{A_2} - \underbrace{\frac{\frac{1}{\alpha_1}}{1 - \frac{s}{\alpha_1}}}_{A_1} \Bigg).$$

For $0 \leq s < \min\{|\alpha_1|, |\alpha_2|\}$, we have the expansion

$$A_2 = \frac{1}{\alpha_2} \sum_{j=0}^{\infty} \left(\frac{s}{\alpha_2}\right)^j = \sum_{j=0}^{\infty} \frac{s^j}{\alpha_2^{j+1}},$$

$$A_1 = \frac{1}{\alpha_1} \sum_{j=0}^{\infty} \left(\frac{s}{\alpha_1}\right)^j = \sum_{j=0}^{\infty} \frac{s^j}{\alpha_1^{j+1}}.$$

Then
$$\mathrm{LHS}(3) = \frac{s^2}{\alpha_1 - \alpha_2} \sum_{j=0}^{\infty} \left(\frac{1}{\alpha_2^{j+1}} - \frac{1}{\alpha_1^{j+1}} \right) s^j = \frac{1}{\alpha_1 - \alpha_2} \sum_{j=0}^{\infty} \left(\frac{1}{\alpha_2^{j+1}} - \frac{1}{\alpha_1^{j+1}} \right) s^{j+2}.$$

Then (3) becomes

$$\frac{1}{\alpha_1 - \alpha_2} \sum_{n=2}^{\infty} \left(\frac{1}{\alpha_1^{n-1}} - \frac{1}{\alpha_2^{n-1}} \right) s^n = \sum_{n=0}^{\infty} P(T_1 = n) s^n \quad \forall 0 \leq s < \min\{|\alpha_1|, |\alpha_2|\}.$$

Thus, $\mathbb{P}(T_1=0)=0$, $\mathbb{P}(T_1=1)=0$,

$$\mathbb{P}(T_1=n)=\frac{1}{\alpha_1-\alpha_2}\left(\frac{1}{\alpha_2^{n-1}}-\frac{1}{\alpha_1^{n-1}}\right)\quad \forall n\geqslant 2. \qquad (4)$$

By (3), $$\sum_{n=0}^{\infty}\mathbb{P}(T_1=n)=\lim_{s\uparrow 1}\sum_{n=0}^{\infty}\mathbb{P}(T_1=n)s^n \quad \text{(Monotone Convergence Theorem)}$$

$$=\lim_{s\uparrow 1}\frac{s^2}{s^2-\mu s+\mu}$$

$$=1.$$

Thus, $\mathbb{P}(T_1=\infty)=0$.

Consider $1\leq \mu<4$

$$\alpha_1=\frac{\mu+i\sqrt{4\mu-\mu^2}}{2},\quad \alpha_2=\frac{\mu-i\sqrt{4\mu-\mu^2}}{2}=\bar{\alpha}_1.$$

Then (4) can be rewritten as

$$\mathbb{P}(T_1=n)=\frac{1}{(\alpha_1\alpha_2)^{n-1}}\,\frac{\alpha_1^{n-1}-\alpha_2^{n-1}}{\alpha_1-\alpha_2}=\frac{1}{\mu^{n-1}}\,\frac{2\,\mathrm{Im}\,\alpha_1^{n-1}}{2\,\mathrm{Im}\,\alpha_1}\quad \forall n\geqslant 2.$$

This implies $\mathrm{Im}\,\alpha_1^{n-1}\geqslant 0$ for all $n\geqslant 2$. Write $\alpha_1=re^{i\theta}$ for $r>0$, $\theta\in[0,2\pi)$. Since $\alpha_1\notin\mathbb{R}$, $\theta\notin\{0,\pi\}$. Thus $\theta\in(0,\pi)$ or $\theta\in(\pi,2\pi)$.

$$0\leq \mathrm{Im}\,\alpha_1^{n-1}=\mathrm{Im}\,r^{n-1}e^{i(n-1)\theta}=r^{n-1}\sin(n-1)\theta \quad \forall n\geqslant 2.$$

Thus, $\sin n\theta\geqslant 0$ for all $n\in\mathbb{N}$. In particular, $\sin\theta\geqslant 0$. Thus $\theta\in(0,\pi)$.

Then $\frac{\pi}{\theta}>1$. There exists $n_0\in\mathbb{N}$ such that $\frac{\pi}{\theta}<n_0<\frac{2\pi}{\theta}$. Then $\pi<n_0\theta<2\pi$. Then $\sin n_0\theta<0$, which is a contradiction.

Consider $\mu > 4$

$$\alpha_1 = \frac{\mu + \sqrt{\mu^2 - 4\mu}}{2}, \qquad \alpha_2 = \frac{\mu - \sqrt{\mu^2-4\mu}}{2}.$$

Then $\mathbb{P}(T_1 = n) = \frac{1}{(\alpha_1\alpha_2)^{n-1}} \cdot \frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2} = \frac{1}{\mu^{n-1}\sqrt{\mu^2-4\mu}}\left[\left(\frac{\mu+\sqrt{\mu^2-4\mu}}{2}\right)^{n-1} - \left(\frac{\mu - \sqrt{\mu^2-4\mu}}{2}\right)^{n-1}\right].$

We get the distribution of the waiting time T_1

$$\begin{cases} \mathbb{P}(T_1=0) = \mathbb{P}(T_1=1) = \mathbb{P}(T_1=\infty) = 0, \\ \mathbb{P}(T_1=n) = \frac{1}{\mu^{n-1}\sqrt{\mu^2-4\mu}}\left[\left(\frac{\mu+\sqrt{\mu^2-4\mu}}{2}\right)^{n-1} - \left(\frac{\mu-\sqrt{\mu^2-4\mu}}{2}\right)^{n-1}\right] \quad \forall n \geq 2 \end{cases} \tag{5}$$

Here $\mu = \frac{1}{q}$.

Consider $\mu = 4$

Then (3) becomes

$$\frac{s^2}{(s-2)^2} = \sum_{n=0}^{\infty} \mathbb{P}(T_1 = n)s^n \quad \forall s \in [0,1). \tag{6}$$

Letting $s \uparrow 1$ and using the Monotone Convergence Theorem on the right hand side, we get $1 = \sum_{n=0}^{\infty} \mathbb{P}(T_1 = n)$. Thus, $\mathbb{P}(T_1 = \infty) = 0$.

We know that

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots = \sum_{j=0}^{\infty} t^j \qquad \forall t \in (-1,1).$$

Take the derivative both sides,

$$\frac{1}{(1-t)^2} = 1 + 2t + 3t^2 + \cdots = \sum_{j=0}^{\infty}(j+1)t^j \qquad \forall t \in (-1,1).$$

Substituting t with $\frac{s}{2}$, we get

$$\frac{4}{(2-s)^2} = \sum_{j=0}^{\infty} \frac{j+1}{2^j} s^j \quad \forall s \in [0,1).$$

Then

$$LHS(6) = \sum_{j=0}^{\infty} \frac{j+1}{2^{j+2}} s^{j+2} = \sum_{n=2}^{\infty} \frac{n-1}{2^n} s^n.$$

Then (6) becomes

$$\sum_{n=2}^{\infty} \frac{n-1}{2^n} s^n = \sum_{n=0}^{\infty} \mathbb{P}(T_1 = n) s^n \quad \forall s \in [0,1).$$

Therefore,

$$\begin{cases} \mathbb{P}(T_1=0) = \mathbb{P}(T_1=1) = \mathbb{P}(T_1=\infty) = 0, \\ \mathbb{P}(T_1 = n) = \frac{n-1}{2^n} \quad \forall n \geq 2. \end{cases} \tag{7}$$

So far we have showed that if a renewal sequence exists, q must be in $(0, \frac{1}{4}]$. Also, in that case the waiting time distribution is given by (5) for $q \in (0, \frac{1}{4})$, and by (7) for $q = \frac{1}{4}$.

Now we show that every $q \in (0, \frac{1}{4}]$ is an admissible value. That is, for a fixed $q \in (0, \frac{1}{4}]$ we determine a renewal sequence $X_0, X_1, X_2, \ldots$ whose potential sequence is $(1, 0, q, q, q, \ldots)$. Let $\tilde{T}_1$ be a $\{0,1,2,\ldots\}$-valued random variable with distribution given by (5) if $q \in (0, \frac{1}{4})$, or given by (7) if $q = \frac{1}{4}$. Then

$$\frac{s^2}{s^2-\mu s+\mu} = \sum_{n=0}^{\infty} P(\tilde{T}_1 = n)\, s^n \quad \forall s \in [0,1),$$

where $\mu = 1/q$. Denote

$$\tilde{\varphi}(s) = \sum_{n=0}^{\infty} P(\tilde{T}_1 = n) s^n = \frac{s^2}{s^2-\mu s+\mu} \quad \forall s \in [0,1). \qquad (8)$$

This is the probability generating function of $\tilde{T}_1$. Let $Z_1 = \tilde{T}_1, Z_2, Z_3, \dots$ be an independent and identically distributed sequence of random variables. Define $\tilde{T}_0 = 0$ and $\tilde{T}_n = Z_1 + Z_2 + \dots + Z_n$ for each $n \geq 2$. Define a sequence of random variables $\tilde{X}_0, \tilde{X}_1, \tilde{X}_2, \dots$ as follows.

$$\tilde{X}_0(\omega) = 1 \quad \forall \omega \in \Omega,$$

$$\tilde{X}_n(\omega) = \begin{cases} 1 & \text{if } n = \tilde{T}_m(\omega) \text{ for some } m \geq 0, \\ 0 & \text{otherwise} \end{cases} \qquad \forall n \geq 1.$$

$\tilde{X}_n$ is indeed a random variable because

$$\{\omega : \tilde{X}_n(\omega) = 1\} = \{\omega : \exists m \geq 0 \text{ such that } n = \tilde{T}_m(\omega)\}$$
$$= \{\omega : \exists m \geq 0 \text{ such that } \omega \in \tilde{T}_m^{-1}(\{n\})\}$$
$$= \bigcup_{m=0}^{\infty} \tilde{T}_m^{-1}(\{n\}),$$

which is a measurable set. We now show that for $r \geq 1$,

$$\tilde{T}_r(\omega) = \inf\{n > \tilde{T}_{r-1}(\omega) : \tilde{X}_n(\omega) = 1\} \quad \text{a.s.}$$

Put $A = \{\omega : \tilde{T}_r(\omega) = \tilde{T}_{r-1}(\omega)\}$. Then $A \subset \{\omega : Z_r(\omega) = 0\}$. Then

$$\mathbb{P}(A) \leq \mathbb{P}(Z_r = 0) = \mathbb{P}(\tilde{T}_1 = 0) = 0.$$

Let $\omega \in A^c$. By the definition of $\tilde{X}_n$, $\tilde{X}_{\tilde{T}_r(\omega)}(\omega) = 1$. Thus $\tilde{T}_r(\omega) \in \{n > \tilde{T}_{r-1}(\omega) : \tilde{X}_n(\omega) = 1\}$. For each $n > \tilde{T}_{r-1}(\omega)$ such that $\tilde{X}_n(\omega) = 1$, there exists $m \geq 0$ such that $n = \tilde{T}_m(\omega)$. Because $\tilde{T}_k$ is an increasing sequence and $n > \tilde{T}_{r-1}(\omega)$, $m > r-1$. Then $m \geq r$. Then $\tilde{T}_m(\omega) \geq \tilde{T}_r(\omega)$. Then $n \geq \tilde{T}_r(\omega)$. Then

$$\tilde{T}_r(\omega) \leq \inf\{n > \tilde{T}_{r-1}(\omega) : \tilde{X}_n(\omega) = 1\}.$$

Thus, $\tilde{T}_r(\omega) = \inf\{n > \tilde{T}_{r-1}(\omega) : \tilde{X}_n(\omega) = 1\}$ a.s.

For $n \geq 2$ and $k_1, k_2, \dots, k_n \geq 1$,

$$\begin{aligned}\mathbb{P}(\tilde{T}_1 - \tilde{T}_0 = k_1, \tilde{T}_2 - \tilde{T}_1 = k_2, \dots, \tilde{T}_n - \tilde{T}_{n-1} = k_n) &= \mathbb{P}(Z_1 = k_1, Z_2 = k_2, \dots, Z_n = k_n) \\ &= \mathbb{P}(Z_1 = k_1)\mathbb{P}(Z_2 = k_2)\dots\mathbb{P}(Z_n = k_n) \\ &= \mathbb{P}(\tilde{T}_1 = k_1)\mathbb{P}(\tilde{T}_1 = k_2)\dots\mathbb{P}(\tilde{T}_1 = k_n).\end{aligned}$$

Thus, $\tilde{X}_0, \tilde{X}_1, \tilde{X}_2, \dots$ is a renewal sequence. Put

$$\tilde{\Psi}(s) = \sum_{n=0}^{\infty} \mathbb{P}(\tilde{X}_n = 1)\, s^n \quad \forall s \in [0,1). \qquad (9)$$

By Theorem 4, Frrestedt-Gray page 493,

$$\tilde{\Psi}(s) = \frac{1}{1 - \tilde{\varphi}(s)}.$$

Replacing $\tilde{\varphi}(s)$ given by (8) into this expression, we get

$$\tilde{\Psi}(s) = \frac{1}{1 - \frac{s^2}{s^2 - \mu s + \mu}} = \frac{s^2 - \mu s + \mu}{-\mu s + \mu} = 1 + \frac{s^2}{-\mu s + \mu} = 1 + \frac{q s^2}{1 - s}$$

$$= 1 + q s^2 (1 + s + s^2 + s^3 + \dots)$$

$$= 1 + \sum_{n=2}^{\infty} q s^n \qquad \forall s \in [0,1).$$

Thus, $\displaystyle\sum_{n=0}^{\infty} \mathbb{P}(\tilde{X}_n = 1) s^n \overset{(9)}{=} 1 + \sum_{n=2}^{\infty} q s^n \qquad \forall s \in [0,1).$

Then

$$\mathbb{P}(\tilde{X}_n = 1) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n = 1, \\ q & \text{if } n \geq 2. \end{cases}$$

Therefore, $(1, 0, q, q, q, \dots)$ is a potential sequence.