**Theory of Probability and Measure Theory – Math 8652**

Homework #4

1) Prove that a sequence of real-valued random variables $\xi_n \geq 0$, $n = 1, ..., N$, such that $\xi_n$ is $\mathcal{F}_n$-measurable and $E|\xi_n| < \infty$ for every $n$, is a submartingale if and only if, for each $1 \leq n \leq N$, we have $\xi_n = E(\eta_n|\mathcal{F}_n)$ a.s. where $\eta_n$ is an increasing sequence of nonnegative random variables such that $\eta_N = \xi_N$.

2) Let $X_n$, $n = 1, ..., N$ be iid standard normal on $(\Omega, \mathcal{F}, P)$ and $w_N = X_1 + ... + X_N$. For each $A \in \mathcal{F}$, define

$$P'(A) = EI_A \exp\left(w_N - \frac{N}{2}\right).$$

Show that $P'$ is a probability measure on $\mathcal{F}$. Also show that $Y_n = X_n - 1$ are iid standard normal on $(\Omega, \mathcal{F}, P')$.

3) Let $(\xi, \xi_1, ..., \xi_n)$ be a Gaussian vector. Prove that for any Borel $f$ such that $E|f(\xi)| < \infty$,

$$E(f(\xi)|\xi_1, ..., \xi_n) = Ef(x + \eta)|_{x=m} \quad \text{a.s.}$$

where $m = E(\xi|\xi_1, ..., \xi_n)$ and $\eta$ is a normal $(0, \sigma^2)$ random variable with $\sigma^2 = E|\xi - m|^2$. Then show that if $\xi \neq m$ then

$$E(f(\xi)|\xi_1, ..., \xi_n) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} f(x) e^{-(x-m)^2/(2\sigma^2)} dx.$$

4) Prove the Pythagorean theorem: For any $X$ such that $EX^2 < \infty$ we have

$$EX^2 = E[E(X|\mathcal{G})]^2 + E[X - E(X|\mathcal{G})]^2.$$