Let X be a Banach space over $\mathbb{C}$, $\mathcal{L}(X)$ be the Banach space of linear continuous map from X to X. For each $A \in \mathcal{L}(X)$, we denote

$$\sigma(A) := \{\lambda \in \mathbb{C} : (\lambda Id_X - A) \text{ is not invertible}\}.$$

We know that $\sigma(A)$ is not empty and is compact. Put $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$

If X is finite dimensional then $\sigma(A)$ is the set of eigenvalues, i.e. the set of roots of the characteristic polynomial. By using the Argument Principle to count the number of roots enclosed in a circle, we can show that the eigenvalues of A depend continuously on A. Hence, $\rho(A)$ depends continuously on A. When X is infinite dimensional, however, this conclusion is not true. A counterexample was constructed by Kakutani, found in the book by C. E. Rickart "General theory of Banach algebras" (1960), page 282:

Let X be a separable Hilbert space over $\mathbb{C}$, with a complete orthonormal sequence (f_m). For each $m \in \mathbb{N}$, we put

$$\alpha_m = \exp(-\ell), \text{ where } m = 2^\ell s \text{ and } s \text{ is odd.}$$

Define a linear map $A : X \to X$, $Af_m = \alpha_m f_{m+1}$ for all $m \in \mathbb{N}$. Because $\alpha_m \in [0,1]$ for all $m \in \mathbb{N}$, A is continuous and $\|A\| \le 1$.

We show that $\rho(A) > 0$.

$$A^2 f_1 = A(\alpha_1 f_2) = \alpha_1 \alpha_2 f_3,$$

$$A^3 f_1 = A(\alpha_1 \alpha_2 f_3) = \alpha_1 \alpha_2 \alpha_3 f_4,$$

$$\cdots\cdots$$

$$A^n f_1 = \alpha_1 \alpha_2 \ldots \alpha_n f_{n+1} \qquad \forall n \in \mathbb{N}.$$

Thus, $\|A^n f_1\| = \alpha_1\alpha_2\cdots\alpha_n$. Take $n = 2^k - 1$. Then

$$\alpha_1, \alpha_2, \ldots, \alpha_n \in \{\exp(-j) : 0 \le j < k\}.$$

We want to count the frequency of occurrence of $\exp(-j)$ in $\alpha_1, \alpha_2, \ldots, \alpha_n$. This is equal to the number of odd numbers s such that $1 \le 2^j s < 2^k$. Such s lie in the set $\{1, 3, 5, \ldots, 2^{k-j} - 1\}$. This set has 2^{k-j-1} elements. Hence, $\exp(-j)$ occurs 2^{k-j-1} times in $\alpha_1, \alpha_2, \ldots, \alpha_n$. Thus,

$$\alpha_1\alpha_2\cdots\alpha_n = \prod_{j=0}^{k-1} (\exp(-j))^{2^{k-j-1}} = \exp\left(-\sum_{j=0}^{k-1} j 2^{k-j-1}\right) = \exp\left(-2^k \sum_{j=0}^{k-1} \frac{j}{2^{j+1}}\right).$$

Then

$$\|A^n\|^{1/n} \ge \|A^n f_1\|^{1/n} = (\alpha_1\alpha_2\cdots\alpha_n)^{1/n} = \exp\left(-\frac{2^k}{2^k - 1} \sum_{j=0}^{k-1} \frac{j}{2^{j+1}}\right).$$

Using Gelfand's formula and taking the limit as $k \to \infty$, we get

$$\rho(A) \ge \exp\left(-\sum_{j=0}^{\infty} \frac{j}{2^{j+1}}\right) = \delta > 0.$$

Now we construct a sequence (A_k) in $\mathcal{L}(X)$ which approximates A. For each $k \in \mathbb{N}$, we define a linear map $A_k : X \to X$,

$$A_k f_m = \begin{cases} 0 & \text{if } m = 2^k s \text{ with } s \text{ odd}, \\ A f_m = \alpha_m f_{m+1} & \text{otherwise}. \end{cases} \qquad (*)$$

Because $\alpha_m \in [0,1]$ for all $m \in \mathbb{N}$, A_k is continuous and $\|A_k\| \le 1$. We show that $A_k \to A$ in $\mathcal{L}(X)$.

$$(A - A_k) f_m = \begin{cases} A f_m = \exp(-k) f_{m+1} & \text{if } m = 2^k s,\ s \text{ odd} \\ 0 & \text{otherwise}. \end{cases}$$

Then $\|A - A_k\| \le \exp(-k)$. Hence $A_k \to A$ in $\mathcal{L}(X)$.

By the definition of A_k at $(*)$, it is a nilpotent operator. In fact,

$A_k^{2^{k+1}} \equiv 0$ because $A_k^{2^{k+1}} f_m = 0$ for all $m \in \mathbb{N}$. We now show that every nilpotent operator has spectrum equal to $\{0\}$. Let $B \in \mathcal{L}(X)$ be a nilpotent, i.e. $B^n = 0$ for some $n \in \mathbb{N}$. For each $\lambda \in \mathbb{C} \setminus \{0\}$,

$$(\lambda - B)(\lambda^{n-1} + \lambda^{n-2} B + \cdots + \lambda B^{n-2} + B^{n-1}) = \lambda^n - B^n = \lambda^n I$$

$$\underbrace{(\lambda^{n-1} + \lambda^{n-2} B + \cdots + \lambda B^{n-2} + B^{n-1})}_{\tilde{B}} (\lambda - B) = \lambda^n - B^n = \lambda^n I$$

Thus, $(\lambda - B)^{-1} = \lambda^{-n} \tilde{B} \in \mathcal{L}(X)$. Thus, $\lambda \in \mathbb{C} \setminus \sigma(B)$. Then $\mathbb{C} \setminus \{0\} \subset \mathbb{C} \setminus \sigma(B)$, which leads to $\sigma(B) \subset \{0\}$. Because $\sigma(B) \neq \phi$, $\sigma(B) = \{0\}$.

By this, we have showed that $\rho(A_k) = 0$ for all $k \in \mathbb{N}$. Thus, (A_k) is a sequence in $\mathcal{L}(X)$ approaching to A but $\lim_{k \to \infty} \rho(A_k) = 0 < \rho(A)$.