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Math 8652: Theory of Probability

Homework #8

① Let $S=\mathbb{Z}$ or $S=\mathbb{Z}^2$, and Σ be the family of all subsets of S. Let $p: S\times\Sigma \to \mathbb{R}$ be a transition kernel, (Ω, F) be the product measurable space $\left(\prod_{n=0}^{\infty} S, \bigotimes_{n=0}^{\infty}\Sigma\right)$, and $X_n : \Omega \to S$ be the map $X_n(x_0,x_1,x_2,\dots) = x_n$.
For each $x\in S$, we denote by P_x the measure on (Ω, F), as constructed in the proof of Ionescu-Tulcea theorem (Homework 7, Problem 4), which turns $(X_n)_{n\geq 0}$ into a Markov chain with initial distribution $\pi_0=\delta_x$, i.e. $\pi_0(\{x\})=1$, and transition kernel p. Denote by E_x the expectation operator with respect to the measure P_x.

Let $u: S\to \mathbb{R}$ be a bounded harmonic function, i.e.

$$u(x) = E_x u(X_1) = \int_S\int_S u(x_1)p(x_0,dx_1)\delta_x(dx_0) = \int_S u(y)p(x,dy) \quad \forall x\in S. \tag{1}$$

We show that u is constant. There exists a number $M>0$ such that $u(x)\geq -M$ for all $x\in S$. The function $x\mapsto u(x)+M$ is also a harmonic function according to the definition (1). By replacing u with $u+M$, we can assume $u(x)\geq 0$ for all $x\in S$. Take any $a\in S$. Put

$$\tau_a = \inf\{n\geq 0 : X_n = a\}.$$

We can induce from the computations in Homework 7, Problem 3, that

$\boxed{2}$

$P(\tau_a < \infty) = 1$. For $n \geq 0$,

$$E_x u(X_{(n+1)\wedge\tau_a}) = \underbrace{E_x u(X_{n+1}) I_{\tau_a > n}}_{\{1\}} + E_x u(X_{\tau_a}) I_{\tau_a \leq n}. \quad (2)$$

$$\{1\} = E_x u(X_{n+1}) I_{X_0, \dots, X_n \neq a}$$

$$= \int_S \int_S \cdots \underbrace{\int_S u(x_{n+1}) I_{x_0, \dots, x_n \neq a}\, p(x_n, dx_{n+1})}_{\{2\}} \cdots p(x_0, dx_1)\, \delta_x(dx_0). \quad (3)$$

We have

$$\{2\} = I_{x_0, x_1, \dots, x_n \neq a} \int_S u(x_{n+1})\, p(x_n, dx_{n+1}) \overset{(1)}{=\!=} I_{x_0, \dots, x_n \neq a}\, u(x_n).$$

Substituting this result into (3), we get

$$\{1\} = \int_S \int_S \cdots \int_S u(x_n) I_{x_0, \dots, x_n \neq a}\, p(x_{n-1}, dx_n) \cdots p(x_0, dx_1)\, \delta_x(dx_0)$$

$$= E_x u(X_n) I_{X_0, \dots, X_n \neq a}$$

$$= E_x u(X_n) I_{\tau_a > n}.$$

Substituting this result into (2), we get

$$E_x u(X_{(n+1)\wedge\tau_a}) = E_x u(X_n) I_{\tau_a > n} + E_x u(X_{\tau_a}) I_{\tau_a \leq n}$$

$$= E_x u(X_{n\wedge\tau_a}) I_{\tau_a > n} + E_x u(X_{n\wedge\tau_a}) I_{\tau_a \leq n}$$

$$= E_x u(X_{n\wedge\tau_a}).$$

Apply this identity many times,

$$E_x u(X_{n\wedge\tau_a}) = E_x u(X_{(n-1)\wedge\tau_a}) = \cdots = E_x u(X_{1\wedge\tau_a}) = E_x u(X_{0\wedge\tau_a}) = E_x u(X_0).$$

Thus,
$$E_x u(X_{n\wedge\tau_a}) = \int_S u(x_0)\,\delta_x(dx_0) = u(x) \quad \forall n \geq 1.$$

Then $u(x) \geq E_x u(X_{n\wedge\tau_a}) I_{\tau_a<\infty}$ for every $n \geq 1$. Then
$$u(x) \geq \liminf_{n\to\infty} E_x u(X_{n\wedge\tau_a}) I_{\tau_a<\infty} \overset{\text{Fatou}}{\geq} E_x\left[\liminf_{n\to\infty} u(X_{n\wedge\tau_a}) I_{\tau_a<\infty}\right]$$
$$= E_x u(X_{\tau_a}) I_{\tau_a<\infty}.$$

By the definition of τ_a, $X_{\tau_a(\omega)}(\omega) = a$ for all $\omega \in (\tau_a < \infty)$. Then
$$u(x) \geq E_x u(X_{\tau_a}) I_{\tau_a<\infty} = E_x u(a) I_{\tau_a<\infty} = u(a) P(\tau_a < \infty)$$
$$= u(a) \quad \forall x \in S.$$

Because $a \in S$ was chosen arbitrarily, we have
$$u(x) \geq u(y) \quad \forall x, y \in S.$$

This implies u is a constant function.

2 Let $(X_n)_{n\geq 0}$ be a simple symmetric random walk in $\mathbb{Z}^d$, where $d=1$ or 2. Let T_1 be the returning time, i.e.
$$T_1 = \inf\{n \geq 1 : X_n = 0\}.$$

We show that $ET_1 = \infty$. Put
$$\Psi(s) = \sum_{n=0}^{\infty} P(X_n = 0)\, s^n, \qquad \forall s \in [0,1)$$
$$\phi(s) = \sum_{n=0}^{\infty} P(T_1 = n)\, s^n.$$

By Theorem 4, Fristedt–Gray page 493, $\Psi(s) = \dfrac{1}{1-\phi(s)}$ for all $s \in [0,1)$.

We induce from the computations in Homework 7, Problem 3, that

$P(T_1=\infty)=0$. Then ϕ is the probability generating function of T_1.

By Theorem 13, Fristedt-Gray page 71,

$$ET_1 = \lim_{s\uparrow 1} \frac{1-\phi(s)}{1-s}.$$

Then $$\frac{1}{ET_1} = \lim_{s\uparrow 1} \frac{1-s}{1-\phi(s))} = \lim_{s\uparrow 1} (1-s)\,\Psi(s), \qquad (1)$$

with the convention that $\frac{1}{\infty}=0$.

At the beginning of page 13, Homework 7, we derived the identity

$$P(X_n=0) = \frac{1}{2^d}\int_{[-1,1]^d} f(\lambda)^n\, d\lambda,$$

where $f(\lambda)=\frac{1}{d}\sum_{k=1}^{d}\cos(\pi\lambda_k)$. Then

$$\sum_{n=0}^{m-1} P(X_n=0)s^n = \frac{1}{2^d}\int_{[-1,1]^d}\sum_{n=0}^{m-1}(sf(\lambda))^n d\lambda = \frac{1}{2^d}\int_{[-1,1]^d}\frac{1-(sf(\lambda))^m}{1-sf(\lambda)}d\lambda. \qquad (2)$$

By the definition of f, we see that $f(\lambda)\in[-1,1]$ for all $\lambda\in[-1,1]^d$. Then

$$0\le \frac{1-(sf(\lambda))^m}{1-sf(\lambda)} \le \frac{2}{1-sf(\lambda)} \le \frac{2}{1-s} \quad \forall\, \lambda\in[-1,1]^d.$$

Also, $$\lim_{m\to\infty}\frac{1-(sf(\lambda))^m}{1-sf(\lambda)} = \frac{1}{1-sf(\lambda)}.$$

By the Dominated Convergence Theorem,

$$\lim_{m\to\infty}\int_{[-1,1]^d}\frac{1-(sf(\lambda))^m}{1-sf(\lambda)}d\lambda = \int_{[-1,1]^d}\frac{1}{1-sf(\lambda)}d\lambda.$$

Taking the limit of both sides of (2) as $m \to \infty$, we get

$$\underbrace{\sum_{n=0}^{\infty} P(X_n=0)s^n}_{\Psi(s)} = \int_{[-1,1]^d} \frac{1}{1-sf(\lambda)}\, d\lambda.$$

Thus,

$$(1-s)\Psi(s) = \int_{[-1,1]^d} \frac{1-s}{1-sf(\lambda)}\, d\lambda. \qquad (3)$$

Because $f(\lambda) \in [-1,1]$, $0 < 1-sf(\lambda) \le 1-s$ for all $s \in [0,1)$, $\lambda \in [-1,1]^d$.

Then

$$0 < \frac{1-s}{1-sf(\lambda)} \le 1 \quad \forall s \in [0,1),\ \lambda \in [-1,1]^d.$$

We see that

$$\begin{aligned} f(\lambda) = 1 &\iff \cos \pi\lambda_1 = \dots = \cos \pi\lambda_d = 1 \\ &\iff \lambda_1 = \dots = \lambda_d = 0 \\ &\iff \lambda = 0. \end{aligned}$$

Thus,

$$\lim_{s \uparrow 1} \frac{1-s}{1-sf(\lambda)} = \frac{0}{1-f(\lambda)} = 0 \quad \forall \lambda \in [-1,1]^d \setminus \{0\}.$$

By the Dominated Convergence Theorem,

$$\lim_{s\uparrow 1} \int_{[-1,1]^d} \frac{1-s}{1-sf(\lambda)}\, d\lambda = 0.$$

Then (3) implies $\lim_{s\uparrow 1}(1-s)\Psi(s) = 0$. Then (1) implies $\frac{1}{ET_1} = 0$. Hence, $ET_1 = \infty$.

(3) Let μ be a distribution on $S = \{0,1,2,\dots\}$ such that $\mu(\{2,3,\dots\}) > 0$.

Denote by Σ the family of all subsets of S. Let $X_n : (\Gamma, \mathcal{G}, Q) \to (S, \Sigma)$,

$n \geq 1$, be an independent and identically distributed sequence of random variables, each having distribution μ. Define a map $p : S \times \Sigma \to [0,1]$,

$$p(n, B) = Q(Y_1 + \dots + Y_n \in B),$$

with the convention that the sum is 0 if it has no summand. Let $(\Omega, \mathcal{F})$ be the product measurable space $\left(\prod_{k=0}^{\infty} S, \bigotimes_{k=0}^{\infty} \Sigma\right)$, and $X_n : \Omega \to S$ be the map $X_n(x_0, x_1, x_2, \dots) = x_n$. For each $n \in S$, we denote by P_n the measure on $(\Omega, \mathcal{F})$, as constructed in the proof of Ionescu-Tulcea theorem (Homework 7, Problem 4), which turns $(X_k)_{k \geq 0}$ into a Markov chain with initial distribution $\pi_n = \delta_n$, i.e. $\pi_n(\{n\}) = 1$, and transition kernel p. Denote by E_n the expectation operator with respect to the measure P_n.

For each random variable X, we denote by $\rho_X : [0,1] \to [0,1]$ the probability generating function of X. The probability generating function of Y_1 will be denoted as ρ.

$$\rho(s) = E s^{Y_1} = \sum_{\ell=0}^{\infty} s^{\ell} Q(Y_1 = \ell),$$

$$\rho_{X_k}(s) = E_1 s^{X_k} = \sum_{\ell=0}^{\infty} s^{\ell} P_1(X_k = \ell),$$

with the convention that $0^0 = 1$. It is known that probability generating functions are continuous. We ~~see~~ show that the number $c = P_1(X_k = 0 \text{ for some } k \geq 0)$ is a fixed point of ρ.

$$\{\omega\in\Omega : X_k = 0 \text{ for some } k\geq 0\} = (X_k = 0 \text{ for some } k\geq 0)$$
$$= (0^{X_k} = 1 \text{ for some } k\geq 0)$$
$$= \bigcup_{k=1}^{\infty} (0^{X_k} = 1). \qquad (1)$$

With the given transition kernel, we see that if $X_k(\omega)=0$ then $X_l(\omega)=0$ for all $l\geq k$. Then the sequence of events $(0^{X_k}=1)$ is increasing. Take the probability of both sides of (1), we get

$$c = P_1(X_k = 0 \text{ for some } k\geq 0) = P_1\Big(\bigcup_{k=1}^{\infty}(0^{X_k}=1)\Big)$$
$$= \lim_{k\to\infty} P_1(0^{X_k}=1)$$
$$= \lim_{k\to\infty} g_{X_k}(0). \qquad (2)$$

For $k\geq 1$,

$$g_{X_k}(s) = E_1 s^{X_k} = \sum_{m=0}^{\infty} E_1 s^{X_k} I_{X_{k-1}=m}$$
$$= \sum_{m=0}^{\infty} E_1\left[E_1(s^{X_k} \mid X_0,\dots,X_{k-1})\, I_{X_{k-1}=m}\right]. \qquad (3)$$

We have

$$E_1(s^{X_k}\mid X_0,\dots,X_{n-1}) = E_1\Big(\sum_{l=0}^{\infty} s^l I_{X_k=l} \mid X_0,\dots,X_{k-1}\Big)$$
$$= \sum_{l=0}^{\infty} s^l E_1(I_{X_k=l}\mid X_0,\dots,X_{k-1})$$
$$= \sum_{l=0}^{\infty} s^l P_1(X_k=l\mid X_0,\dots,X_{k-1})$$
$$= \sum_{l=0}^{\infty} s^l p(X_{k-1}, \{l\}).$$

Then (3) becomes

$$g_{X_k}(s) = \sum_{m=0}^{\infty} E_1\left[\sum_{l=0}^{\infty} s^l p(X_{k-1}, \{l\}) I_{X_{k-1}=m}\right]$$

$$= \sum_{m,l=0}^{\infty} s^l E_1\left[p(X_{k-1}, \{l\}) I_{X_{k-1}=m}\right]$$

$$= \sum_{m,l=0}^{\infty} s^l \underbrace{p(m, \{l\})}_{=Q(Y_1+\dots+Y_m=l)} P_1(X_{k-1}=m)$$

$$= \sum_{m=0}^{\infty} \left[\sum_{l=0}^{\infty} s^l Q(Y_1+\dots+Y_m=l)\right] P_1(X_{k-1}=m)$$

$$= \sum_{m=0}^{\infty} g_{Y_1+\dots+Y_m}(s)\, P_1(X_{k-1}=m)$$

$$= \sum_{m=0}^{\infty} g_{Y_1}(s) g_{Y_2}(s) \cdots g_{Y_m}(s)\, P_1(X_{k-1}=m)$$ (because $Y_1, \dots, Y_m$ are independent)

$$= \sum_{m=0}^{\infty} g(s)^m P_1(X_{k-1}=m)$$

$$= g_{X_{k-1}}(g(s)). \qquad (4)$$

Applying (4) many times, we get

$$g_{X_k}(s) = g_{X_{k-1}}(g(s)) = g_{X_{k-2}}(g(g(s))) = \dots = g_{X_0}(g(g(\dots(s)\dots)).$$

Because $X_0 = 1$ almost surely, $g_{X_0}(s) = s$. Then $g_{X_k}(s) = \underbrace{g\circ g\circ\dots\circ g}_{k \text{ times}}(s)$.

Then

$$g_{X_k}(s) = g(g_{X_{k-1}}(s)) \qquad \forall s \in [0,1).$$

In particular, $\rho_{X_k}(0) = \rho(\rho_{X_{k-1}}(0))$. Letting $k \to \infty$ and using (2), we get $c = \rho(c)$. That is, c is a fixed point of ρ.

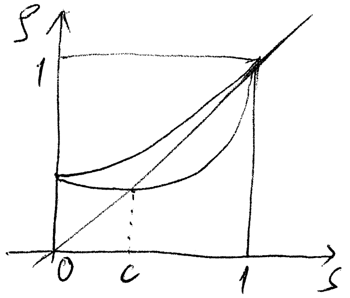
We show that c is the smallest fixed point of ρ. For each $n \geq 0$, denote $\pi_{\{0\}}(n) = P_n(X_k = 0$ for some $k \geq 0)$. This is the probability of extinction of a population which originally has n individuals. Then $c = \pi_{\{0\}}(1)$. Let $d \in [0,1]$ be a fixed point of ρ. For $k \geq 0$,

$$
\begin{aligned}
E_k d^{X_1} &= \sum_{l=0}^{\infty} d^l P_k(X_1 = l) = \sum_{l=0}^{\infty} d^l P_k(X_1 = l \mid X_0) \\
&= \sum_{l=0}^{\infty} d^l p(k, \{l\}) \\
&= \sum_{l=0}^{\infty} d^l Q(Y_1 + \cdots + Y_k = l) \\
&= \rho_{Y_1 + \cdots + Y_k}(d) \\
&= \rho_{Y_1}(d)\, \rho_{Y_2}(d) \cdots \rho_{Y_k}(d) \\
&= \rho_{Y_1}(d)^k = d^k .
\end{aligned}
$$

This identity shows that the function $k \in S \mapsto d^k \in \mathbb{R}$ is harmonic on S. By Theorem 8, Fristedt-Gray page 522, $d^k \geq \pi_{\{0\}}(k)$ for every $k \in S$. In particular, $d \geq \pi_{\{0\}}(1) = c$. Thus, c is the smallest fixed point of ρ.

We show that $c = 1$ if and only if $EY_1 \leq 1$. By Theorem 13, Fristedt-Gray page 71, $EY_1 = \lim_{s \uparrow 1} \dfrac{1 - \rho(s)}{1 - s}$.

Assume $c=1$. Then ρ has no fixed point in $[0,1)$. Because the map $s\in[0,1] \mapsto \rho(s)-s \in \mathbb{R}$ is continuous and has no ~~fixed~~ zero in $[0,1)$, it does not change its sign therein. On the other hand,



$\rho(0)-0=\rho(0)\geq 0$. Thus, $\rho(s)-s>0$ for all $s\in[0,1)$. Then

$$0\leq \frac{1-\rho(s)}{1-s}<1 \qquad \forall s\in[0,1).$$

Taking the limit as $s\uparrow 1$, we get $E\,Y_1\leq 1$.

Conversely, assume $EY_1\leq 1$. Suppose by contradiction that $c<1$. By Theorem 14, Fristedt-Gray page 73, ρ is equal to its Maclaurin series on $[0,1)$. This allows us to take derivative of ρ (as a series) term by term.

$$\rho(s)=\sum_{l=0}^{\infty} s^l Q(Y_1=l),$$

$$\rho'(s)=\sum_{l=1}^{\infty} l\, s^{l-1} Q(Y_1=l),$$

$$\rho''(s)=\sum_{l=2}^{\infty} l(l-1)\, s^{l-2}\, \underbrace{Q(Y_1=l)}_{\mu(\{l\})}.$$

Because $\mu(\{2,3,\dots\})>0$, there exists $l_0\geq 0$ such that $\mu(\{l_0\})>0$. Then $\rho''(s)>0$ for all $s\in(0,1)$. For each $s\in(c,1)$, there exists $\theta_s\in(s,1)$ such that $\dfrac{1-\rho(s)}{1-s}=\rho'(\theta_s)$.

Since ρ' is ~~strongl~~ strictly increasing in $(0,1)$, $\rho'(\theta_s) > \rho'(s)$. Then

$$\frac{1-\rho(s)}{1-s} > \rho'(s) \quad \forall s \in (c,1).$$

Then

$$\left(\frac{1-\rho(s)}{1-s}\right)' = \frac{-\rho'(s)(1-s)+(1-\rho(s))}{(1-s)^2} > 0 \qquad \forall s \in (c,1).$$

Then $\frac{1-\rho(s)}{1-s}$ is strictly increasing on $(c,1)$. Consequently,

$$\frac{1-\rho(s)}{1-s} > \frac{1-\rho(c)}{1-c} = \frac{1-c}{1-c} = 1 \qquad \forall s \in (c,1).$$

Then

$$EY_1 = \lim_{s\uparrow 1} \frac{1-\rho(s)}{1-s} > 1.$$

This is a contradiction.

4. Fix $d \geq 2$ and let $S = \{1,2,\dots,d\}$. For $1 \leq i,j \leq d$, we denote

$$p(i,j) = \begin{cases} \frac{1}{2} & \text{if } 1<i<d,\ j=i\pm 1, \\ 1 & \text{if } (i,j)=(1,1) \text{ or } (d,d), \\ 0 & \text{otherwise.} \end{cases}$$

Let Σ be the family of all subsets of S. For each $i \in S$, $B \in \Sigma$, we define

$$p(i,B) = \sum_{j\in B} p(i,j).$$

This makes $p(i,\cdot)$ a measure on (S,Σ). In addition,

$$p(i,S) = \sum_{j\in S} p(i,j) = \sum_{j=1}^{d} p(i,j) = \begin{cases} p(i,i-1)+p(i,i+1) & \text{if } 1<i<d, \\ p(1,1) & \text{if } i=1, \\ p(d,d) & \text{if } i=d \end{cases}$$

$$= 1.$$

Thus, $p(i,\cdot)$ is a probability measure on (S,Σ). Then p is a transition kernel.

Using the method to prove the Ionescu-Tulcea theorem, we can construct a Markov chain $(X_n)_{n\geq 0}$ with transition kernel p. The construction is as follows. Let $(\Omega, \mathcal{F})$ be the product measureable space $\left(\prod_{k=0}^{\infty} S, \bigotimes_{k=0}^{\infty} \Sigma\right)$. Let $X_n : \Omega \to S$, $n \geq 0$, be the map $X_n(x_0, x_1, x_2, \dots) = x_n$. For $n \geq 0$, let $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$. For each probability measure π on (S,Σ), there exists a unique probability measure P_π on $(\Omega, \mathcal{F})$ such that

$$P_\pi(A) = \int_S \int_S \cdots \int_S I_B(x_0, x_1, \dots, x_{n-1}, x_n)\, p(x_{n-1}, dx_n) \cdots p(x_0, dx_1)\, \pi(dx_0)$$

for all $A \in \mathcal{F}$ of the form $A = B \times S \times S \times \cdots$, $B \in \bigotimes_{k=0}^{n} \Sigma$ for some $n \geq 0$.

Beside this abstract construction, we have a more explicit construction as follows. Let $(Y_n)_{n\geq 1}$ be an independent and identically distributed sequence of random variables such that $P(Y_1 = 1) = P(Y_1 = -1) = \frac{1}{2}$.

Define

$$X_0 = \left[\frac{d+1}{2}\right] \quad \left(\text{the integer part of } \frac{d+1}{2}\right),$$

$$X_{n+1} = X_n + Y_{n+1} I_{1 < X_n < d} \qquad \forall n \geq 0.$$

The number $\left[\frac{d+1}{2}\right]$ is not significant. It can be replaced by any number in S. We show that $(X_n)_{n\geq 0}$ is a Markov chain with respect to the filtration $\mathcal{F}_n = \sigma(X_0, Y_1, Y_2, \dots, Y_n)$ with transition kernel p. Because

$$X_1 = \left[\frac{d+1}{2}\right] + Y_1 I_{1 < \left[\frac{d+1}{2}\right] < d},$$

X_1 is $\mathcal{F}_1$-measurable. Suppose X_n is $\mathcal{F}_n$-measurable for some $n \geq 1$. Then

$$X_{n+1} = \underbrace{X_n}_{\mathcal{F}_n\text{-meas.}} + \underbrace{Y_{n+1}}_{\mathcal{F}_{n+1}\text{-meas.}} \underbrace{I_{1<X_n<d}}_{\mathcal{F}_n\text{-meas.}}$$

is $\mathcal{F}_{n+1}$-measurable. For $1 \leq k \leq d$, we show that $P(X_{n+1}=k \mid \mathcal{F}_n) = p(X_n, k)$.

That is to show

$$E(I_{X_{n+1}=k} \mid \mathcal{F}_n) = p(X_n, k) \quad \forall\, 1 \leq k \leq d. \qquad (1)$$

- $\underline{2 < k < d-1}$

$$\begin{aligned}
E(I_{X_{n+1}=k} \mid \mathcal{F}_n) &= E(I_{X_{n+1}=k} I_{X_n=k-1} \mid \mathcal{F}_n) + \\
&\qquad E(I_{X_{n+1}=k} I_{X_n=k+1} \mid \mathcal{F}_n) \\
&= E(I_{Y_{n+1}=1} I_{X_n=k-1} \mid \mathcal{F}_n) + E(I_{Y_{n+1}=-1} I_{X_n=k+1} \mid \mathcal{F}_n) \\
&= E(I_{Y_{n+1}=1} \mid \mathcal{F}_n) I_{X_n=k-1} + E(I_{Y_{n+1}=-1} \mid \mathcal{F}_n) I_{X_n=k+1} \\
&= E(I_{Y_{n+1}=1}) I_{X_n=k-1} + E(I_{Y_{n+1}=-1}) I_{X_n=k+1} \\
&= \frac{1}{2}\left(I_{X_n=k-1} + I_{X_n=k+1}\right).
\end{aligned}$$

On the other hand,

$$p(X_n, k) = p(k-1, k) I_{X_n=k-1} + p(k+1, k) I_{X_n=k+1} =$$

$$= \frac{1}{2}\left(I_{X_n = k-1} + I_{X_n = k+1}\right).$$

Thus, (1) is true.

- $\underline{k=2}$

$$E(I_{X_{n+1}=2} \mid \mathcal{F}_n) = E(I_{X_{n+1}=2}\, I_{X_n=3} \mid \mathcal{F}_n)$$

$$= E(I_{Y_{n+1}=-1}\, I_{X_n=3} \mid \mathcal{F}_n) = E(I_{Y_{n+1}=-1} \mid \mathcal{F}_n) I_{X_n=3}$$

$$= E(I_{Y_{n+1}=-1})\, I_{X_n=3} = \frac{1}{2} I_{X_n=3}.$$

$$p(X_n, 2) = p(3,2)\, I_{X_n=3} = \frac{1}{2} I_{X_n=3}.$$

Thus, (1) is true.

- $\underline{k=1}$

$$E(I_{X_{n+1}=1} \mid \mathcal{F}_n) = E\left(I_{X_{n+1}=1}\, I_{X_n=1} \mid \mathcal{F}_n\right) + E\left(I_{X_{n+1}=1}\, I_{X_n=2} \mid \mathcal{F}_n\right)$$

$$= E(I_{X_n=1} \mid \mathcal{F}_n) + E(I_{Y_{n+1}=-1}\, I_{X_n=2} \mid \mathcal{F}_n)$$

$$= I_{X_n=1} + \underbrace{E(I_{Y_{n+1}=-1} \mid \mathcal{F}_n)}_{= E(I_{Y_{n+1}=-1})} I_{X_n=2}$$

$$= I_{X_n=1} + \frac{1}{2} I_{X_n=2}.$$

On the other hand,

$$p(X_n, 1) = p(1,1)\, I_{X_n=1} + p(2,1)\, I_{X_n=2} = I_{X_n=1} + \frac{1}{2} I_{X_n=2}.$$

Thus, (1) is true.

The case $k = d-1$ is similar to $k=2$. The case $k=d$ is similar to

$k=d$. Therefore, (1) is true for all $k=1,2,\dots,d$.

Next, we determine all invariant probability distributions on S. That is to determine all d-tuples $(a_1, a_2, \dots, a_d) \in [0,1]^d$ such that

$$\begin{cases} \displaystyle\sum_{k=1}^{d} a_k = 1, & (2) \\ \displaystyle\sum_{i=1}^{d} a_i \cdot p(i,j) = a_j \quad \forall 1 \le j \le d. & (3) \end{cases}$$

Such a tuple determines an invariant distribution on S, given by

$$\pi(X_0 = i) = a_i \quad \forall\, 1 \le i \le d.$$

If $d=2$ then $(a_1, a_2) = (a_1, 1-a_2)$ for some $a_1 \in [0,1]$. Consider $d \geq 3$.

For $j=1$, (3) becomes $a_1 + \frac{1}{2} a_2 = a_1$. Then $a_2 = 0$.

For $j=d$, (3) becomes $\frac{1}{2} a_{d-1} + a_d = a_d$. Then $a_{d-1} = 0$.

- If $d=3$ then $a_2 = 0$. The condition (3) is satisfied for all $j=1,2,3$. The condition (2) becomes $a_1 + a_3 = 1$. We get $(a_1, a_2, a_3) = (a_1, 0, 1-a_1)$ for some $a_1 \in [0,1]$.
- If $d=4$ then $a_2 = 0$, $a_3 = a_{d-1} = 0$. The condition (3) is satisfied for all $j=1,2,3,4$. The condition (2) becomes $a_1 + a_4 = 1$. We get $(a_1, a_2, a_3, a_4) = (a_1, 0, 0, 1-a_1)$ for some $a_1 \in [0,1]$.
- Consider the case $d \geq 5$.

For $j=2$, (3) becomes $\frac{1}{2} a_3 = a_2 = 0$. Then $a_3 = 0$. For $2 < j < d-1$,

(3) becomes $\frac{1}{2}a_{j-1}+\frac{1}{2}a_{j+1}=a_j$. Then $a_{j+1}-a_j=a_j-a_{j-1}$. Then

$$a_{d-1}-a_{d-2}=a_{d-2}-a_{d-3}=\cdots=a_3-a_2 \quad (=0)$$

Thus, $a_2=a_3=\cdots=a_{d-1}=0$. The condition (2) becomes $a_1+a_d=1$. Therefore,

$$(a_1,a_2,\ldots,a_{d-1},a_d)=(a_1,0,\ldots,0,1-a_1)$$

for arbitrary choice of $a_1\in[0,1]$. This result holds for all $d\geq 2$.