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Math 8652: Theory of Probability

Homework #7

(1) Let $\xi_0, \xi_1, \xi_2, \ldots$ be a real-valued random sequence on $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$ be a filtration with $\mathcal{F}_0 = \{\phi, \Omega\}$. Suppose each ξ_n is $\mathcal{F}_n$-measurable and $E|\xi_n| < \infty$. We show that $(\xi_n, \mathcal{F}_n)_{n \geq 0}$ is a submartingale if and only if there exist random sequences $A_0, A_1, A_2, \ldots$ and $m_0, m_1, m_2, \ldots$ such that

(i) $\xi_n = A_n + m_n \quad \forall n \geq 0$,

(ii) $(m_n, \mathcal{F}_n)_{n \geq 0}$ is a martingale,

(iii) $0 = A_0 \leq A_1 \leq A_2 \leq \ldots$

(iv) A_n is $\mathcal{F}_{n-1}$-measurable for every $n \geq 2$.

($\Leftarrow$) Suppose there exist random sequences $(A_n)_{n \geq 0}$ and $(m_n)_{n \geq 0}$ that satisfy (i), (ii), (iii), (iv). For each $n \geq 0$,

$$E(\xi_{n+1} | \mathcal{F}_n) = E(A_{n+1} + m_{n+1} | \mathcal{F}_n) = E(A_{n+1} | \mathcal{F}_n) + E(m_{n+1} | \mathcal{F}_n). \quad (1)$$

Since A_{n+1} is $\mathcal{F}_n$-measurable, $E(A_{n+1} | \mathcal{F}_n) = A_{n+1}$. Since $(m_k, \mathcal{F}_k)_{k \geq 0}$ is a martingale, $E(m_{n+1} | \mathcal{F}_n) = m_n$. Then (1) becomes

$$E(\xi_{n+1} | \mathcal{F}_n) = A_{n+1} + m_n \geq A_n + m_n = \xi_n.$$

Therefore, $(\xi_n, \mathcal{F}_n)_{n \geq 0}$ is a submartingale.

($\Rightarrow$) Suppose $(\xi_n, \mathcal{F}_n)_{n \geq 0}$ is a submartingale. Set $A_0 = 0$ and

$$A_n = \sum_{k=1}^{n} \left[E(\xi_k | \mathcal{F}_{k-1}) - \xi_{k-1} \right] \quad \forall n \in \mathbb{N}. \quad (2)$$

Each term in the sum is well-defined because ξ_{k-1} is real-valued. A sequence

$(A_n)_{n\geq 0}$ satisfying (i), (ii), (iii), (iv), if exists, is unique and given by (2).

Indeed,

$$E(\xi_{n+1}|F_n) = A_{n+1}+m_n = (A_{n+1}-A_n)+(A_n+m_n) = (A_{n+1}-A_n)+\xi_n,$$

which implies

$$A_{n+1} = \sum_{k=0}^{n}(A_{k+1}-A_k) = \sum_{k=0}^{n}[E(\xi_{k+1}|F_k)-\xi_k].$$

Because

$$E(\xi_k|F_{k-1}) \geq \xi_{k-1} \quad a.s. \quad \forall k\in\mathbb{N},$$

the formula (2) implies $0=A_0\leq A_1\leq A_2\leq\cdots$ For $1\leq k\leq n$, $E(\xi_k|F_{k-1})-\xi_{k-1}$ is F_{k-1}-measurable. Since $F_{k-1}\subset F_{n-1}$, $E(\xi_k|F_{k-1})-\xi_{k-1}$ is also F_{n-1}-measurable. Then A_n is the sum of F_{n-1}-measurable functions. Thus, A_n is F_{n-1}-measurable.

Next, we show that $EA_n<\infty$ for each $n\in\mathbb{N}$. Denote $\eta_k=E(\xi_k|F_{k-1})$ for each $k\in\mathbb{N}$. By the definition of conditional expectations,

$$E\eta_k I_A = E\xi_k I_A \quad \forall A\in F_{k-1}.$$

For $A=\{\omega:\eta_k(\omega)>0\}$, we get $E(\eta_k)_+ = E\xi_k I_{\eta_k>0}$. For $A=\{\omega:\eta_k(\omega)<0\}$, we get $E(\eta_k)_- = E\xi_k I_{\eta_k<0}$. Then

$$E|\eta_k| = E(\eta_k)_+ + E(\eta_k)_- = E\xi_k I_{\eta_k>0} + E\xi_k I_{\eta_k<0} = E\xi_k I_{\eta_k\neq 0}$$
$$\leq E|\xi_k|<\infty.$$

Then

$$EA_n = E\sum_{k=1}^{n}(\eta_k-\xi_{k-1}) \leq E\sum_{k=1}^{n}(|\eta_k|+|\xi_{k-1}|) = \sum_{k=1}^{n}(E|\eta_k|+E|\xi_{k-1}|)<\infty.$$

Then $A_n<\infty$ almost surely. This allows to set $m_n=\xi_n-A_n$. We show that $(m_n,F_n)_{n\geq 0}$ is a martingale.

$$E|m_n| = E|\xi_n-A_n| \leq E|\xi_n|+EA_n<\infty.$$

Since A_n is $\mathcal{F}_{n-1}$-measurable, it is also $\mathcal{F}_n$-measurable. Then m_n is $\mathcal{F}_n$-measurable.

For $n \geq 0$,

$$E(m_{n+1} | \mathcal{F}_n) = E(\xi_{n+1} | \mathcal{F}_n) - \underbrace{E(A_{n+1} | \mathcal{F}_n)}_{= A_{n+1} \text{ because } A_{n+1} \text{ is } \mathcal{F}_n\text{-measurable}}$$

$$= E(\xi_{n+1} | \mathcal{F}_n) - A_{n+1}$$

$$\overset{(2)}{=} E(\xi_{n+1} | \mathcal{F}_n) - \sum_{k=1}^{n+1} [E(\xi_k | \mathcal{F}_{k-1}) - \xi_{k-1}]$$

$$= E(\xi_{n+1} | \mathcal{F}_n) - (E(\xi_{n+1} | \mathcal{F}_n) - \xi_n) - \underbrace{\sum_{k=1}^{n} [E(\xi_k | \mathcal{F}_{k-1}) - \xi_{k-1}]}_{= A_n}$$

$$= \xi_n - A_n$$

$$= m_n.$$

Therefore, $(m_n, \mathcal{F}_n)_{n\geq 0}$ is a martingale.

(2) Let $X_1, X_2, X_3, \ldots$ be an independent and identically distributed sequence of random variables, each having uniform distribution on $[0,2]$. Set

$$\xi_n = \prod_{k=1}^{n} X_k \qquad \forall n \in \mathbb{N}.$$

We show that $\xi_n \to 0$ a.s. as $n \to \infty$. It can be done by using either the Strong Law of Large Numbers or the Martingale Convergence Theorem.

<u>Method 1</u> (using the Strong Law of Large Numbers)

Because X_n is uniformly distributed on $[0,2]$, $X_n > 0$ almost surely. Set $Y_n = \log X_n$ and $S_n = Y_1 + Y_2 + \cdots + Y_n$. Then $S_n = \log \xi_n$ almost surely. Because the sequence $X_1, X_2, X_3, \ldots$ is independent, the sequence $Y_1 = \log X_1,$

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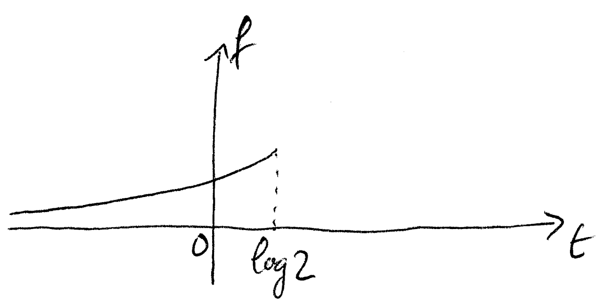
$Y_2 = \log X_2,\ Y_3 = \log X_3, \dots$ is also independent. For each $t \in \mathbb{R}$,

$$P(Y_n \le t) = P(\log X_n \le t) = P(X_n \le e^t) = \begin{cases} \frac{1}{2}e^t & \text{if } e^t \le 2, \\ 1 & \text{if } e^t > 2 \end{cases}$$

$$= \begin{cases} \frac{1}{2}e^t & \text{if } t \le \log 2, \\ 1 & \text{if } t > \log 2. \end{cases}$$

Thus, $Y_1, Y_2, Y_3, \dots$ have the same distribution whose density function is

$$f(t) = \frac{d}{dt} P(Y_n \le t) = \begin{cases} \frac{1}{2} e^t & \text{if } t \le \log 2, \\ 0 & \text{if } t > \log 2. \end{cases}$$



Then
$$E Y_n = \int_{-\infty}^{\infty} t f(t)\, dt = \int_{-\infty}^{\log 2} \frac{1}{2} t e^t \, dt = \frac{1}{2}(t-1)e^t \Big|_{t \to -\infty}^{t=\log 2} = \log 2 - 1.$$

Denote $c = \log 2 - 1 < 0$. Then $EY_n = c$. By the Strong Law of Large Numbers,

$$\frac{S_n}{n} = \frac{Y_1 + Y_2 + \dots + Y_n}{n} \longrightarrow c \quad \text{a.s.} \quad \text{as } n \to \infty.$$

Put $A = \{\omega \in \Omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = c\}$. Then $P(A) = 1$. For each $\omega \in A$, there exists $n_0 \in \mathbb{N}$ such that $\frac{S_n(\omega)}{n} < \frac{c}{2}$ for all $n > n_0$. Then $S_n(\omega) < \frac{cn}{2}$ for all $n > n_0$. Then $\lim_{n\to\infty} S_n(\omega) = -\infty$. Thus, $\lim_{n\to\infty} S_n = -\infty$ a.s. Then

$$\xi_n = e^{S_n} \to 0 \quad \text{a.s.} \quad \text{as } n\to\infty.$$

Method 2 (using the Martingale Convergence Theorem)

Put $\mathcal{F}_0 = \{\phi, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ for each $n \in \mathbb{N}$. It is clear that $(\mathcal{F}_n)_{n\geq 0}$ is an increasing sequence of σ-fields and that ξ_n is $\mathcal{F}_n$-measurable. Because each X_n is uniformly distributed on $[0,2]$, $EX_n = \frac{0+2}{2} = 1$. Then

$$\begin{aligned} E\xi_n &= E(X_1X_2\cdots X_n) \\ &= (EX_1)(EX_2)\cdots(EX_n) \quad (\text{since } X_1, X_2, \dots, X_n \text{ are independent}) \\ &= 1. \end{aligned}$$

In particular, $E|\xi_n| < \infty$. For $A \in \mathcal{F}_n$,

$$\begin{aligned} E\xi_{n+1} I_A = E(X_1X_2\cdots X_{n+1} I_A) &= E(X_1X_2\cdots X_n I_A X_{n+1}) \\ &= E(X_1X_2\cdots X_n I_A)\, EX_{n+1} \quad (\text{because } X_{n+1} \text{ and } \mathcal{F}_n \text{ are independent}) \\ &= E(X_1X_2\cdots X_n I_A) \\ &= E\xi_n I_A. \end{aligned}$$

Thus, $\xi_n = E(\xi_{n+1} | \mathcal{F}_n)$. We have showed that $(\xi_n, \mathcal{F}_n)_{n\geq 0}$ is a martingale.

$$\sup_{n\in\mathbb{N}} E|\xi_n| = \sup_{n\in\mathbb{N}} E\xi_n = 1 < \infty.$$

By the Martingale Convergence Theorem, the sequence (ξ_n) converges almost surely. Denote $\xi_\infty = \lim_{n\to\infty} \xi_n \geq 0$. We show that $\xi_\infty = 0$ almost surely.

Let $A=\{\omega : \xi_\infty(\omega)>0\}$. For each $\omega_0 \in A$,

$$\lim_{n\to\infty} X_n(\omega_0) = \lim_{n\to\infty} \frac{\xi_n(\omega_0)}{\xi_{n-1}(\omega_0)} = \frac{\xi_\infty(\omega_0)}{\xi_\infty(\omega_0)} = 1.$$

There exists $n_0 \in \mathbb{N}$ such that $X_n(\omega_0) \in \left(\frac{3}{4}, \frac{5}{4}\right) := I$ for all $n \geq n_0$.

Put
$$A_n = \{\omega \in \Omega : X_m(\omega) \in I \ \ \forall m \geq n\}.$$

Then $\omega_0 \in A_{n_0} \subset \bigcup_{n=1}^{\infty} A_n$. Hence, $A \subset \bigcup_{n=1}^{\infty} A_n$.

$$\begin{aligned}
P(A_n) &= P(X_n \in I, X_{n+1} \in I, X_{n+2} \in I, \dots) \\
&\leq P(X_n \in I, X_{n+1} \in I, \dots, X_{n+k} \in I) \quad \forall k \in \mathbb{N} \\
&= P(X_n \in I)\, P(X_{n+1} \in I) \cdots P(X_{n+k} \in I) \quad \forall k \in \mathbb{N} \\
&= [P(X_1 \in I)]^{k+1} \quad \forall k \in \mathbb{N} \\
&= \left(\frac{|I|}{2}\right)^{k+1} = \left(\frac{1}{4}\right)^{k+1} \quad \forall k \in \mathbb{N}.
\end{aligned}$$

Thus, $P(A_n) = 0$. Then
$$P(A) \leq \sum_{n=1}^{\infty} P(A_n) = 0,$$

which implies $P(A) = 0$.

3 Let $Y_1, Y_2, Y_3, \dots$ be an independent and identically distributed sequence of $\mathbb{Z}^d$-valued random variables. Let $(e_1, e_2, \dots, e_d)$ be the standard basis of $\mathbb{Z}^d$.

Suppose $d \geq 3$ and
$$P(Y_1 = e_k) = P(Y_1 = -e_k) = \frac{1}{2d} \quad \forall 1 \leq k \leq d.$$

Consider the simple symmetric random walk $(X_n)_{n\geq 0}$ in $\mathbb{Z}^d$:

$$X_0 = 0, \quad X_n = Y_1 + Y_2 + \dots + Y_n \quad \forall n \in \mathbb{N}.$$

We show that the probability to return to the origin (after the first step) is strictly less than one.

First, we show that $Z_n = I_{X_n = 0}$ $(n \geq 0)$ is a renewal sequence. Put $T_0 = 0$, $T_m = \inf\{n > T_{m-1} : Z_n = 1\}$ for all $m \in \mathbb{N}$. For $m \in \mathbb{N}$,

$$\begin{aligned}
\{\omega : T_1(\omega) > m\} = (T_1 > m) &= (Z_1 = 0, Z_2 = 0, \dots, Z_m = 0) \\
&= (X_1 \neq 0, X_2 \neq 0, \dots, X_m \neq 0 \\
&= (Y_1 \neq 0, Y_1 + Y_2 \neq 0, \dots, Y_1 + Y_2 + \dots + Y_m \neq 0) \\
&= \bigcup_{r \in A_m} (Y_1 = r_1, Y_2 = r_2, \dots, Y_m = r_m)
\end{aligned}$$

where $A_m = \{r = (r_1, r_2, \dots, r_m) \in (\mathbb{Z}^d)^m : r_1 \neq 0, r_1 + r_2 \neq 0, \dots, r_1 + r_2 + \dots + r_m \neq 0\}$.

Then

$$\begin{aligned}
P(T_1 > m) &= \sum_{r \in A_m} P(Y_1 = r_1, Y_2 = r_2, \dots, Y_m = r_m) \\
&= \sum_{r \in A_m} P(Y_1 = r_1) P(Y_2 = r_2) \cdots P(Y_m = r_m) \\
&= \sum_{r \in A_m} P(Y_1 = r_1) P(Y_1 = r_2) \cdots P(Y_1 = r_m).
\end{aligned}$$

This implies that the distribution of T_1 depends only on the distribution of Y_1.

Now let $n \geq 2$ and $k_1, k_2, \dots, k_n \geq 1$. Denote $l_1 = k_1$, $l_2 = k_1 + k_2, \dots, l_n = k_1 + k_2 + \dots + k_n$. We have

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$$\{\omega\in\Omega : T_1(\omega)-T_0(\omega)=k_1,\ T_2(\omega)-T_1(\omega)=k_2,\dots,\ T_n(\omega)-T_{n-1}(\omega)=k_n\}$$

$$= (T_1-T_0=k_1,\ T_2-T_1=k_2,\dots,\ T_n-T_{n-1}=k_n)$$

$$= (T_1=\ell_1,\ T_2=\ell_2,\dots,\ T_n=\ell_n)$$

$$= (Z_1=\dots=Z_{\ell_1-1}=0,\ Z_{\ell_1}=1,\ Z_{\ell_1+1}=\dots=Z_{\ell_2-1}=0,\ Z_{\ell_2}=1,\ \dots,$$
$$Z_{\ell_{n-1}+1}=\dots=Z_{\ell_n-1}=0,\ Z_{\ell_n}=1)$$

$$= (X_1,\dots,X_{\ell_1-1}\neq 0,\ X_{\ell_1}=0,\ X_{\ell_1+1},\dots,X_{\ell_2-1}\neq 0,\ X_{\ell_2}=0,\dots,\ X_{\ell_{n-1}+1},\dots,$$
$$X_{\ell_n-1}\neq 0,\ X_{\ell_n}=0)$$

$$= \underbrace{(X_1,\dots,X_{\ell_1-1}\neq 0,\ X_{\ell_1}=0,\dots,\ X_{\ell_{n-2}+1},\dots,X_{\ell_{n-1}-1}\neq 0,\ X_{\ell_{n-1}}=0)}_{A}\underbrace{(X_{\ell_{n-1}+1},\dots,X_{\ell_n-1}\neq 0,\ X_{\ell_n}=0)}_{B} \qquad (1)$$

The event A is $\sigma(Y_1,Y_2,\dots,Y_{\ell_{n-1}})$-measurable and

$$A=(T_1-T_0=k_1,\dots,T_{n-1}-T_{n-2}=k_{n-1}),$$

$$AB = A(X_{\ell_{n-1}+1},\dots,X_{\ell_n-1}\neq 0,\ X_{\ell_n}=0)$$

$$= A\underbrace{(Y_{\ell_{n-1}+1},\ Y_{\ell_{n-1}+1}+Y_{\ell_{n-1}+2},\dots,\ Y_{\ell_{n-1}+1}+\dots+Y_{\ell_n-1}\neq 0,\ Y_{\ell_{n-1}+1}+\dots+Y_{\ell_n}=0)}_{C}$$

(because $X_{\ell_{n-1}}=0$).

The event C is $\sigma(Y_{\ell_{n-1}+1},\dots,Y_{\ell_n})$-measurable. By the independence of the sequence $(Y_n)_{n\geq 1}$, $\mathbb{P}(AC)=\mathbb{P}(A)\,\mathbb{P}(C)$. Then (1) yields

$$\mathbb{P}(T_1-T_0=k_1, T_2-T_1=k_2, \dots, T_n-T_{n-1}=k_n) = \mathbb{P}(AB) = \mathbb{P}(AC) = \mathbb{P}(A)\mathbb{P}(C)$$
$$\overset{(2)}{=\!=} \mathbb{P}(T_1-T_0=k_1, \dots, T_{n-1}-T_{n-2}=k_{n-1})\mathbb{P}(C). \tag{3}$$

Define $Y_j' = Y_{\ell_{n-1}+j}$ for $j \in \mathbb{N}$,

$$X_j' = \begin{cases} 0 & \text{if } j=0, \\ Y_1'+\dots+Y_j' & \text{if } j \geq 1, \end{cases}$$

$$T_1' = \inf\{j \geq 1 : X_j' = 0\}.$$

Then $(X_j')_{j\geq 0}$ is a random walk in $\mathbb{Z}^d$ with waiting time T_1'. Also,

$$C = (X_1' \neq 0, X_2' \neq 0, \dots, X_{k_n-1}' \neq 0, X_{k_n}' = 0) = (T_1' = k_n).$$

We showed earlier that the distribution of the waiting time depends only on the distribution of the single step, which is $Y_1' = Y_{\ell_{n-1}+1}$. Since Y_1' and Y_1 have the same distribution, T_1' and T_1 have the same distribution. Then $\mathbb{P}(C) = \mathbb{P}(T_1' = k_n) = \mathbb{P}(T_1 = k_n)$. Then (3) is rewritten as

$$\mathbb{P}(T_1-T_0=k_1, \dots, T_{n-1}-T_{n-2}=k_{n-1}, T_n-T_{n-1}=k_n)$$
$$= \mathbb{P}(T_1-T_0=k_1, \dots, T_{n-1}-T_{n-2}=k_{n-1})\,\mathbb{P}(T_1=k_n). \tag{4}$$

The equation

$$\mathbb{P}(T_1-T_0=k_1, \dots, T_n-T_{n-1}=k_n) = \mathbb{P}(T_1=k_1)\cdots\mathbb{P}(T_1=k_n) \tag{5}$$

is true for $n=1$. By (4) and an induction argument in $n \geq 1$, we conclude that (5) is true for all $n \geq 1$. We have showed that $Z_n = I_{X_n = 1}$ is a renewal sequence.

Next, we show that $P(T_1 < \infty) < 1$. Put

$$\Psi(s) = \sum_{n=0}^{\infty} P(X_n = 0)\, s^n,$$

$$\phi(s) = \sum_{n=0}^{\infty} P(T_1 = n)\, s^n. \qquad \forall s \in [0,1)$$

By Theorem 4, Fristedt-Gray page 493, $\Psi(s) = \dfrac{1}{1-\varphi(s)}$ for all $s \in [0,1)$.

We have

$$\begin{aligned}
P(T_1 < \infty) &= \sum_{n=0}^{\infty} P(T_1 = n) \\
&= \lim_{s \uparrow 1} \sum_{n=0}^{\infty} P(T_1 = n)\, s^n \quad \text{(by the Monotone Convergence Theorem)} \\
&= \lim_{s \to 1^-} \phi(s) \\
&= \lim_{s \to 1^-} \frac{\Psi(s) - 1}{\Psi(s)}.
\end{aligned}$$

To show $P(T_1 < \infty) < 1$, it suffices to show $\lim_{s \to 1^-} \Psi(s)$ exists and belongs to $[1, \infty)$. On the other hand,

$$\begin{aligned}
\lim_{s \to 1^-} \Psi(s) &= \lim_{s \uparrow 1} \sum_{n=0}^{\infty} P(X_n = 0)\, s^n \\
&= \sum_{n=0}^{\infty} P(X_n = 0) \quad \text{(by the Monotone Convergence Theorem)}
\end{aligned}$$

This series is greater than or equal to 1 because $P(X_0 = 0) = 1$. We only need to show $\sum_{n=0}^{\infty} P(X_n=0) < \infty$.

Denote by $(\cdot,\cdot)$ the usual inner product in $\mathbb{R}^d$. We first show that

$$\frac{1}{2^d}\int_{[-1,1]^d} e^{i\pi(\lambda,x)}\,d\lambda = \begin{cases} 1 & \text{if } x=0, \\ 0 & \text{if } x\in\mathbb{Z}^d\setminus\{0\}. \end{cases} \qquad (6)$$

Write $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_d)\in\mathbb{R}^d$, $x=(x_1,x_2,\dots,x_d)\in\mathbb{Z}^d$. Then

$$\text{LHS}(6) = \frac{1}{2^d}\int_{[-1,1]^d} e^{i\pi(\lambda_1x_1+\dots+\lambda_dx_d)}\,d\lambda$$

$$= \frac{1}{2^d}\int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1} e^{i\pi\lambda_1x_1}\cdots e^{i\pi\lambda_dx_d}\,d\lambda_1\,d\lambda_2\dots d\lambda_n$$

$$= \left(\frac{1}{2}\int_{-1}^{1} e^{i\pi\lambda_1x_1}d\lambda_1\right)\cdots\left(\frac{1}{2}\int_{-1}^{1} e^{i\pi\lambda_dx_d}d\lambda_d\right). \qquad (7)$$

We have

$$\frac{1}{2}\int_{-1}^{1} e^{i\pi\lambda_kx_k}d\lambda_k = \begin{cases} 1 & \text{if } x_k=0, \\ \frac{1}{2i\pi x_k}\left(e^{i\pi x_k}-e^{-i\pi x_k}\right) & \text{if } x_k\neq 0 \end{cases}$$

$$= \begin{cases} 1 & \text{if } x_k=0, \\ \frac{\sin\pi x_k}{\pi x_k} = 0 & \text{if } x_k=0. \end{cases}$$

Then (7) yields

$$\text{LHS}(6) = \begin{cases} 1 & \text{if } x_1=x_2=\dots=x_d=0, \\ 0 & \text{if } x_k\neq 0 \text{ for some } 1\le k\le d. \end{cases}$$

We have proved (6). Substituting X_n for x in (6), we get

$$I_{X_n=0} = \frac{1}{2^d}\int_{[-1,1]^d} e^{i\pi(\lambda,X_n)}\,d\lambda.$$

Take the expectation of both sides,

$$\mathbb{P}(X_n = 0) = \frac{1}{2^d} E \int_{[-1,1]^d} e^{i\pi(\lambda, X_n)} d\lambda.$$

The integrand is a bounded function in $(\lambda,\omega) \in [-1,1]^d \times \Omega$. Then Fubini's theorem allows us to switch the order of integration. Then

$$\mathbb{P}(X_n = 0) = \frac{1}{2^d} \int_{[-1,1]^d} E e^{i\pi(\lambda, X_n)} d\lambda. \qquad (8)$$

We have

$$E e^{i\pi(\lambda, X_n)} = E\left[e^{i\pi(\lambda, Y_1)} e^{i\pi(\lambda, Y_2)} \cdots e^{i\pi(\lambda, Y_n)} \right]$$

$$= \left(E e^{i\pi(\lambda, Y_1)}\right)\left(E e^{i\pi(\lambda, Y_2)}\right) \cdots \left(E e^{i\pi(\lambda, Y_n)}\right) \quad \text{(because } Y_1, \dots, Y_n \text{ are independent)}$$

$$= \left(E e^{i\pi(\lambda, Y_1)}\right)^n \quad \text{(because } Y_1, \dots, Y_n \text{ have the same distribution).}$$

We compute

$$E e^{i\pi(\lambda, Y_1)} = \sum_{k=1}^{d} e^{i\pi(\lambda, e_k)} \mathbb{P}(Y_1 = e_k) + \sum_{k=1}^{d} e^{i\pi(\lambda, -e_k)} \mathbb{P}(Y_1 = -e_k)$$

$$= \frac{1}{2d} \sum_{k=1}^{d} e^{i\pi\lambda_k} + \frac{1}{2d} \sum_{k=1}^{d} e^{-i\pi\lambda_k}$$

$$= \frac{1}{2d} \sum_{k=1}^{d} \left(e^{i\pi\lambda_k} + e^{-i\pi\lambda_k}\right)$$

$$= \frac{1}{d} \sum_{k=1}^{d} \cos(\pi\lambda_k).$$

Put $f(\lambda) = \frac{1}{d} \sum_{k=1}^{d} \cos(\pi\lambda_k)$. Then $E e^{i\pi(\lambda, X_n)} = \left(E e^{i\pi(\lambda, Y_1)}\right)^n = f(\lambda)^n$.

Then (8) becomes
$$P(X_n=0) = \frac{1}{2^d}\int_{[-1,1]^d} f(\lambda)^n\,d\lambda.$$

Then
$$\sum_{n=0}^{\infty} P(X_n=0) = \frac{1}{2^d}\sum_{n=0}^{\infty}\int_{[-1,1]^d} f(\lambda)^n\,d\lambda = \frac{1}{2^d}\lim_{m\to\infty}\int_{[-1,1]^d}\sum_{n=0}^{m-1} f(\lambda)^n\,d\lambda$$
$$= \frac{1}{2^d}\lim_{m\to\infty}\int_{[-1,1]^d}\frac{1-f(\lambda)^m}{1-f(\lambda)}\,d\lambda. \qquad (9)$$

By the definition of f, $f(\lambda)\in(-1,1)$ for almost every $\lambda\in[-1,1]^d$. Then the integrand is nonnegative and is bounded from above by $\frac{2}{1-f(\lambda)}$. Thus, (9) implies
$$\sum_{n=0}^{\infty} P(X_n=0) \le \frac{1}{2^d}\int_{[-1,1]^d}\frac{2}{1-f(\lambda)}\,d\lambda$$
$$= \frac{1}{2^{d-1}}\int_{[-1,1]^d}\frac{1}{1-\frac{\cos\pi\lambda_1+\dots+\cos\pi\lambda_k}{d}}\,d\lambda$$
$$= \underbrace{\frac{d}{2^{d-1}}\int_{[-1,1]^d}\frac{1}{(1-\cos\pi\lambda_1)+\dots+(1-\cos\pi\lambda_d)}\,d\lambda}_{\{1\}}.$$

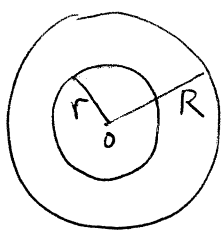
We need to show $\{1\}<\infty$. The function $g:[-\pi,\pi]\to\mathbb{R}$,
$$g(t) = \begin{cases}\frac{1-\cos t}{t^2} & \text{if } t\in[-\pi,\pi]\setminus\{0\},\\ \frac{1}{2} & \text{if } t=0\end{cases}$$
is continuous and (strictly) positive in $[-\pi,\pi]$. Thus $\min_{[-\pi,\pi]} g(t)=\varepsilon>0$.

Consequently, $1-\cos(\pi\lambda_k) \geq \varepsilon(\pi\lambda_k)^2$ for all $1 \leq k \leq d$. Then

$$\{1\} \leq \int_{[-1,1]^d} \frac{1}{\varepsilon\pi^2(\lambda_1^2+\cdots+\lambda_d^2)}\,d\lambda$$

$$\leq \frac{1}{\varepsilon\pi^2} \underbrace{\int_{B_R} \frac{1}{\lambda_1^2+\cdots+\lambda_d^2}\,d\lambda}_{\{2\}},$$

where B_R is a ball in $\mathbb{R}^d$, centered at the origin with radius $R \in (0,\infty)$, such that $[-1,1]^d \subset B_R$. It remains to show $\{2\} < \infty$.



$$\{2\} = \int_0^R \int_{\partial B_r} \frac{1}{r^2}\,dS\,dr = \int_0^R \frac{|\partial B_r|}{r^2}\,dr$$

$$= \int_0^R \frac{|\partial B_1|\,r^{d-1}}{r^2}\,dr = |\partial B_1| \underbrace{\int_0^R r^{d-3}\,dr}_{<\infty \text{ because } d \geq 3.}$$

④ Let (S,Σ) be a measurable space, $p(x,B)$, $x \in S$, $B \in \Sigma$, be a transition kernel, and π_0 be a probability measure on (S,Σ). We define

$$\Omega = \prod_{k=0}^{\infty} S = S \times S \times S \times \cdots,$$

$$\mathcal{F} = \bigotimes_{k=0}^{\infty} \Sigma = \Sigma \otimes \Sigma \otimes \Sigma \otimes \cdots,$$

$$\mathcal{F}_n = \underbrace{\Sigma \otimes \cdots \otimes \Sigma}_{n+1} \otimes \{\phi, S\} \otimes \{\phi, S\} \otimes \cdots \quad \forall n \geq 0,$$

$$\mathcal{G}_n = \{B \times S \times S \times \cdots : B \in \underbrace{\Sigma \otimes \cdots \otimes \Sigma}_{n+1}\} \quad \forall n \geq 0,$$

$$\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{G}_n .$$

Some immediate facts are

$(\Omega, \mathcal{F})$ is a measurable space;

$(\mathcal{F}_n)_{n\geq 0}$ is an increasing sequence of σ-subfields of $\mathcal{F}$;

$\mathcal{E}$ is a subfield of $\mathcal{F}$ and $\mathcal{F} = \sigma(\mathcal{E})$.

Each $A \in \mathcal{E}$ can be written as $A = B \times S \times S \times \cdots$ for some $B \in \bigotimes_{k=0}^{n} \Sigma$.

The choice of B is not unique. We define a map $P : \mathcal{E} \to [0, \infty]$,

$$P(A) = \int_S \int_S \cdots \int_S I_B(x_0, x_1, \ldots, x_{n-1}, x_n)\, p(x_{n-1}, dx_n) \cdots p(x_0, dx_1)\, \pi_0(dx_0). \tag{1}$$

We show that P is well-defined. Suppose

$$A = B_1 \times S \times S \times \cdots \quad \text{for } B_1 \in \bigotimes_{k=0}^{n_1} \Sigma,$$

$$A = B_2 \times S \times S \times \cdots \quad \text{for } B_2 \in \bigotimes_{k=0}^{n_2} \Sigma.$$

$B_1 \neq B_2$ only if $n_1 \neq n_2$. Assume without loss of generality that $n_1 < n_2$.

Then $B_2 = B_1 \times \underbrace{S \times \cdots \times S}_{n_2 - n_1}$.

$$\int_S \int_S \cdots \int_S I_{B_2}(x_0, x_1, \ldots, x_{n_2-1}, x_{n_2})\, p(x_{n_2-1}, dx_{n_2}) \cdots p(x_0, dx_1)\, \pi_0(dx_0)$$

$$= \int_S \int_S \cdots \int_S \cdots \underbrace{\int_S I_{B_1}(x_0, x_1, \ldots, x_{n_1})\, p(x_{n_2-1}, dx_{n_2})}_{= I_{B_1}(x_0, x_1, \ldots, x_{n_1})} \cdots p(x_0, dx_1)\, \pi_0(dx_0)$$

$$= \int_S \int_S \cdots \int_S \cdots \underbrace{\int_S I_{B_1}(x_0, x_1, \dots, x_{n_1}) \, p(x_{n_2-2}, dx_{n_2-1})}_{= I_{B_1}(x_0, x_1, \dots, x_{n_1})} \cdots p(x_{n_1}, dx_{n_1+1}) \cdots p(x_0, dx_1)\, \pi_0(dx_0)$$

$$= \cdots\cdots$$

$$= \int_S \int_S \cdots \int_S I_{B_1}(x_0, x_1, \dots, x_{n_1}) \, p(x_{n_1-1}, dx_{n_1}) \cdots p(x_0, dx_1)\, \pi_0(dx_0).$$

Thus, $P(A)$ does not depend on different choices of the form for A. We have showed that P is well-defined. Since $\Omega = S \times S \times S \times \cdots$,

$$P(\Omega) = \int_S I_S(x_0)\, \pi_0(dx_0) = \pi_0(S) = 1.$$

Let A_1 and A_2 be disjoint elements of $\mathcal{E}$. Write $A_k = B_k \times S \times S \times \cdots$ where $B_k \in \bigotimes_{j=0}^{\ell} \Sigma$. Then B_1 and B_2 are disjoint and

$$A_1 \cup A_2 = (B_1 \cup B_2) \times S \times S \times \cdots$$

By the definition of P,

$$P(A_1 \cup A_2) = \int_S \int_S \cdots \int_S \underbrace{I_{B_1 \cup B_2}(x_0, x_1, \dots, x_\ell)}_{= I_{B_1} + I_{B_2}} \, p(x_{\ell-1}, dx_\ell) \cdots p(x_0, dx_1)\, \pi_0(dx_0)$$

$$= P(A_1) + P(A_2).$$

Thus, P is additive on $\mathcal{E}$. Assume P is σ-additive on $\mathcal{E}$. By Theorem 14, Fristedt-Gray page 94, P extends to a probability measure on $(\Omega, \mathcal{F})$.

For each $n \geq 0$, we define $X_n : \Omega \to S$, $X_n(x_0, x_1, x_2, \ldots) = x_n$. We show that $(X_n)_{n\geq 0}$ is a Markov chain relative to $(\mathcal{F}_n)_{n\geq 0}$ with transition kernel $p(x, B)$. For each $B \in \Sigma$,

$$\{(x_0, x_1, x_2, \ldots) \in \Omega : X_n(x_0, x_1, x_2, \ldots) \in B\}$$

$$= \{(x_0, x_1, x_2, \ldots) \in \Omega : x_n \in B\}$$

$$= \underbrace{S \times \cdots \times S}_{n} \times B \times S \times S \times \cdots$$

which belongs to $\mathcal{F}_n$. Thus, X_n is $\mathcal{F}_n$-measurable.

Let $n \geq 0$ and $B \in \Sigma$. By the definition of conditional probability, $P(X_{n+1} \in B \mid \mathcal{F}_n) = E(I_{X_{n+1} \in B} \mid \mathcal{F}_n)$. We want to show $E(I_{X_{n+1}\in B} \mid \mathcal{F}_n) = p(X_n, B)$. That is to show

$$E(I_{X_{n+1} \in B}\, I_A) = E[p(X_n, B) I_A] \quad \forall A \in \mathcal{F}_n. \qquad (2)$$

Take $A \in \mathcal{F}_n$. Then $A = B' \times \underbrace{S \times S \times \cdots}_{\text{could be finite}}$ for some $B' \in \bigotimes_{k=0}^{n} \Sigma$. Then

$$(X_{n+1} \in B) A = \{(x_0, x_1, \ldots) \in \Omega : x_{n+1} \in B,\ (x_0, x_1, \ldots, x_n) \in B'\}$$

$$= \{(x_0, x_1, \ldots) \in \Omega : (x_0, x_1, \ldots, x_n, x_{n+1}) \in B' \times B\}$$

$$= \underbrace{(B' \times B)}_{\in \bigotimes_{k=0}^{n+1} \Sigma} \times S \times S \times \cdots$$

Taking the probability of this event, we get

$$\mathrm{LHS}(2) = P((X_{n+1}\in B)A) = P((B'\times B)\times S\times S\times\cdots)$$

$$= \int_S\int_S\cdots\int_S \underbrace{\int_S I_{B'\times B}(x_0,x_1,\dots,x_n,x_{n+1})\,p(x_n,dx_{n+1})}_{=I_{B'}(x_0,x_1,\dots,x_n)\,p(x_n,B)}\,p(x_{n-1},dx_n)\cdots p(x_0,dx_1)\,\pi_0(dx_0)$$

$$= \int_S\int_S\cdots\int_S I_A(x_0,x_1,\dots,x_n)\,p(x_n,B)\,p(x_{n-1},dx_n)\cdots p(x_0,dx_1)\,\pi_0(dx_0).$$

Put $Y = p(X_n,B)$. It is $\mathcal{F}_n$-measurable because X_k is $\mathcal{F}_n$-measurable and $p(\cdot,B)$ is Σ-measurable. To verify (2), we need to show

$$E\,Y I_A = \int_S\int_S\cdots\int_S I_A(x_0,x_1,\dots,x_n)\,Y(x_0,x_1,\dots,x_n)\,p(x_{n-1},dx_n)\cdots p(x_0,dx_1)\,\pi_0(dx_0). \tag{3}$$

Because Y is nonnegative, there exists a sequence of simple functions (s_m) such that $0\le s_1\le s_2\le\cdots\le Y$ and $\lim s_n = Y$. For each $A'\in\mathcal{F}_n$,

$$E I_{A'} I_A = P(A'A) = \int_S\int_S\cdots\int_S \underbrace{I_{A'A}(x_0,x_1,\dots,x_n)}_{I_{A'}I_A}\,p(x_{n-1},dx_n)\cdots p(x_0,dx_1)\,\pi_0(dx_0)$$

Because each s_m is a linear combination of indicator functions, we have

$$E\,s_m I_A = \int_S\int_S\cdots\int_S I_A(x_0,x_1,\dots,x_n)\,s_m(x_0,\dots,x_n)\,p(x_{n-1},dx_n)\cdots p(x_0,dx_1)\,\pi_0(dx_0).$$

Letting $m\to\infty$ and applying the Monotone Convergence Theorem, we get (3).

Therefore, $(X_n)_{n\geq 0}$ is a Markov chain relative to $(\mathcal{F}_n)_{n\geq 0}$ with transition kernel $p(x,B)$. We show that π_0 is the initial distribution of this chain, i.e. $\mathbb{P}(X_0\in B) = \pi_0(B)$ for all $B\in\Sigma$.

$$(X_0\in B) = \{(x_0,x_1,\dots)\in\Omega : x_0\in B\} = B\times S\times S\times\cdots$$

By the definition of $\mathbb{P}$ at (1),

$$\mathbb{P}(X_0\in B) = \int_S I_B(x_0)\,\pi_0(dx_0) = \pi_0(B).$$