Theory of Probability and Measure Theory – Math 8652

Homework #5

1) Let $X_1, ..., X_n \in L^1(\Omega, \mathcal{F}, P)$. Define $\text{Span}(X_1, ..., X_n)$ as the collection of elements X of $L^1(\Omega, \mathcal{F}, P)$ for each of which there exist real numbers $\alpha_1, ...\alpha_n$ such that

$$\left\| X - \sum_{k=1}^{n} \alpha_k X_k \right\| = 0.$$

Prove that $\text{Span}(X_1, ..., X_n)$ is a closed subset of $L^1(\Omega, \mathcal{F}, P)$.

2) Let $\Omega = \bigcup_n A_n$ be a partition of Ω into disjoint sets $A_n \in \mathcal{F}$, $n = 1, 2, ...$ such that $P(A_n) > 0$ for any n. Let $\mathcal{G} = \sigma(A_n, n = 1, 2, ...)$. Assume that $E|\xi| < \infty$. Prove that

$$E(\xi|\mathcal{G}) = \frac{1}{P(A_n)} E\xi I_{A_n}$$

on A_n for any n.

3) Prove that any nonnegative quadratic function attains its minimum at at least one point, that is, if we are given numbers a_{ij}, b_i, c, $i, j = 1, ..., d$, such that the function on $\mathbb{R}^d$

$$f(x) = \sum_{i,j=1}^{d} a_{ij} x_i x_j + 2 \sum_{i=1}^{d} b_i x_i + c = (ax, x) + 2(b, x) + c$$

is nonnegative, then there exists a point $\bar{x} \in \mathbb{R}^d$ such that

$$f(\bar{x}) = \inf_{x \in \mathbb{R}^d} f(x).$$

4) We know that for simple symmetric random walk S_n, $n = 0, 1, 2, ...$, starting from the origin, the process

$$\xi_n = \left(\frac{5}{4}\right) 2^{S_n}$$

is a martingale. Prove that $\xi_n \to 0$ a.s. and conclude from this that

$$E \sup_{n \geq 0} \xi_n = \infty.$$