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Math 8301, Manifolds and Topology

Homework 1   16/20 (missing #5)

(1) Show that any open subset of a manifold is a manifold.

Proof Let M be an n-dimensional manifold, and A be an open subset of M. We'll show that A is also an n-dimensional manifold. Take $x \in A$, we'll show that x has an open neighborhood in A that is homeomorphic to $\mathbb{R}^n$. Since M is a manifold, there exists an open neighborhood of x in M, called U, such that $U \cong \mathbb{R}^n$. Thus, there is a homeomorphism $f: U \to \mathbb{R}^n$. Put $V = A \cap U$. Then V is open in U since A is open in M.

We have $x \in V$.

Since f is continuous at x, there exists an open ball $B(f(x),r)$ in $\mathbb{R}^n$

such that $f^{-1}(B(f(x),r)) \subset V$. Put $W = f^{-1}(B(f(x),r))$.

Then W is open in U and contains x. Moreover $W \overset{f}{\cong} B(f(x),r)$.

then $W \cong \mathbb{R}^n$ by the following chain

$$W \overset{f}{\cong} B(f(x),r) \xrightarrow[\cong]{z \mapsto z - f(x)} B(0,r) \xrightarrow[\cong]{z \mapsto \frac{1}{r}z} B(0,1) \xrightarrow[\cong]{z \mapsto \frac{z}{1-\|z\|}} \mathbb{R}^n$$

Here we'll ~~prove~~ verify the last homeomorphism:

$$\phi : B(0,1) \longrightarrow \mathbb{R}^n$$
$$z \longmapsto \frac{z}{1-\|z\|}$$

- $\underline{\phi \text{ is injective}}$

If $\phi(z_1) = \phi(z_2)$ then $\frac{z_1}{1-\|z_1\|} = \frac{z_2}{1-\|z_2\|}$ $(*)$ then

$$\frac{\|z_1\|}{1-\|z_1\|} = \frac{\|z_2\|}{1-\|z_2\|}, \text{ or } \|z_1\|(1-\|z_2\|) = \|z_2\|(1-\|z_1\|), \text{or}$$

$$\|z_1\| = \|z_2\|$$

Thus $(*)$ implies $z_1 = z_2$.

- $\underline{\phi \text{ is surjective}}$

For each $y \in \mathbb{R}^n$, we have $\phi\left(\frac{y}{1+\|y\|}\right) = y$. Thus ϕ is surjective. Thus ϕ is bijective and

$$\phi^{-1}(y) = \frac{y}{1+\|y\|}$$

- Since the identity map on ~~B(0,1)~~ $\mathbb{R}^n$ and the norm are continuous, we have ϕ and ϕ^{-1} are continuous. Thus ϕ is a homeomorphism.

Since $V = A \cap U$ open in A, $f(A \cap U)$ open in $\mathbb{R}^n$ (since f is a homeomorphism)
$\Rightarrow$ locally Euclidean

In conclusion, W is an open neighborhood of x in A that is homeomorphic to $\mathbb{R}^n$. Therefore, A is ~~an n-manifold~~ locally Euclidean.

Finally, to confirm that A is n-manifold, we need to check if A is Hausdorff and second countable. Since $\underset{M?}{\mathbb{R}^n}$ is Hausdorff, the induced topology on A is also Hausdorff. ~~For~~

We know that $\mathbb{Q}^n$ is countable and dense in $\mathbb{R}^n$. Thus we can label the elements of $\mathbb{Q}^n$ as a sequence $(r_n)_{n\in\mathbb{N}}$. For each $n, k \in \mathbb{N}$, we denote $B_{nk} = B(r_n, \frac{1}{k})$. Then $\{B_{nk}\}_{n,k\in\mathbb{N}}$ is a topological basis of $\mathbb{R}^n$ (which implies that $\mathbb{R}^n$ is second countable). Thus, the family $\{B_{nk} \cap A : n, k \in \mathbb{N}\}$ is a topological basis of A, which concludes that A is second countable.

From now on, if we need to check if a topological space M is an n-manifold while we know that M is contained in some $\mathbb{R}^m$, we ~~do not~~ ~~ne~~ know that M is already Hausdorff and second countable. In other words, we only need to check the "Local Euclidean" property. 4/4

(2) For each value $t \in \mathbb{R}$, decide whether the space

$$\{(x,y,z) \in \mathbb{R}^3 : xyz = t\}$$

is a manifold, and explain why or why not.

Proof We consider two cases: $t \neq 0$ and $t = 0$.

Case 1: If $t \neq 0$, then we define a function

$$f: \mathbb{R}^3 \to \mathbb{R}$$
$$f(x,y,z) = xyz - t$$

Then f is continuously differentiable and $\nabla f(x,y,z) = (yz, zx, xy)$.

If $f(x,y,z) = 0$ then $xyz = t \neq 0$; consequently $x, y, z \neq 0$ and $\nabla f \neq 0$.

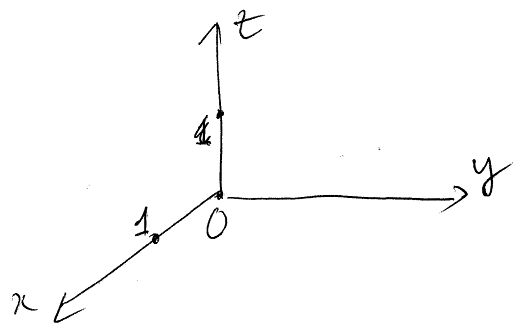
Thus (by Implicit Function Theorem), the set $\{(x,y,z) : f(x,y,z) = 0\}$ is a 2-manifold. In other words, the set

$\{(x,y,z) \in \mathbb{R}^3 : xyz = t\}$ is a 2-manifold.

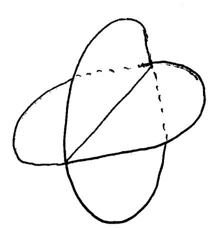
Case 2: If $t=0$, then

$$\{(x,y,z): xyz=0\} = \{(x,y,z): x=0 \text{ or } y=0 \text{ or } z=0\},$$

which is the union of three planes $x=0$, $y=0$ and $z=0$.



Consider the point of coordinate $(x,y,z)=\cancel{(0,0,1)}\ (1,0,0)$. ~~Then~~ It has an open neighborhood looking like a star fruit:

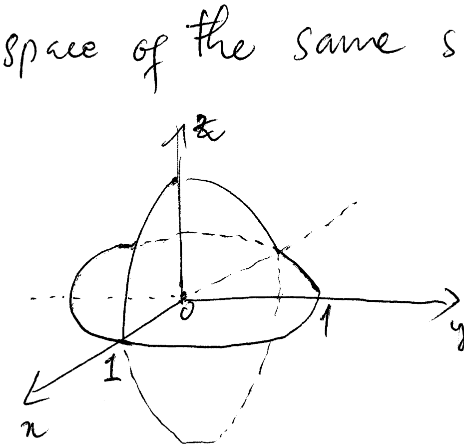


This shape is formed by taking the intersection

$$\underbrace{B((1,0,0),r)}_{\text{ball in } \mathbb{R}^3,\text{ with any } r<1.} \cap \{(x,y,z): x=0 \text{ or } y=0 \text{ or } z=0\}$$

We will this shape is not homeomorphic to $\mathbb{R}^n$, for any $n\in\mathbb{N}$.

Before doing so, we see that the "starfruit" is homeomorphic to the space of the same shape, called M, which is made by



two ~~di~~ open unit disks:

$$B_1 = \{(x,y,z): z=0,\ x^2+y^2<1\}$$

$$B_2 = \{(x,y,z): y=0,\ x^2+z^2<1\}$$

and $M = B_1 \cup B_2$.

$\boxed{6}$

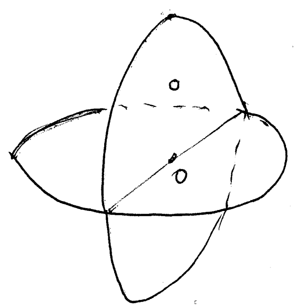
B_1 and B_2 can be thought as the open unit ball $B_2(0,1)$ in $\mathbb{R}^2$ because the ~~topology~~ topologies on them are the same. Suppose by contradiction that $M \cong \mathbb{R}^n$ for some $n \in \{0,1,2,\dots\}$.

- If $n=0$: $M \cong \mathbb{R}^0$

  Then each singleton is open in M, which is not true because each open set in M must contain infinitely many elements.

- If $n=1$: $M \cong \mathbb{R}^1$

  By removing one point in M, for instance $(0,0,\frac{1}{2})$, we still have a homemorphism from



  $M \setminus (0,0,\frac{1}{2})$ to $\mathbb{R}^1 \setminus \{a\}$

  $\uparrow$ Some point $a \in \mathbb{R}$

  We see that $M \setminus (0,0,\frac{1}{2})$ is path-connected, while $\mathbb{R}^1 \setminus \{a\}$ is not path-connected. This is a contradiction.

- If $n=2$: $M \cong \mathbb{R}^2$

  Let $f: M \longrightarrow \mathbb{R}^2$ be a homeomorphism. Then we define $g_1 : B_1 \to f(B_1)$ such that $g_1(x) = f(x)$ $\forall x \in B_1$. Then g_1 is bijective. We'll show that

g_1 is continuous and g_1^{-1} is also continuous. Equivalently, we show that $g_1(U)$ is open in $f(B_1)$ if U is open in B_1, and $g_1^{-1}(V)$ is open in B_1 if V is open in $f(B_1)$. Let U be open in B_1. Then there exists U' open in M such that $U = U' \cap B_1$. Since f is a homeomorphism, $f(U')$ is open in $\mathbb{R}^2$. Thus $f(U' \cap B_1) = f(U') \cap f(B_1)$ is open in $f(B_1)$. Thus $g_1(U) = f(U)$ is open in $f(B_1)$. Similarly, $g_1^{-1}(V)$ is open in B_1 if V is open in $f(B_1)$. Therefore g_1 is a homeomorphism. Symmetrically,

$$g_2 : B_2 \to f(B_2)$$
$$g_2(x) = f(x)$$

is also a homeomorphism. Now we apply the "Invariance of domain" theorem, which says that:

[If two subsets in $\mathbb{R}^n$ are homeomorphic, and one of them is open, then so is the other.]

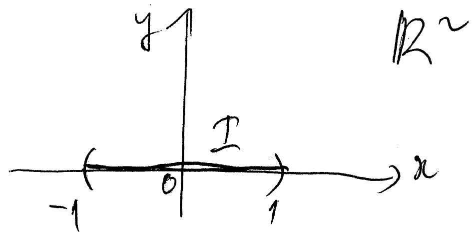
Here B_1 and $f(B_1)$ are ~~hom~~ subsets of $\mathbb{R}^2$, and they are homeomorphic, and B_1 is open in $\mathbb{R}^2$. Thus $f(B_1)$ is open in $\mathbb{R}^2$. Likewise, $f(B_2)$ is open in $\mathbb{R}^2$. Thus,

$$f(B_1 \cap B_2) = f(B_1) \cap f(B_2) \text{ is also open in } \mathbb{R}^2$$

Put $I = B_1 \cap B_2$, then I is an ~~inter~~ open interval in $\mathbb{R}$.

$$I = \{(x,y,z) : y = z = 0,\ -1 < x < 1\}$$

~~In another~~ On the other hand, I can be thought as a segment in $\mathbb{R}^2$:



$$I \cong \{(x,y) : y = 0,\ -1 < x < 1\}$$

Then I and $f(I)$ are subsets in $\mathbb{R}^2$, which are homeomorphic. Moreover, $f(I)$ is open in $\mathbb{R}^2$ as we've shown above. Thus I must also be open in $\mathbb{R}^2$. This is a contradiction because no disc of center O in $\mathbb{R}^2$ is contained in I, whence the proof completes.

(b) ~~$n \geq 3$: $M \cong \mathbb{R}^n$~~ $n = 3$: $M \cong \mathbb{R}^3$

We see that M and $\mathbb{R}^3$ are subsets of $\mathbb{R}^3$, and $\mathbb{R}^3$ is open in $\mathbb{R}^3$. Thus, M must also be open in $\mathbb{R}^3$. This is a contradiction because M contains no open ball in $\mathbb{R}^3$.

(c) $n \geq 4$: $M \cong \mathbb{R}^n$

We can consider M as a subset of $\mathbb{R}^n$ because $\mathbb{R}^3$ can be imbedded into $\mathbb{R}^n$. We see that M and $\mathbb{R}^n$ are homeomorphic subsets in $\mathbb{R}^n$. Thus M must be open in $\mathbb{R}^n$. This is a contradiction ~~by~~ because M contains no open ball in $\mathbb{R}^n$.

Easier: removing the x-axis from this starfruit gives 4 connected components. This is impossible in $\mathbb{R}^2$ (removing a line gives $\underline{2}$ connected components at most)

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(3) For which values $t \in \mathbb{R}$ is the space

$$\{(x,y) \in \mathbb{R}^2 \mid x^2 + xy + ty^2 = 1\}$$

a closed manifold?

Proof

Let us put $M = \{(x,y) \in \mathbb{R}^2 \mid x^2 + xy + ty^2 = 1\}$. Then M is a topological space with the topology induced by that of $\mathbb{R}^2$. Again, as mentioned in Problem (1), M is Hausdorff and second countable. We only need to concern the local Euclidean property of M. We see that

$$x^2 + xy + ty^2 = \left(x + \frac{y}{2}\right)^2 + \left(t - \frac{1}{4}\right)y^2$$

Now we consider <u>three</u> cases ~~to~~ and decide in each case whether M is a closed manifold. Hereby we confirm again that a closed manifold is a compact manifold without boundary.

<u>Case 1</u>: $t < 1/4$

We put $c = \frac{1}{4} - t > 0$. For each $n \in \mathbb{N}$, we choose $x = n$ and solve y such that $(x, y) \in M$. We have

$$1 = \cancel{x^2 + xy +} \left(x + \frac{y}{2}\right)^2 - cy^2 = x^2 + xy + \left(\frac{1}{4} - c\right) y^2$$

Replace x by n. We get

$$\left(\frac{1}{4} - c\right) y^2 + ny + (n^2 - 1) = 0$$

$$\Delta = n^2 - 4\left(\frac{1}{4} - c\right)(n^2 - 1) = n^2 - (1 - 4c)(n^2 - 1)$$

$$= n^2 - (1 - 4c)n^2 + (1 - 4c)$$

$$= 4cn^2 + (1 - 4c)$$, which is positive provided that n is sufficiently large. One solution for y is

$$y_n = \frac{-n + \sqrt{4cn^2 + (1 - 4c)}}{2\left(\frac{1}{4} - c\right)}$$

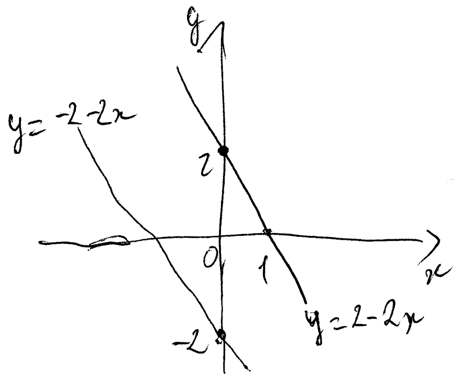
Therefore, the sequence $\{(x_n, y_n)\}$, where $x_n = n$, belongs to M ~~for~~ from some index $n_0 \in \mathbb{N}$. However, this sequence has no convergent subsequence because $x_n = n \to \infty$ as $n \to \infty$. Thus M is not a compact manifold, and consequently not a closed manifold.

Case 2: $t = 1/4$

Then $M = \{(x,y)\in\mathbb{R}^2 : (x + \frac{y}{2})^2 = 1\}$

$= \{(x,y)\in\mathbb{R}^2 : x+\frac{y}{2} = 1$ or $x + \frac{y}{2} = -1\}$

$= \{(x,y)\in\mathbb{R}^2 : y = 2-2x\} \cup \{(x,y)\in\mathbb{R}^2 : y = -2-2x\}$



For each $n \in \mathbb{N}$, we put

$x_n = n$ and $y_n = 2-2n$

Then the sequence $\{(x_n, y_n)\}$ is in M, which has no convergent subsequence since $x_n \to \infty$.

Thus M is not compact, and subsequently not a closed manifold.

Case 3: $t > 1/4$

We can put $c = \sqrt{t - \frac{1}{4}} > 0$. Then

$$M = \{(x,y) \in \mathbb{R}^2 : (x + \frac{y}{2})^2 + c^2y^2 = 1\}$$

We'll show that $M \cong S^1$ and then show that S^1 is a closed 1-manifold.

First, let's consider the following linear transformation from $\mathbb{R}^2$ to itself

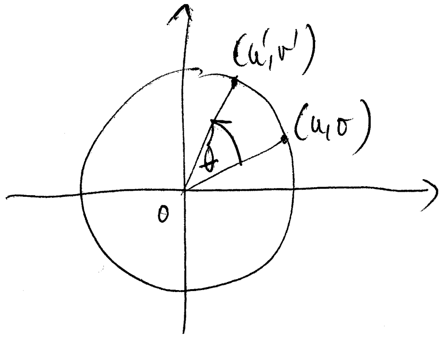
$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Since $\det \begin{pmatrix} 1 & 1/2 \\ 0 & c \end{pmatrix} = c \neq 0$, the transformation is bijective, linear and thus continuous in $\mathbb{R}^2$. The inverse transformation is also continuous. Thus we get a homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. We see that

$$\left(x + \frac{y}{2}\right)^2 + c^2 y^2 = 1 \Leftrightarrow u^2 + v^2 = 1$$

Thus, in this transformation, M is mapped to S^1. Hence $M \cong S^1$.

Now the last step to do is to show that S^1 is a closed 1-manifold. To do so, we'll show that each element in S^1 has a neighborhood that is homeomorphic to $\mathbb{R}$.



We have the following linear tranformation from S^1 to itself

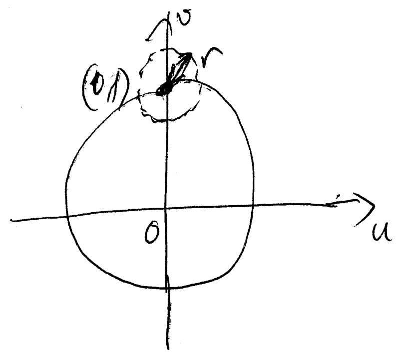
$$\phi_\theta : S^1 \longrightarrow S^1$$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix}$$

This rotation is also a homeomorphism which can map one point in S^1

to any other point in S^1 by selecting appropriate angle θ.

Thus, we only need to show that the point $(u,v)=(0,1)$ has a neighborhood in S that is homeomorphic to $\mathbb{R}$.



Let $r \in (0,1)$. Then the point $\vec{z} = (0,1)$ in S^1 has a neighborhood

$$V = S^1 \cap \underbrace{B_2(\vec{z}, r)}_{\text{ball in } \mathbb{R}^2}$$

We have $V = \{(u,v) : u^2+v^2 = 1 \text{ and } u^2 + (v-1)^2 < r^2\}$

and

$$\begin{cases} u^2+v^2 = 1 \\ u^2+(v-1)^2 < r^2 \end{cases} \iff \begin{cases} v^2 = 1-u^2 \\ (u^2+v^2) - 2v + 1 < r^2 \end{cases}$$

$$\iff \begin{cases} v^2 = 1-u^2 \\ 2-2v < r^2 \end{cases} \iff \begin{cases} v^2 = 1-u^2 \\ v > \frac{2-r^2}{2} \end{cases}$$

$$\iff \begin{cases} v = \sqrt{1-u^2} \\ v > \frac{2-r^2}{2} \end{cases} \iff \begin{cases} v = \sqrt{1-u^2} \\ \sqrt{1-u^2} > \alpha \end{cases}, \text{ where } \alpha = \frac{2-r^2}{2}$$

$$\iff \begin{cases} v = \sqrt{1-u^2} \\ |u| < \sqrt{1-\alpha^2} \end{cases}$$

$$\iff \begin{cases} v = \sqrt{1-u^2} \\ u \in (-\beta, \beta) \end{cases}, \text{ where } \beta = \sqrt{1-\alpha^2} > 0$$

Thus $V = \{(u,v) : v = \sqrt{1-u^2} \text{ and } u \in (-\beta,\beta)\}$

$= \{(u, \sqrt{1-u^2}) : u \in (-\beta,\beta)\}$

We define a map $\phi : (-\beta,\beta) \longrightarrow V$

$u \longmapsto (u, \sqrt{1-u^2})$

Then ϕ is injective, surjective and continuous (since each component function $u \mapsto u$, $u \mapsto \sqrt{1-u^2}$ is continuous). Moreover, the inverse map of ϕ is $\phi^{-1}(u,v) = u \quad \forall (u,v) \in V$. This is also a continuous function. Thus ϕ is a homeomorphism. Thus

$$V \cong (-\beta,\beta) \xrightarrow[\cong]{x \mapsto \frac{x}{\beta - |x|}} \mathbb{R}^1$$

Easier: $x^2+xy+ty^2=1$ is a conic. Use the discriminant to classify the graph in terms of t.

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~~Thus~~ In conclusion, M is a closed manifold if and only if $t > \frac{1}{4}$.

④ If M is any surface and S^2 is the 2-sphere, explain why $S^2 \# M \cong M$.

Proof In the proof, we will use the Invariance of the Boundary theorem, which says:

[If M is a manifold with boundary, then a point of M cannot be both a boundary point and an interior point. Thus ∂M and $\operatorname{Int} M$ are disjoint subsets whose union is M.]

(*) A corollary of this is: Let M and N be two homeomorphic topological ~~vector~~ spaces ~~and~~ through $f: M \to N$. Suppose that M is a manifold with boundary. Then N is also a manifold with boundary and $f(\partial M) = \partial N$.

*Proof of the corollary:

For each $x \in \partial M$, x has a neighborhood in M, say U that is homeomorphic to the closed upper half plane $\mathbb{H}^n$, where $x \mapsto \in \partial \mathbb{H}^n$. We put $V = f(U)$. Then V is a neighborhood of $f(x)$ in N and $V \cong U \cong \mathbb{H}^n$. Thus $f(x) \in \partial N$. Thus $f(\partial M) \subset \partial N$.

Now take $y \in \partial N$. We'll show that $y \in f(\partial M)$. Suppose by contradiction that $y \notin f(\partial M)$. Then there exists $x \in \operatorname{int} M$ such that $y = f(x)$. Because $x \in \operatorname{int} M$, x has a neighborhood in M, called U', that is homeomorphic to $\mathbb{R}^n$. Thus $V' = f(U') \cong \mathbb{R}^n$ is a neighborhood of $f(x) = y$ in N. Thus y is also an interior point of N, which contradicts the Invariance of the Boundary theorem. ∎

Now in order to prove the problem, we claim the following without proof:

$\bar{B}_2$ (the closed unit ball in $\mathbb{R}^2$) and $\mathbb{R}^2 \setminus B_2$ are manifolds with boundary and $\partial \bar{B}_2 = \partial(\mathbb{R}^2 \setminus B_2) = S^1$

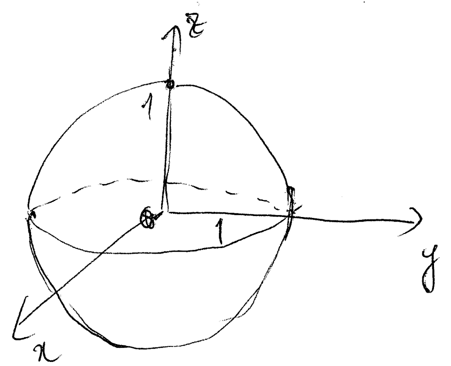
We'll follow the following steps, one by one, to the conclusion:

1) ~~Show that~~ Let U be an open subset of M which is homeomorphic to $\mathbb{R}^2$ and

$$f : \mathbb{R}^n \to U$$

be a homeomorphism. Put $B = f(B_2)$ (B is called coordinate ball). Then $M \setminus B$ is a manifold with boundary and $\partial(M \setminus B) = f(\partial \overline{B}_2)$. We ~~2)~~ then denote $\overline{B} = f(\overline{B}_2)$. By the Corollary (*), we have $\partial \overline{B} = \partial(M \setminus B)$.

2) The sphere S^2 after being removed the part $S^2 \cap B_3(\vec{z}_0, \sqrt{2})$, where $\vec{z}_0 = (0,0,1)$, is homeomorphic to $\overline{B}_2$. In other words, there exists a homeomorphism $g : S \to \overline{B}_2$, where $S = S^2 \setminus B_3(\vec{z}_0, \sqrt{2})$.



Consequently, S is a manifold with boundary and $\partial S \cong \partial \overline{B}_2$

3) Then we recall the definition of connected sum:

$$M \# S^2 := (M \setminus B) \amalg_r S$$

where r is any homeomorphism from $\partial(M \setminus B)$ to ∂S. The notation $\amalg_r$ indicates the ~~dis~~ quotient topology on the disjoint union of $M \setminus B$

and S, the equivalence relation on $(M\setminus B)\amalg S$ identifies x and $r(x)$, for each $x\in\partial(M\setminus B)$. Since $S \overset{g}{\cong} \bar{B}_2 \overset{f}{\cong} \bar{B}$, ~~wh~~ we can choose $r = f\circ g$.

$$\begin{tikzcd} \bar{B} & \bar{B}_2 \arrow[l, "f"'] \\ & S \arrow[u, "g"'] \end{tikzcd}$$

Then we want $(M\setminus B)\amalg_r S \cong (M\setminus B)\amalg_f \bar{B}_2$.

4) Once we obtain the above homeomorphism, we'll show that

$$(M\setminus B)\amalg_f \bar{B}_2 \cong M$$, which completes the proof.

Giving detail for each step

Step 1: We have 2 things to prove

- $M\setminus B$ is a manifold with boundary and $\partial(M\setminus B) = f(\partial\bar{B}_2)$
- $\partial\bar{B} = \partial(M\setminus B)$

The second assertion will be obtained if we have the first. Indeed, suppose that we have $\partial(M\setminus B) = f(\partial\bar{B}_2)$. Then $\bar{B} = f(\bar{B}_2) = f(B_2\cup\partial B_2)$

$$= f(B_2)\cup f(\partial B_2)$$

By corollary $(*)$, f maps boundary to boundary. Thus $\partial\bar{B} = f(\partial B_2)$, which is equal to $\partial(M\setminus B)$. Therefore, we only have to verify the first assertion.

Because $f(\mathbb{R}^2 \setminus B_2) = U \setminus B$, by Corollary (*) we have

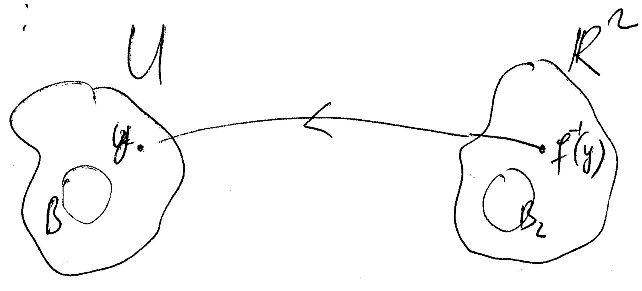
$$\cancel{f(\mathbb{R}^2} \; f(\partial(\mathbb{R}^2 \setminus B_2)) = \partial(U \setminus B), \text{ or}$$

equivalently $\partial(U\setminus B) = f(\partial \bar{B}_2)$. For each $x \in \partial(U \setminus B)$, we'll show that $x \in \partial(M \setminus B)$. Indeed, there exists an ~~neigh~~ open neighborhood of x in $U \setminus B$, say V, such that $V \cong \mathbb{H}^2$ in the way that x is mapped to $\partial \mathbb{H}^2$. We have $V = \mathcal{O} \cap (U \setminus B)$ where $\mathcal{O}$ is some open set in M. Thus

$$V = \left(\mathcal{O} \cap (U \setminus B)\right) \cap (M \setminus B) = \underbrace{\mathcal{O} \cap U}_{\text{open in } M} \cap (M \setminus B)$$

Thus V is an open neighborhood of x in $M \setminus B$, which is homeomorphic to $\mathbb{H}^2$ in the way that x is mapped to $\partial \mathbb{H}^2$. Thus $x \in \partial(M \setminus B)$, and $\partial(U \setminus B) \subset \partial(M \setminus B)$. In other words, $f(\partial \bar{B}_2) \subset \partial(M \setminus B)$. Now we'll show that $\partial(M \setminus B) \subset f(\partial \bar{B}_2)$. Suppose ~~the~~ by contradiction that there exists $y \in \partial(M \setminus B) \setminus f(\partial \bar{B}_2)$. We consider 2 cases:

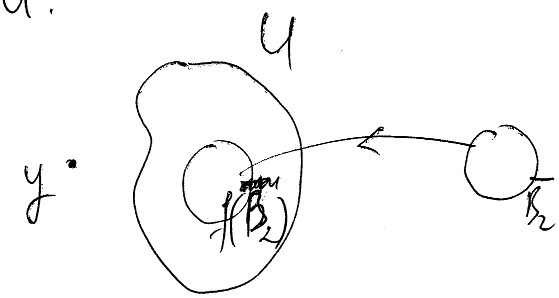
- $y \in U$:



we have ~~Since~~ $y \in \partial(M \setminus B) \subset (M \setminus B)$ (Invariance of the Boundary theorem)

Thus $y \in U \setminus B$. Since $y \in \partial(M \setminus B)$, there exists an open neighborhood of y in $M \setminus B$, say V, that's homeomorphic to $\mathbb{H}^2$ the way that maps y to $\partial \mathbb{H}^2$. Since $U \setminus B$ is open in $M \setminus B$, $V \cap (U \setminus B)$ is a neighborhood of y in $U \setminus B$ that contains another neighborhood of y which is homeomorphic to $\mathbb{H}^2$ the way that maps y to $\partial \mathbb{H}^2$. Thus $y \in \partial(U \setminus B)$ and therefore $y \in f(\partial \bar{B}_2)$, which is a contradiction.

- $y \in M \setminus U$:



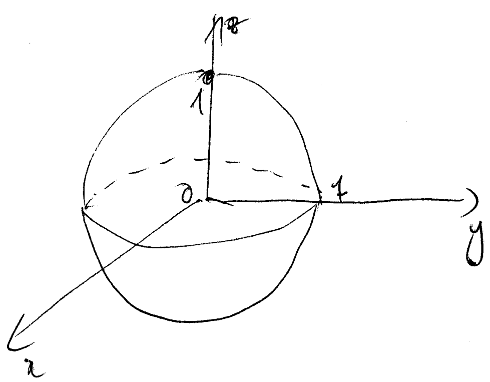
Because $\bar{B}_2$ is a compact set, $f(\bar{B}_2)$ is also compact. We see that $y \in M \setminus U$ and ~~thus~~ $f(\bar{B}_2) \subset U$. Thus $y \notin f(\bar{B}_2)$. Since M is a Hausdorff space, we can separate y from the compact set $f(\bar{B}_2)$ by open sets. Thus there exists an open set V in M which contains y and $V \cap f(\bar{B}_2) = \emptyset$. Thus $V \subset \underbrace{M \setminus f(\bar{B}_2)}_{\text{open in } M}$. ~~Thus~~ $\subset M \setminus f(B_2) = M \setminus B$

Thus V is an open neighborhood of y in $M \setminus B$. Since M is 2-manifold, V

(20)

contains an open set V' in M which is homeomorphic to $\mathbb{R}^n$. Thus y must be an interior point of $M \setminus B$. This contradicts the fact that $y \in \partial(M \setminus B)$.

Step 2:



$$S = S^2 \setminus B_3(\vec{z_0}, \sqrt{2}) \text{ where } \vec{z_0} = (0,0,1)$$

$$= \cancel{\{(x,y,z): x^2+y^2+z^2=1, (z-1)^2 < 2\}}$$

$$= \cancel{\{(x,y,z): x^2+y^2+z^2=1, -\sqrt{2}+1 < z < 1+\sqrt{2}\}}$$

$$= \{(x,y,z): x^2+y^2+z^2 = 1,\ x^2+y^2+(z-1)^2 \geq 2\}$$

$$= \{(x,y,z): x^2+y^2+z^2 = 1,\ 2-2z \geq 2\}$$

$$= \{(x,y,z) : x^2+y^2+z^2 = 1,\ z \leq 0\}$$

Define the map $g: S \to \overline{B_2}$

$$(x,y,z) \mapsto (x,y)$$

~~then γ is injective because~~

- g is injective:

if $g(x,y,z) = g(x',y',z')$ then $x = x'$ and $y = y'$. Moreover,

$$z = -\sqrt{1-(x^2+y^2)} = -\sqrt{1-(x'^2+y'^2)} = z'$$

- γ is surjective:

For each $(x,y) \in \overline{B_2}$, we have $x^2+y^2 \leq 1$. Define $z = -\sqrt{1-(x^2+y^2)}$

Then $x^2+y^2+z^2=1$ and $z\leq 0$ and $g(x,y,z)=(x,y)$.

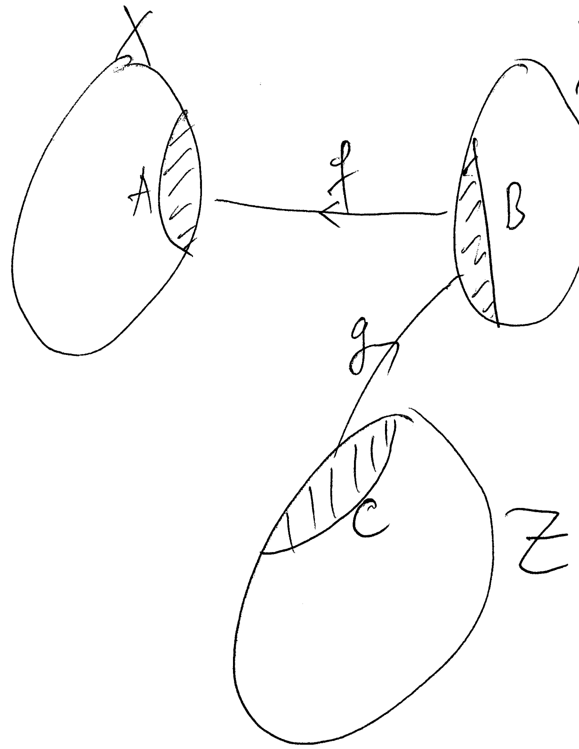
- g is continuous: immediately because g is just the natural projection.

$g^{-1}: \bar{B}_2 \to S$

$(x,y) \mapsto (x,y,-\sqrt{1-(x^2+y^2)})$

Hence g^{-1} is continuous. Therefore g is a homeomorphism from S to $\bar{B}_2$.

Step 3 To show that $(M\setminus B) \coprod_{F=f\circ g} S \cong (M\setminus B) \coprod_f \bar{B}_2$, we will work on a more general (abstract) set up.



Let X, Y, Z be three topological spaces; A, B, C be subsets of X, Y, Z respectively. ~~Then~~ Suppose there exist homeomorphisms

$f: B \to A$,

$g: Z \to Y$ that satisfies $g(C)=B$

Then we'll prove that $X \coprod_f Y \cong X \coprod_{f\circ g} Z$.

Indeed, by definition of adjunction topologies,

$X \coprod_f Y := (X \amalg Y)/\sim_f$

$$X \amalg_f Y := (X \amalg Y)/\sim_f$$

$$= \underbrace{\{\{y, f(y)\} : y \in B\}}_{①_f} \cup \underbrace{\{\{x\} : x \in X \setminus A\}}_{②_f} \cup \underbrace{\{\{y\} : y \in Y \setminus B\}}_{③_f}$$

We put $h = f \circ (g|_C)$.

$$X \amalg_h Z := (X \amalg Z)/\sim_h$$

$$= \underbrace{\{\{z, h(z)\} : z \in C\}}_{①_h} \cup \underbrace{\{\{x\} : x \in X \setminus A\}}_{②_h} \cup \underbrace{\{\{z\} : z \in Z \setminus C\}}_{③_h}$$

We are trying to find a homeomorphism between these two sets.

$$\begin{tikzcd}
X \amalg Z \arrow[r, "q_2"] \arrow[d, "k"] & (X \amalg Z)/\sim_h \arrow[d, dashed, "\phi"] \\
X \amalg Y \arrow[r, "q_1"] & (X \amalg Y)/\sim_f
\end{tikzcd}$$

Define the map $k : X \amalg Z \to X \amalg Y$

$$k(u) = \begin{cases} u & \text{if } u \in X \\ g(u) & \text{if } u \in Z \end{cases}$$

Then k is continuous on the disjoint topology $X \amalg Z$ because $k|_X = id_X$ and $k|_Z = g$ are continuous. Similarly, k^{-1} is also continuous. Thus k is a homeomorphism.

Let q_1 and q_2 be the canonical projection as shown in the diagram. Now we define $\phi : (X \amalg Z)/\sim_h \longrightarrow (X \amalg Y)/\sim_f$

such that

$$\phi(\{z, h(z)\}) = \{g(z), f(g(z))\} \qquad \forall z \in C$$

$$\phi(\{x\}) = \{x\} \qquad \forall x \in X \setminus A$$

$$\phi(\{z\}) = \{g(z)\} \qquad \forall z \in Z \setminus C$$

We can see very easily that ϕ maps type $①_h$ to $①_f$, $②_h$ to $②_f$, $③_h$ to $③_f$ and ϕ is bijective. Now we'll show that ϕ is continuous and ϕ^{-1} is also continuous. Recall that the characteristic property of ~~quotient topology~~ in the general form is

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, dashed] & Y \arrow[d, "f"] \\ & Z \end{tikzcd}$$

If q is a quotient map from X to Y. Then a map $f : Y \to Z$ is continuous if and only if $f \circ q$ is continuous $X \to Z$

Therefore, to show that ϕ is continuous, we only need to show that $\phi \circ q_2$ is continuous. We will show that $\phi \circ q_2 = q_1 \circ k$ (commutative diagram), which will lead immediately to that $\phi \circ q_2$ is continuous.

For each $u \in X \amalg Z$, we'll show that $\phi \circ q_2(u) = q_1 \circ k(u)$. We have four cases for u: $u \in X \setminus A$, $u \in Z \setminus C$, $u \in A$, $u \in C$

- $u \in X \setminus A$: $u \xrightarrow{k} u \xrightarrow{q_1} \{u\}$

  $u \xrightarrow{q_2} \{u\} \xrightarrow{\phi} \{u\}$

- $u \in Z \setminus C$: $u \xrightarrow{k} g(u) \xrightarrow{q_1} \{g(u)\}$

  $u \xrightarrow{q_2} \{u\} \xrightarrow{\phi} \{g(u)\}$

- $u \in A$: $u \xrightarrow{k} u \xrightarrow{q_1} \{u, f^{-1}(u)\}$

  $u \xrightarrow{q_2} \{u, h^{-1}(u)\} \xrightarrow{\phi} \{\underbrace{g(h^{-1}(u))}_{f^{-1}(u)}, f(\underbrace{g(h^{-1}(u))}_{u})\}$

- $u \in C$: $u \xrightarrow{k} g(u) \xrightarrow{q_1} \{g(u), f(g(u))\}$

  $u \xrightarrow{q_2} \{u, h(u)\} \xrightarrow{\phi} \{g(u), f(g(u))\}$

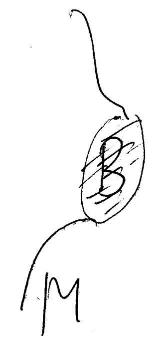
Thus $q_1 \circ k = \phi \circ q_2$, and this gives us the continuity of ϕ. Similarly, we have $q_2 \circ k^{-1} = \phi^{-1} \circ q_1$, which gives us the continuity of ϕ^{-1}. Hence ϕ is an homeomorphism. (Note that ϕ is very natural!!)

Step 4 We'll show that $(M \setminus B) \amalg_f \overline{B}_2 \cong M$.

By using the ~~pro~~ lemma proved in step 3, we know that

$$(M \setminus B) \amalg_f \overline{B_2} \cong (M \setminus B) \amalg_{id} \overline{B}$$

Thus we only need to show $(M\setminus B)\amalg_{id}\bar{B} \cong M$.



We define the map $\psi : (M\setminus B)\amalg_{id}\bar{B} \to M$

$\{x\} \mapsto x \quad \forall x \in (M\setminus B)\amalg \bar{B}$

It's obvious that ψ is bijective.

$$\begin{tikzcd}
(M\setminus B)\amalg \bar{B} \arrow[r, "q"] \arrow[dr, dashed] & (M\setminus B)\amalg_{id}\bar{B} \arrow[d, "\psi"] \\
 & M
\end{tikzcd}$$

we have $\psi\circ q(u) = u \quad \forall u \in (M\setminus B)\cup\bar{B}$. Since $\psi\circ q|_{M\setminus B}$ and $\psi\circ q|_{\bar{B}}$ are continuous, $\psi\circ q$ is continuous. Thus ψ is continuous (Characteristic of quotient space). Now we'll show ψ^{-1} is continuous. To show that, we need to show that every point $z \in (M\setminus B)\amalg_{id}\bar{B}$ has an open neighborhood (contained in any given neighborhood of z) that is homeomorphic to $\mathbb{R}^2$. There is no problem if $z \in \mathrm{int}(M\setminus B)$ or $z \in \mathring{B}$. Now if $z \in \partial(M\setminus B) = \partial(\bar{B})$, then there exists a neighborhood of z in $(M\setminus B)$ that is homeomorphic to $\mathbb{H}^2$ the way that maps z to $\partial\mathbb{H}^2$; also there exists a neighborhood of z in $\bar{B}$ that is homeomorphic to $\mathbb{H}^2$ the way that maps z to ∂H^2. Then (after taking homeomorphism), there exists a neighborhood of z in M ~~such that~~ which homeomorphic to $\mathbb{H}^2 \amalg \mathbb{H}^2 = \mathbb{R}^2$, which means z is an interior point of M.

Note: We have used "hand waving" approach in the last arguments. It's quite complicated to make regorous. The idea to do so may be the detail description of what open sets on $(M \setminus B) \amalg_{\partial} \bar{B}$ are, and then what open sets on $(M \setminus B) \amalg_{id} \bar{B}$ are (written in explicit form from a open set of M). 4/4

This is much more work than is necessary: just explain that $S^2 - D^2 \cong D^2$, so in forming $S^2 \# M$, you are gluing a copy of D^2 onto M after cutting a copy of D^2 out of M, so you aren't changing anything.

#5? 0/4