Theory of Probability and Measure Theory – Math 8652

Homework #7

1) (Doob's decomposition. Problem 24.10 in the textbook) Prove that a sequence of real-valued random variables ξ_n, $n = 1, ..., N$, such that ξ_n is $\mathcal{F}_n$-measurable and $E|\xi_n| < \infty$ for every n, is a submartingale if and only if $\xi_n = A_n + m_n$, where m_n is an $\mathcal{F}_n$-martingale and A_n is an increasing sequence such that $A_1 = 0$ and A_n is $\mathcal{F}_{n-1}$-measurable for every $n \geq 2$.

2) Let $X_1, X_2, X_3, ...$ be an iid sequence of random variables, each having uniform distribution on [0,2]. Show that

$$\xi_n = \prod_{k=1}^{n} X_k$$

converges to 0 almost surely.

3) Let $Y_1, Y_2, Y_3, ...$ be iid $\mathbb{Z}^d$-valued random variables such that

$$P(Y_k = \pm e_i) = \frac{1}{2d},$$

$i = 1, ..., d$, where $e_1, ..., e_d$ is the standard basis in $\mathbb{R}^d$. Suppose $d \geq 3$. Show that the probability for the random walk $X_0 = 0$, $X_n = Y_1 + ... + Y_n$ to ever return to the origin is strictly less than 1.

4) In the proof of Ionescu-Tulcea theorem given in Lecture 29, assume there is a probability measure on $\mathcal{F}$ which agrees with P on $\mathcal{E}$. Prove that $X_0(\omega), X_1(\omega), ...$ is a Markov chain relative to $\{\mathcal{F}_n\}$ with transition kernel $p(x, B)$ and with initial distribution π_0.