Name: Tuan Pham | Math 8301: Topology & Manifolds | 1

ID: 4652218 | Homework 10

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① Let $f: A \to B$ and $g: B \to C$ be maps of abelian groups. We'll show that there is an exact sequence:

$$0 \to \ker f \to \ker gf \to \ker g \to \operatorname{coker} f \to \operatorname{coker} gf \to \operatorname{coker} g \to 0.$$

By definition, we have $\operatorname{coker} f = B/\operatorname{Im} f$, $\operatorname{coker} gf = C/\operatorname{Im} gf$, $\operatorname{coker} g = C/\operatorname{Im} g$.

Thus, our goal is to define group morphisms $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ such that the following sequence is exact.

$$0 \to \underset{①}{\ker f} \xrightarrow{\varphi_1} \underset{②}{\ker gf} \xrightarrow{\varphi_2} \underset{③}{\ker g} \xrightarrow{\varphi_3} \underset{④}{B/\operatorname{Im} f} \xrightarrow{\varphi_4} \underset{⑤}{C/\operatorname{Im} gf} \xrightarrow{\varphi_5} \underset{⑥}{C/\operatorname{Im} g} \to 0$$

We will define $\varphi_1(x) = x$ ✓

$\varphi_2(x) = f(x)$ ✓

$\varphi_3(y) = y + \operatorname{Im} f$ ✓

$\varphi_4(y + \operatorname{Im} f) = g(y) + \operatorname{Im} gf$, for all $y \in B$. ✓

$\varphi_5(z + \operatorname{Im} gf) = z + \operatorname{Im} g$, for all $z \in C$. ✓

First we'll check that all maps above are well-defined.

- Check if φ_1 is well-defined:

Take

To check if φ_1 is well-defined, we will verify that $\ker f \subset \ker gf \subset A$. T

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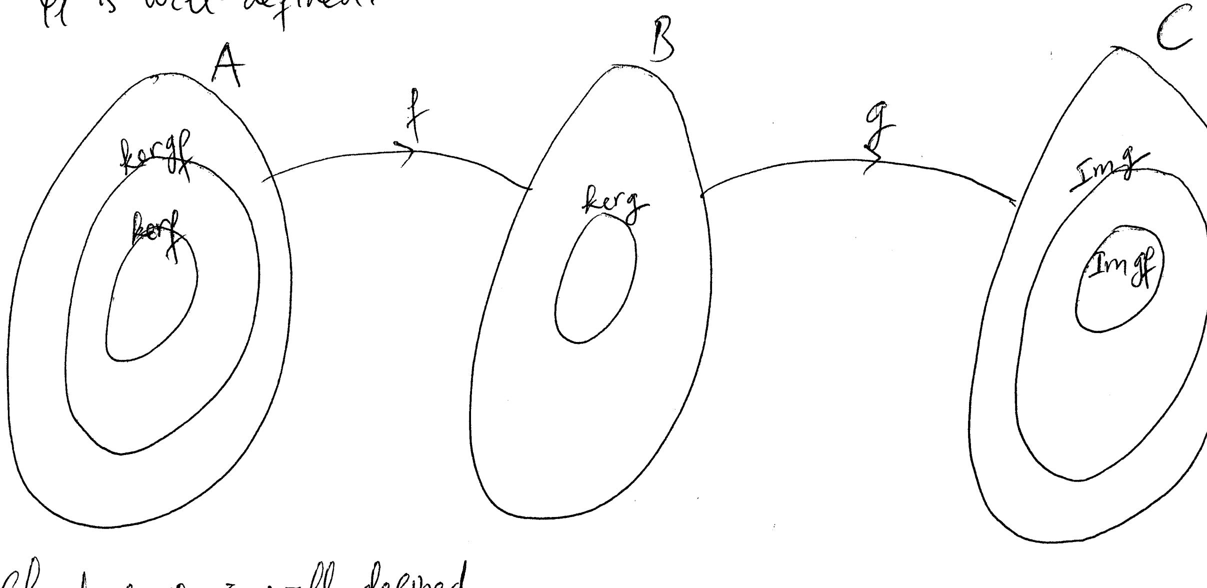
some $x \in \ker f$. Then $f(x) = 0$. Then $g(f(x)) = 0$. Then $x \in \ker gf$. Thu

φ_1 is well-defined.



- **Check if φ_2 is well-defined**

Take $x \in \ker gf$. We want to show that $\varphi_2(x) \in \ker g$. We have $gf(x) = 0$.

Thus $f(x) \in \ker g$. Thus $\varphi_2(x) \in \ker g$.

- **Check if φ_3 is well-defined**

Each $y \in \ker g$ is viewed as an element in B. On B, we have a canonical homomorphism $B \longrightarrow B/\mathrm{Im} f$. Thus $\varphi_3(y)$ was defined to be the image of y under this homomorphism. Thus φ_3 is well-defined.

- **Check if φ_4 is well-defined**

We defined
$$\varphi_4 : B/\mathrm{Im} f \longrightarrow C/\mathrm{Im}(gf)$$
$$y + \mathrm{Im} f \longmapsto g(y) + \mathrm{Im}\, gf$$

Suppose that $y + \mathrm{Im} f = y' + \mathrm{Im} f$ for some $y, y' \in B$. We want to show that $g(y) + \mathrm{Im}\, gf = g(y') + \mathrm{Im}\, gf$. We have $y - y' \in \mathrm{Im} f$. Thus, there exists $x \in A$ such that $y - y' = f(x)$. Then $g(y) - g(y') = g(y - y') = gf(x)$.

Thus $g(y)-g(y') \in \mathrm{Im}\, gf$ and therefore $g(y)+\mathrm{Im}\, gf = g(y')+\mathrm{Im}\, gf$.

- Check if φ_5 is well-defined

We defined
$$\varphi_5 : C/\mathrm{Im}\, gf \longrightarrow C/\mathrm{Im}\, g$$
$$z+\mathrm{Im}\, gf \longmapsto z+\mathrm{Im}\, g$$

Suppose that $z+\mathrm{Im}\, gf = z'+\mathrm{Im}\, gf$ for some $z, z' \in C$. We want to show that $z+\mathrm{Im}\, g = z'+\mathrm{Im}\, g$. Having $z-z' \in \mathrm{Im}\, gf$, we'll show that $z-z' \in \mathrm{Im}\, g$. There exists $x \in A$ ~~$y \in B$~~ such that $z-z' = g(f(x))$. Thus $z-z' \in \mathrm{Im}\, g$.

Now that $\varphi_1, \varphi_2, \dots, \varphi_5$ are well-defined, they are naturally group homomorphisms by definition. We'll show that the sequence is exact at 1, 2, ..., 6. The exactness at 1 is equivalent to the injectivity of φ_1, which holds because $\varphi_1(x) = x$. The exactness at 6 is equivalent to the surjectivity of φ_5, which holds because $\varphi_5(z+\mathrm{Im}\, gf) = z+\mathrm{Im}\, g$ for all $z \in C$.

- Check the exactness at 2

Since φ_1 is injective, $\mathrm{Im}\, \varphi_1 = \ker f$. We need to show that $\ker \varphi_2 = \ker f$. Because $\varphi_2(x) = f(x)$, $\ker \varphi_2 \subset \ker f$. Conversely, since $\ker f \subset \ker gf$ and $\varphi_2(x) = f(x) = 0$ for all $x \in \ker f$, we get $\ker f \subset \ker \varphi_2$. Therefore $\ker \varphi_2 = \ker f$.

- Check the exactness at 3

We want to show that $\mathrm{Im}\, \varphi_2 = \ker \varphi_3$. We have

$$
\begin{aligned}
\operatorname{Im}\varphi_2 &= \{f(x) : x \in \ker gf\} \\
&= \{f(x) : x \in A,\ g(f(x)) = 0\} \\
&= \{y \in B : g(y) = 0 \text{ and } \exists\, x \in A \text{ with } y = f(x)\} \\
&= (\ker g) \cap (\operatorname{Im} f).
\end{aligned}
$$

We have

$$
\begin{aligned}
\ker\varphi_3 &= \{y \in \ker g : \varphi_3(y) = 0\}, \\
&= \{y \in \ker g : y + \operatorname{Im} f = 0\} \\
&= \{y \in \ker g : y \in \operatorname{Im} f\} \\
&= (\ker g) \cap (\operatorname{Im} f).
\end{aligned}
$$

Therefore $\operatorname{Im}\varphi_2 = \ker\varphi_3$.

- <u>Check the exactness at ④</u>

We have $\operatorname{Im}\varphi_3 = \{y + \operatorname{Im} f : y \in \ker g\} = \cancel{(\ker g)/\operatorname{Im} f}\ \operatorname{Im} f + \ker g$.

$$\ker\varphi_4 = \{y + \operatorname{Im} f : y \in B \text{ and } \underbrace{g(y) + \operatorname{Im} gf = 0}_{(*)}\}$$

We have

$$
\begin{aligned}
(*) &\Leftrightarrow g(y) \in \operatorname{Im} gf \\
&\Leftrightarrow \exists\, x \in A : g(y) = g(f(x)) \\
&\Leftrightarrow \exists\, x \in A : g(y - f(x)) = 0 \\
&\Leftrightarrow \exists\, x \in A : y - f(x) \in \ker g \\
&\Leftrightarrow \exists\, x \in A : y \in f(x) + \ker g \\
&\Leftrightarrow y \in \operatorname{Im} f + \ker g.
\end{aligned}
$$

Therefore $\ker\varphi_4 = \operatorname{Im} f + \ker g = \operatorname{Im}\varphi_3$.

• Check the exactness at ⑤

We have $\mathrm{Im}\,\varphi_4 = \{g(y) + \mathrm{Im}\,gf : y \in B\} = \mathrm{Im}\,g + \mathrm{Im}\,gf$.

$$\begin{aligned}\ker\varphi_5 &= \{z + \mathrm{Im}\,gf : z + \mathrm{Im}\,g = 0\}\\ &= \{z + \mathrm{Im}\,gf : z \in \mathrm{Im}\,g\}\\ &= \mathrm{Im}\,g + \mathrm{Im}\,gf.\end{aligned}$$

Therefore, $\mathrm{Im}\,\varphi_4 = \ker\varphi_5$.

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③ We'll show that there exists a natural transformation $\varepsilon : C_0(X) \to \mathbb{Z}$ satisfying $\varepsilon \circ \partial = 0$ and sending any point of X to 1.

First, we will figure out what $C_0(X)$ means. By definition,

$$C_0X = \Big\{ \sum_{f\cdot n} n_\sigma \sigma \;\Big|\; \sigma : \Delta_0 \to X \text{ continuous} \Big\},$$

where $\Delta_0 = \{0\}$ is the standard 0-simplex. Therefore, any map from $\{0\}$ to X is continuous and can be identified with $\sigma(0) \in X$. Thus

$$C_0X = \bigoplus_{\substack{\sigma\text{-singular}\\ 0\text{-simplices}}} \mathbb{Z}\sigma \underset{\substack{\uparrow\\ \text{can be identified}\\ \text{with}}}{\equiv} \bigoplus_{x \in X} \mathbb{Z}\{x\} = \mathbb{Z}\langle X\rangle,$$ which is the abelian group generated by X.

By identifying C_0X with $\mathbb{Z}\langle X\rangle$, we see that C_0X is a free abelian group generated by elements of X. To define a group morphism $\varepsilon : C_0X \to \mathbb{Z}$, it suffices to define it on the basis of C_0X. We define ε as follow.

$$\varepsilon\Big(\sum_{f.n} n_x \{x\}\Big) := \sum_{f.n} n_x$$

Then ε is well-defined and being a groupmorphism from C_0X to $\mathbb{Z}$. We see that $\varepsilon(\{x\}) = 1$ for all $x \in X$. Consequently, ε is surjective.

Next, we'll show that $\varepsilon \circ \partial = 0$. Since C_1X is a free abelian group generated by singular 1-simplices, it suffices to show that $\varepsilon(\partial\sigma) = 0$ for all singular 1-simplices σ. By definition, a singular 1-simplex σ is a continuous map $\sigma : \Delta_1 = [0,1] \longrightarrow X$. We have $\partial\sigma = \sum_{i=0}^{1} (-1)^i \sigma \circ d_1^i$ where $d_1^i : \Delta_0 \longrightarrow \Delta_1$ was defined as an affine map. Specifically, $d_1^0 : \{0\} \to [0,1]$ satisfies $d_1^0(0) = 1$ and $d_1^1 : \{0\} \to [0,1]$ satisfies $d_1^1(0) = 0$. Thus, $\partial\sigma = \sigma(1) - \sigma(0)$. Thus,

$$\begin{aligned} \varepsilon(\partial\sigma) &= \varepsilon(\sigma(1) - \sigma(0)) \\ &= \varepsilon(\sigma(1)) - \varepsilon(\sigma(0)) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

You haven't shown that ε is a natural transformation, though!

Therefore, $\varepsilon \circ \partial = 0$. Consequently, we obtain the following chain complex.

$$\cdots \xrightarrow{\partial} C_2X \xrightarrow{\partial} C_1X \xrightarrow{\partial} C_0X \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

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(4) The chain complex above is now called the reduced singular chain complex of X. We ~~dno~~ denote by $\widetilde{H}_n(X)$ to be the homotopy group of this complex. The usual homology groups of X are still denoted by $H_n(X)$. We see that the reduced

chain complex and the usual chain complex of X,

$$\cdots \xrightarrow{\partial} C_2X \xrightarrow{\partial} C_1X \xrightarrow{\partial} C_0X \longrightarrow 0,$$

are the same on the left of C_0X. Thus $\tilde{H}_nX = H_nX$. What might be the difference is $\tilde{H}_0X$ and H_0X. We will show that if $X \neq \phi$ then $H_0X \simeq \mathbb{Z} \oplus \tilde{H}_0X$.

In case $X \neq \phi$, every singular n-chain ~~are~~ is well-defined, i.e. without using any convention. We denote $B = \ker\varepsilon$, $B' = \mathrm{Im}(\partial : C_1X \to C_0X)$. Then B' is a subgroup of B, and B is a subgroup of C_0X. Take $x_0 \in X$ (this is possible because $X \neq \phi$). We'll show that $C_0X = B \oplus \mathbb{Z}\{x_0\}$, where C_0X was viewed as the free abelian group generated by elements of X. This statement is just a special case of a more general result, namely:

Let $f : A \to A'$ be a surjective morphism of abelian groups and A' a free abelian group. Then there exists a subgroup C of A such that $f|_C : C \to A'$ is an isomorphism and $A = (\ker f) \oplus C$.

We can apply this result for $f = \varepsilon$, $A = C_0X$, $A' = \mathbb{Z}$, $C = \mathbb{Z}\{x_0\}$. However, we'll give a proof for our special case. The first step is to show that $C_0X = B + \mathbb{Z}\{x_0\}$. We have $B + \mathbb{Z}\{x_0\} \subset C_0X$. Moreover, each element in C_0X is of the form $z = \sum_{x \in J} n_x\{x\}$ where J is a finite set. We have

$$z = \sum_{x \in J} n_x\{x\} = \sum_{x \in J} \underbrace{n_x(\{x\} - \{x_0\})}_{\in \ker\varepsilon} + \underbrace{\sum_{x \in J} n_x\{x_0\}}_{\in \mathbb{Z}\{x_0\}} \in (\ker\varepsilon) + \mathbb{Z}\{x_0\} = B + \mathbb{Z}\{x_0\}.$$

Therefore, $C_0X = B \oplus \mathbb{Z}\{x_0\}$. The next step is to show that $B \cap \mathbb{Z}\{x_0\} = \{0\}$. Take $z \in B \cap \mathbb{Z}\{x_0\}$. Then $z \in \ker \varepsilon$ and there exists $n \in \mathbb{Z}$ with $z = n\{x_0\}$. We have $\varepsilon(z) = \varepsilon(n\{x_0\}) = n\,\varepsilon(\{x_0\}) = n$. Since $z \in \ker\varepsilon$, $\varepsilon(z) = 0$. Thus $n=0$. Thus $z = 0$.

By definition, $H_0X = C_0X/B'$ and $\tilde{H}_0X = B/B'$. Using the relation $C_0X = B \oplus \mathbb{Z}\{x_0\}$, we'll show that $(C_0X)/B' \simeq (B/B') \oplus (\mathbb{Z}\{x_0\} + B')/B'$. We'll show instead an even more general statement as follow.

[Let G be an abelian group and H, K subgroups of G such that $G = H \oplus K$. Let N be a subgroup of H. Then $G/N \simeq (H/N) \oplus ((K+N)/N)$.]

Proof. First we'll show that $(H/N) \cap (K+N)/N) = \{0\}$. Let $x+N$ be in the intersection with some $x \in G$. Since $x+N \in H/N$, $x \in H$. Since $x+N \in (K+N)/N$, $x \in K+N$. Thus there exist $x_1 \in K$ and $x_2 \in N$ such that $x = x_1 + x_2$. Since $x_2 \in N \subset H$, $x_1 = x - x_2 \in H$. Thus $x_1 \in (H \cap K)$. Thus $x_1 = 0$. Thus $x = x_2 \in N$. Hence, $x + N = 0$.

Next, we see that each $x \in G$ can be written uniquely as $x = x_1 + x_2$ with $x_1 \in H$ and $x_2 \in K$. We define the following map.

$$\varphi : G/N \longrightarrow (H/N) \oplus ((K+N)/N)$$

$$x+N \longmapsto (x_1+N) + (x_2+N)$$

Then φ is well-defined and being a group morphism. We'll show that φ is bijective. First, we check the injectivity. Suppose that $\varphi(x+N)=0$. Then $(x_1+N)+(x_2+N)=0$. Since we have the direct sum of H/N and $(K+N)/N$, it follows $x_1+N=0$ and $x_2+N=0$. Thus $x_1, x_2 \in N$. Thus $x=x_1+x_2 \in N$ and hence $x+N=N=0$. Next, we check the surjectivity. Take any $y \in H$ and $z \in K+N$. We put $x=y+z$. Then $\varphi(x)=(y+N)+(z+N)$. Thus φ is surjective. We know that $(K+N)/N \simeq K/(K\cap N)$. Thus a consequence of the above general statement is that $G/N \simeq (H/N) \oplus K/(K\cap N)$. ∎

Applying the above result for $G=C_0X$, $H=B$, $K=\mathbb{Z}\{x_0\}$, $N=B'$, we get $(C_0X)/B' \simeq (B/B') \oplus [(\mathbb{Z}\{x_0\})/(\mathbb{Z}\{x_0\} \cap B')]$. Since $B' \subset B$ and $B \cap \mathbb{Z}\{x_0\}$, we get $\mathbb{Z}\{x_0\} \cap B' = \{0\}$. Thus $(C_0X)/B' \simeq (B/B') \oplus \mathbb{Z}\{x_0\}$. Therefore,

$$H_0X \simeq \tilde{H}_0X \oplus \mathbb{Z}.$$

This statement is true for every space $X \neq \phi$. More specifically,

$$H_nX = \begin{cases} \tilde{H}_nX & \text{for all } n \geq 1, \\ \simeq \tilde{H}_0X \oplus \mathbb{Z} & \text{for } n=0. \end{cases}$$

In case $X=\phi$, all singular n-chains are conventionally zero. Thus $C_n(\phi)=\{0\}$ for all $n \geq 0$. Thus $H_n(\phi)=\{0\}$ for all ~~$n\in\mathbb{Z}$~~ $n \geq 0$. Moreover, since $\varepsilon: C_0X \to \mathbb{Z}$, ε must be also zero. Thus $\tilde{H}_0(\phi)=\{0\}$. Thus $\tilde{H}_n(\phi)=\{0\}$

There's an extra reduced homology group : $\tilde{H}_{-1}(\emptyset) \cong \mathbb{Z}$!

for all $n \geq 0$.

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(5) Let $(\mathcal{V}, \mathcal{F})$ be a simplicial complex that has the property that for all vertices $v \in \mathcal{V}$, there are only finitely many faces $\sigma \in \mathcal{F}$ containing v (then we say $(\mathcal{V}, \mathcal{F})$ is locally finite). First, we'll try to define the groups C_n^{BM} whose elements are arbitrary sums of n-simplices. We denote by $\mathbb{Z}\sigma$ the free cyclic group generated by a face $\sigma \in \mathcal{F}$. Then we define C_n^{BM} to be the <u>product</u> of the abelian groups $\mathbb{Z}\sigma$, where σ's are the n-dimensional faces of $\mathcal{F}$. That is, $C_n^{BM} = \prod_{\substack{\sigma \in \mathcal{F} \\ |\sigma| = n+1}} \mathbb{Z}\sigma$.

We put $I_n = \{\sigma \in \mathcal{F} : |\sigma| = n+1\}$ for all $n \geq 0$. Then there are two other ways to write C_n^{BM}:

1) $C_n^{BM} = \left\{ \sum_{\sigma \in I_n} m_\sigma \sigma \;\middle|\; m_\sigma \in \mathbb{Z} \right\}$. This is the notation we used in case that $\mathcal{V}$ is finite.

2) $C_n^{BM} = \mathrm{Map}(I_n, \mathbb{Z})$, which is the abelian group of maps from I_n to $\mathbb{Z}$, under the ~~com~~ pointwise addition.

Next, we'll define a boundary operator $\partial : C_n^{BM} \to C_{n-1}^{BM}$ for $n \geq 1$. The first way to write C_n^{BM} gives us a hint. We define

$$\partial\left(\sum_{\sigma \in I_n} m_\sigma \sigma\right) := \sum_{\sigma \in I_n} m_\sigma \partial\sigma,$$

where the boundary of an n-face is defined as usual:

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \{v_0, \dots, \underset{\substack{\uparrow \\ \text{drop this vertex}}}{\hat{v}_i}, \dots, v_n\}$$

where $\sigma = \{v_0, v_1, \dots, v_n\}$. We need to show that our definition is good. Each face $\sigma' \in I_{n-1}$ is contained in only finitely many faces $\sigma \in I_n$ because (V, F) is locally finite. Thus, from the idea of writing

$$\sum_{\sigma \in I_n} m_\sigma \partial\sigma = \sum_{\sigma \in I_n} m_\sigma \sum_{i=0}^{n} (-1)^i \{v_0, \dots, \hat{v}_i, \dots, v_n\},$$

we then group the coefficients of each face $\sigma' \in I_{n-1}$ together. Then we get a formal definition for $\partial : C_n^{BM} \to C_{n-1}^{BM}$. The problem is that it's too messy to write explicitly this definition due to the double sum above. Hence, we'll use the second way to write C_n^{BM}. That will make our later ~~notion~~ notations neat, but it comes at the expense of a certain degree of intuition. However, since we have discussed the idea of what $\partial : C_n^{BM} \to C_{n-1}^{BM}$ should be, the following notations don't really matter.

If $\sigma = \{v_0, v_1, \dots, v_n\}$ and $\beta = \{v_0, \dots, \hat{v}_{i_0}, \dots, v_n\}$ then we denote $i_{\sigma,\beta} := i_0$. Let $h \in \mathrm{Map}(I_n, \mathbb{Z})$, we'll define $\partial h \in \mathrm{Map}(I_{n-1}, \mathbb{Z})$ as follow.

$$(\partial h)(\beta) = \sum_{\substack{\sigma \in I_n \\ \sigma \supset \beta}} (-1)^{i_{\sigma,\beta}} h(\sigma) \quad \text{for all } \beta \in I_{n-1}.$$

Then ∂ is a map from $\mathrm{Map}(I_n, \mathbb{Z})$ to $\mathrm{Map}(I_{n-1}, \mathbb{Z})$. It comes naturally from this definition that ∂ is a group homomorphism. Indeed, with h_1, h_2 in $Map(I_n, \mathbb{Z})$, we have

$$\partial(h_1+h_2)(\beta) = \sum_{\substack{\sigma \in I_n \\ \sigma \supset \beta}} (-1)^{i_{\sigma,\beta}} (h_1+h_2)(\sigma) = \sum_{\substack{\sigma \in I_n \\ \sigma \supset \beta}} (-1)^{i_{\sigma,\beta}} (h_1(\sigma)+h_2(\sigma))$$

$$= \sum_{\substack{\sigma \in I_n \\ \sigma \supset \beta}} (-1)^{i_{\sigma,\beta}} h_1(\sigma) + \sum_{\substack{\sigma \in I_n \\ \sigma \supset \beta}} (-1)^{i_{\sigma,\beta}} h_1(\sigma)$$

$$= (\partial h_1)(\beta) + (\partial h_2)(\beta).$$

Next, we'll show that $\partial \circ \partial = 0$. Let $n \geqslant 1$. We have

$$Map(I_{n+1}, \mathbb{Z}) \xrightarrow{\partial} Map(I_n, \mathbb{Z}) \xrightarrow{\quad\partial\quad} Map(I_{n-1}, \mathbb{Z})$$

Take $g \in Map(I_{n+1}, \mathbb{Z})$ and $\alpha \in I_{n-1}$, we will show that $\partial(\partial g)\alpha = 0$.

By definition,

$$\partial(\partial g)\alpha = \sum_{\substack{\beta \in I_n \\ \beta \supset \alpha}} (-1)^{i_{\beta,\alpha}} (\partial g)(\beta) = \sum_{\substack{\beta \in I_n \\ \beta \supset \alpha}} (-1)^{i_{\beta,\alpha}} \sum_{\substack{\sigma \in I_{n+1} \\ \sigma \supset \beta}} (-1)^{i_{\sigma,\beta}} g(\sigma)$$

$$= \sum_{\substack{\beta \in I_n,\ \sigma \in I_{n+1}, \\ \sigma \supset \beta \supset \alpha}} (-1)^{i_{\beta,\alpha} + i_{\sigma,\beta}} g(\sigma)$$

$$= \sum_{\substack{\sigma \in I_{n+1} \\ \sigma \supset \alpha}} \underbrace{\left(\sum_{\substack{\beta \in I_n, \\ \alpha \subset \beta \subset \sigma}} (-1)^{i_{\beta,\alpha} + i_{\sigma,\beta}} \right)}_{(*)} g(\sigma)$$

We will show that $(*) = 0$. Since σ is two-element more than α, there are only two summands in $(*)$, corresponding to β_1 and β_2. To show that $(*) = 0$, we'll show that $(-1)^{i_{\beta_1,\alpha} + i_{\sigma,\beta_1}} + (-1)^{i_{\beta_2,\alpha} + i_{\sigma,\beta_2}} = 0$. In other words, we'll show that $i_{\beta_1,\alpha} + i_{\sigma,\beta_1}$ and $i_{\beta_2,\alpha} + i_{\sigma,\beta_2}$ are of different parity. Since we only care about parities of the positions, we can assume $\sigma = \{0, 1, \dots, n+1\}$ and $\alpha = \sigma \setminus \{i, j\}$ with $i < j$. Then $\beta_1 = \{0, \dots, \hat{i}, \dots, n+1\}$ and $\beta_2 = \{0, \dots, \hat{j}, \dots, n+1\}$. We can even assume that $n = 7$. We'll consider four cases:

- <u>i and j are even</u> We can assume $i = 2$ and $j = 6$.

σ: 0, 1, 2, 3, 4, 5, 6, 7, 8

β_1: 0, 1, —, 3, 4, 5, 6, 7, 8 $\Rightarrow i_{\sigma,\beta_1} = 2$, (even)

β_2: 0, 1, 2, 3, 4, 5, —, 7, 8 $\Rightarrow i_{\sigma,\beta_2} = 6$ (even)

α: 0, 1, —, 3, 4, 5, —, 7, 8 $\Rightarrow i_{\beta_1,\alpha} = 5$ (odd), $i_{\beta_2,\alpha} = 2$ (even)

Thus $i_{\beta_1,\alpha} + i_{\sigma,\beta_1}$ is odd and $i_{\beta_2,\alpha} + i_{\sigma,\beta_2}$ is even.

- <u>i and j are odd</u> we can assume $i = 3$ and $j = 7$.

σ: 0, 1, 2, 3, 4, 5, 6, 7, 8

β_1: 0, 1, 2, —, 4, 5, 6, 7, 8 $\Rightarrow i_{\sigma,\beta_1} = 3$ (odd)

β_2: 0, 1, 2, 3, 4, 5, 6, —, 8 $\Rightarrow i_{\sigma,\beta_2} = 7$ (odd)

α: 0, 1, 2, —, 4, 5, 6, —, 8 $\Rightarrow i_{\beta_1,\alpha} = 6$ (even), $i_{\beta_2,\alpha} = 3$ (odd)

Thus $i_{\beta_1,\alpha} + i_{\sigma,\beta_1}$ is odd and $i_{\beta_2,\alpha} + i_{\sigma,\beta_2}$ is even.

<u>i is even and j is odd</u> We can assume $i = 2$ and $j = 5$.

$\sigma:\ 0, 1, 2, 3, 4, 5, 6, 7, 8$

$\beta_1:\ 0, 1, -, 3, 4, 5, 6, 7, 8 \Rightarrow i_{\sigma,\beta_1} = 2$ (even)

$\beta_2:\ 0, 1, 2, 3, 4, -, 6, 7, 8 \Rightarrow i_{\sigma,\beta_2} = 5$ (odd)

$\alpha:\ 0, 1, -, 3, 4, -, 6, 7, 8 \Rightarrow i_{\beta_1,\alpha} = 4$ (even), $i_{\beta_2,\alpha} = 2$ (even)

Thus $i_{\beta_1,\alpha} + i_{\sigma,\beta_1}$ is even and $i_{\beta_2,\alpha} + i_{\sigma,\beta_2}$ is odd.

<u>i is odd and j is even</u> We can assume $i = 3$ and $j = 6$.

$\sigma:\ 0, 1, 2, 3, 4, 5, 6, 7, 8$

$\beta_1:\ 0, 1, 2, -, 4, 5, 6, 7, 8 \Rightarrow i_{\sigma,\beta_1} = 3$ (odd)

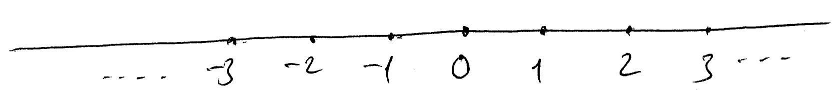
$\beta_1:\ 0, 1, 2, 3, 4, 5, -, 7, 8 \Rightarrow i_{\sigma,\beta_2} = 6$ (even)

$\alpha:\ 0, 1, 2, -, 4, 5, -, 7, 8 \Rightarrow i_{\beta_1,\alpha} = 5$ (odd), $i_{\beta_2,\alpha} = 3$ (odd)

Thus $i_{\beta_1,\alpha} + i_{\sigma,\beta_1}$ is even and $i_{\beta_2,\alpha} + i_{\sigma,\beta_2}$ is odd.

Therefore, we conclude that $\partial \circ \partial = 0$.

Next, we'll find the associated homology groups H^{BM}_{*} for a "triangulation" of $\mathbb{R}$. First, we view $\mathbb{R}$ as a simplicial complex with $V = \{k : k \in \mathbb{Z}\}$ and $F = \{\{k, k+1\} : k \in \mathbb{Z}\} \cup \{\{k\} : k \in \mathbb{Z}\}$.



Because each vertex $k \in V$ belongs to exactly three faces, $\{k\}$, $\{k-1, k\}$, and $\{k, k+1\}$, we get a locally finite simplicial complex (V, F). Therefore, we can define BM-chains and boundary operator as above for this case.

Since F has no n-dimensional faces for $n \geq 2$, we have $C_n^{BM} = \{0\}$. Thus, we ~~only~~ get a very short chain complex $0 \longrightarrow C_1^{BM} \xrightarrow{\partial} C_0^{BM} \longrightarrow 0$.

We have $H_n^{BM} = \{0\}$ for all $n \geq 2$ and $H_1^{BM} = \ker(\partial)$ and $H_0^{BM} = C_0^{BM}/Im(\partial)$.

We have
$$C_0^{BM} = \Big\{\sigma = \sum_{k \in \mathbb{Z}} n_k \{k\} \;\Big|\; n_k \in \mathbb{Z}\Big\},$$
$$C_1^{BM} = \Big\{\tau = \sum_{k \in \mathbb{Z}} m_k \{k, k+1\} \;\Big|\; m_k \in \mathbb{Z}\Big\}.$$

By definition, $\partial\tau = \sum m_k \partial\{k, k+1\} = \sum m_k(\{k+1\} - \{k\}) = \sum (m_{k-1} - m_k)\{k\}$.

Thus,
$$\begin{aligned}
\ker(\partial) &= \Big\{\tau = \sum m_k \{k, k+1\} \;\Big|\; m_{k-1} - m_k = 0 \text{ for all } k \in \mathbb{Z}\Big\} \\
&= \Big\{\tau = \sum m_k \{k, k+1\} \;\Big|\; m_k = m \text{ for all } k \in \mathbb{Z}\Big\} \\
&= \Big\{\tau = m \sum \{k, k+1\} \text{ for some } m \in \mathbb{Z}\Big\} \\
&= \mathbb{Z}\tau_0, \quad \text{where } \tau_0 = \sum_{k \in \mathbb{Z}} \{k, k+1\}.
\end{aligned}$$

Therefore, $H_1^{BM} = \mathbb{Z}\tau_0 \simeq \mathbb{Z}$. Thus, τ_0 is the fundamental cycle of $\mathbb{R}$ at dimension $n = 1$. Next, we have

$$Im(\partial) = \Big\{\sigma = \sum n_k \{k\} \;\Big|\; \text{there exists a sequence } (m_k)_{k \in \mathbb{Z}} \text{ such that } n_k = m_{k-1} - m_k \text{ for all } k \in \mathbb{Z}\Big\}.$$

For each sequence $(n_k)_{k\in\mathbb{Z}}$ in $\mathbb{Z}$, we can define $m_0 = 0$ and

$$m_k = \begin{cases} m_{k-1} - n_k & \text{for } k \geq 1, \\ n_{k+1} + m_{k+1} & \text{for } k \leq -1. \end{cases}$$

Then we get $n_k = m_{k-1} - m_k$. Therefore,

$$\mathrm{Im}\,\partial = \left\{ \sigma = \sum_{k\in\mathbb{Z}} n_k \{k\} \;\middle|\; n_k \in \mathbb{Z} \right\} = C_0^{BM}.$$

Thus $H_0^{BM} = C_0^{BM}/\mathrm{Im}\,\partial = \{0\}$. In conclusion,

$$H_n^{BM} = \begin{cases} 0 & \text{if } n \neq 1, \\ \mathbb{Z}\tau_0 \simeq \mathbb{Z} & \text{if } n = 1. \end{cases}$$

4/4

② Let M be the Möbius strip and ∂M its boundary. We will calculate the relative homology groups $H_*(M, \partial M)$.

The chain complex of M, which is $\cdots \xrightarrow{\partial} C_2 M \xrightarrow{\partial} C_1 M \xrightarrow{\partial} C_0 M \xrightarrow{\partial} 0$, induces a relative chain complex $\cdots \xrightarrow{\partial} C_2(M,\partial M) \xrightarrow{\partial} C_1(M,\partial M) \xrightarrow{\partial} C_0(M,\partial M) \to 0$, where $C_n(M,\partial M) := C_n(M)/C_n(\partial M)$. Now assume that the Möbius strip has a triangulation (V, F) such that F has no faces of dimension higher than two. Then $C_n(M) = 0$ for all $n \geq 3$ and the chain complex of M remains $0 \xrightarrow{\partial} C_2(M) \xrightarrow{\partial} C_1(M) \xrightarrow{\partial} C_0(M) \xrightarrow{\partial} 0$. Consequently, the relative chain complex remains

$$0 \to C_2(M,\partial M) \xrightarrow{\partial} C_1(M,\partial M) \xrightarrow{\partial} C_0(M,\partial M) \to 0.$$

The relative homology groups are

$$H_n(M,\partial M) := \frac{\ker(C_n(M,\partial M)\to C_{n-1}(M,\partial M))}{\operatorname{Im}(C_{n+1}(M,\partial M)\to C_n(M,\partial M))} = \frac{Z_n(M,\partial M)}{B_n(M,\partial M)}.$$

By the definition of relative boundary operator, we have $Z_n(M,\partial M) = Z_n(M) + C_n(\partial M)$ for all $n \geq 0$. We also have $H_n(M,\partial M) = 0$ for all $n \geq 3$. We'll find $H_0(M,\partial M)$, $H_1(M,\partial M)$ and $H_2(M,\partial M)$ by computing $Z_n(M,\partial M)$ and $B_n(M,\partial M)$.

We put $I_n = \{\sigma : \Delta^n \to M \text{ continuous}\}$ for all $n \geq 0$. By definition, we get

$$Z_n(M,\partial M) = \left\{ \tau = \sum_{\sigma\in I_n} m_\sigma \sigma + C_n(\partial M) \;\middle|\; \sum_{\sigma \in I_n} m_\sigma \partial\sigma + C_{n-1}(\partial M) = 0 \right\},$$

$$B_n(M,\partial M) = \left\{ \tau = \sum_{\sigma\in I_{n+1}} m_\sigma \partial\sigma + C_n(\partial M) \right\},$$

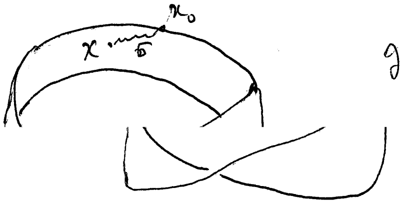
where m_σ's are implied to be integers and being zero for all but at finitely many σ's.

<u>For $n=0$</u> $\quad Z_0(M,\partial M) = \ker(C_0(M,\partial M) \to 0) = C_0(M,\partial M)$,

$$B_0(M,\partial M) = \left\{ \tau = \sum_{\sigma \in I_1} m_\sigma \partial\sigma + C_0(\partial M) \right\}.$$

We want to show that $B_0(M,\partial M) = C_0(M,\partial M)$. The latter group is generated by elements of the form $\{x\} + C_0(\partial M)$ with $x \in M$. Thus it's suffices to show that $\{x\} + C_0(\partial M) \in B_0(M,\partial M)$. Let x_0 be a point on ∂M and σ be

…the path connecting x_0 to x. Then $\partial\sigma + C_0(\partial M) \in B_0(M,\partial M)$ and



$$\partial\sigma + C_0(\partial M) = \sigma(1) - \sigma(0) + C_0(\partial M)$$
$$= \{x\} - \underbrace{\{x_0\}}_{\in \partial M} + C_0(\partial M) = \{x\} + C_0(\partial M).$$

Thus, $\{x\} + C_0(\partial M) \in B_0(M,\partial M)$. Thus, $Z_0 = B_0$

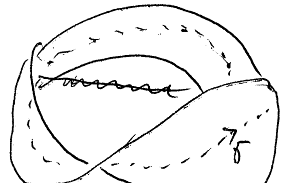
Therefore, $H_0(M,\partial M) = \{0\}$.

<u>For $n=1$</u> We have $Z_1(M,\partial M) = Z_1(M) + C_1(\partial M)$. This means of M relative to ∂M equals (a cycle in M) + (a path in ∂M) Speaking, $B_1(M,\partial M)$ is the set of cycles in M relative to ∂M that is of a 2-dimensional subspace. We want to show that $B_1(M,\partial M)$

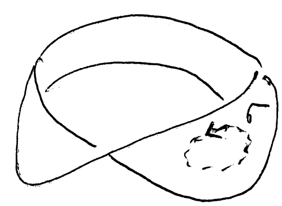
In the following, we'll use natural language, not mathematical

Thus, we'll only suggest the idea, not a rigorous proof.

Take a cycle $\sigma \in Z_1(M)$. We see that $\sigma \cup (\partial M)$ is of the space $M \setminus (\sigma \cup (\partial M))$.



σ is not null-homotopic



σ is null-homotopic

Thus, $B_1(M,\partial M) = Z_1(M,\partial M)$. Therefore, $H_1(M,\partial M) = \{0\}$. X

$= C_0(M,\partial M)$.

, a cycle

). Roughly

the boundary

$= Z_1(M,\partial M)$.

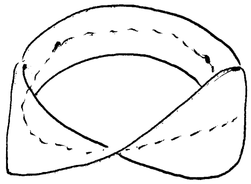
l, to show this.

the boundary

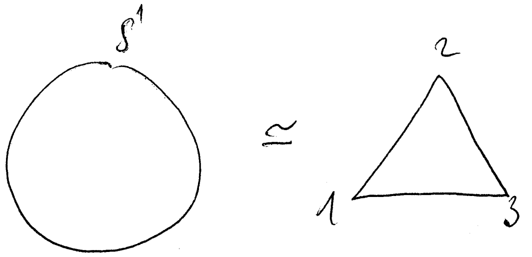
It's $\mathbb{Z}/2$

<u>For $n=2$</u> $\quad 0 \longrightarrow C_2(M,\partial M) \xrightarrow{\partial} C_1(M,\partial M) \xrightarrow{\partial} C_0(M,\partial M) \longrightarrow 0.$

We have $B_2(M,\partial M) = 0 + C_2(\partial M)$, and $Z_2(M,\partial M) = Z_2(M) + C_2(\partial M)$. Since the chain complex of M is $0 \to C_2M \to C_1M \to C_0M \to 0$, we get $Z_2(M) \simeq H_2(M)$.



We see that the Möbius strip M can be retracted onto the center line as in the figure. Thus, M is homotopically equivalent to S^1. Thus, $H_2(M) \simeq H_2(S^1)$. Since S^1 is just a one-dimensional complex, $C_2(S^1) = \{0\}$. Thus $H_2(S^1) = 0$. Thus,



$V = \{1,2,3\}$

$F = \{\{1,2\},\{2,3\},\{1,3\},\{1\},\{2\},\{3\}\}$

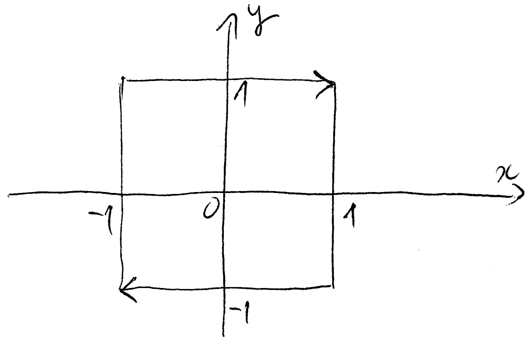
$H_2(M) = 0$. Thus $Z_2(M) = 0$. Thus, $Z_2(M,\partial M) = 0 + C_2(\partial M)$. Thus, $B_2(M,\partial M) = Z_2(M,\partial M)$. Therefore, $H_2(M,\partial M) = \{0\}$.

In conclusion, $H_n(M,\partial M) = \{0\}$ for all $n \geq 0$.

$\boxed{\text{Second approach}}$ We try to use exact sequences to solve the problem (finding $H_n(M,\partial M)$).

~~We'll first show the following four lemmas~~.

*) First, we'll use the following identifications:



$$S^1 \equiv (\{0\} \times [-1,1]) / (0,-1) \sim (0,1),$$

$$M \equiv ([-1,1] \times [-1,1]) / (t,-1) \sim (-t,1) \quad -1 \leq t \leq 1$$

$$\partial M \equiv (\{-1,1\} \times [-1,1]) / \begin{matrix} (-1,1) \sim (1,-1) \\ (-1,-1) \sim (1,1) \end{matrix}$$

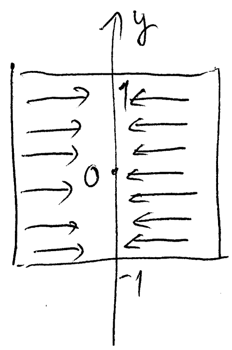
We'll show the following four lemmas:

1) ~~That~~ S^1 is a deformation retract of M. That is, there exists a retraction $r: M \to S^1$ which is homotopically equivalent to $id_M: M \to M$. Consequently, we get the isomorphisms of homology groups $r_*: H_* M \to H_* S^1$.

<u>Proof</u> We define $r: M \to S^1$
$(u,v) \mapsto (0,v)$ for all $-1 \leq u,v \leq 1$.

The homotopy $H: M \times [0,1] \to M$ is define as $H((u,v),t) = ((1-t)u, v)$ for all $-1 \leq u,v \leq 1$, $0 \leq t \leq 1$.

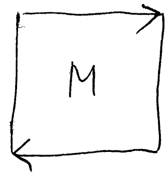
Then $H(x,0) = x$ and $H(x,1) = r(x)$.



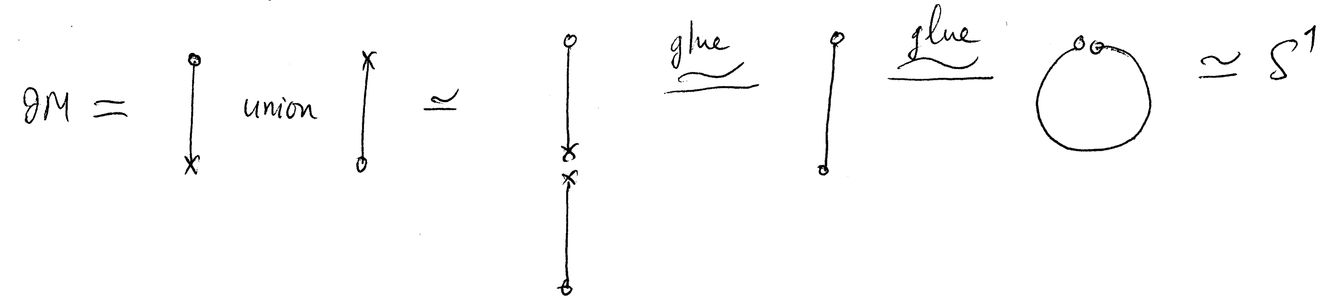
Moreover, we see that $r(\partial M) = S^1$.

2) ∂M and S^1 are homeomorphic. Consequently, we get the isomorphisms between homology groups $H_n \partial M \simeq H_n S^1$ for all $n \geq 0$.

Proof We have the picture



Thus, a "proof" by picture is straightforward:



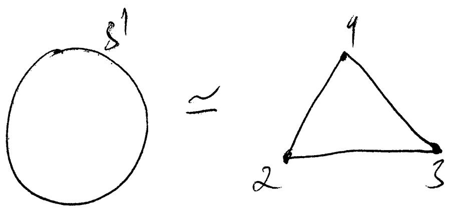
3) For any topological space X and $n \geq 0$, $H_n(X,X) = 0$.

Proof By definition of relative chain complexes, $C_n(X,X) = C_n(X)/C_n(X) \simeq 0$.

Thus $Z_n(X,X) = B_n(X,X) = 0$. Thus $H_n(X,X) = Z_n(X,X)/B_n(X,X) = 0$.

4)
$$H_n S^1 \cong \begin{cases} \mathbb{Z} & \text{for } n=0 \text{ or } n=1 \\ 0 & \text{for all } n \geq 2 \end{cases}$$

Proof We consider S^1 as a simplicial complex. $S^1 \simeq (V, F)$, where



$V = \{1,2,3\}$

$F = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$

Thus, $C_n = 0$ for all $n \geq 2$. Consequently, $H_n = 0$ for all $n \geq 2$.

We have $C_1 = \{ a\{1,2\} + b\{1,3\} + c\{2,3\} \mid a,b,c \in \mathbb{Z} \}$

$C_0 = \{ x\{1\} + y\{2\} + z\{3\} \mid x,y,z \in \mathbb{Z} \}$

* Compute H_0: we have $Z_0 = C_0$.

$$B_0 = \{ a(\{2\}-\{1\}) + b(\{3\}-\{1\}) + c(\{3\}-\{2\}) \mid a,b,c \in \mathbb{Z} \}$$

$$= \{ (-a-b)\{1\} + (a-c)\{2\} + (b+c)\{3\} \mid a,b,c \in \mathbb{Z} \}$$

$$= \{ x\{1\} + y\{2\} + z\{3\} \mid x+y+z = 0 \}$$

We define the map $\phi : C_0 \to \mathbb{Z}$ such that

$$\phi(x\{1\} + y\{2\} + z\{3\}) = x+y+z$$

Then ϕ is a group morphism. Moreover ϕ is surjective and $\ker\phi = B_0$.

Thus, that $C_0/\ker\phi \simeq \operatorname{Im}\phi$ implies $C_0/B_0 \simeq \mathbb{Z}$. Thus $\cancel{\&}$ $Z_0/B_0 \simeq \mathbb{Z}$.

Therefore $H_0 \simeq \mathbb{Z}$.

* Compute H_1: we have $B_1 = 0$. Thus $H_1 \simeq Z_1$.

$$Z_1 = \{ a\{1,2\} + b\{1,3\} + c\{2,3\} \mid a(\{2\}-\{1\}) + b(\{3\}-\{1\}) + c(\{3\}-\{2\}) = 0 \}$$

$$= \{ a\{1,2\} + b\{1,3\} + c\{2,3\} \mid \underbrace{-a-b = 0,\ a-c = 0,\ b+c = 0}_{\iff (a,b,c) = (a,-a,a)} \}$$

$$= \{ a(\{1,2\} - \{1,3\} + \{2,3\}) \mid a \in \mathbb{Z} \}$$

$$\simeq \mathbb{Z}.$$

Thus $H_1 \simeq \mathbb{Z}$.

Now we return to the problem. The retract $r : M \to S^1$ having $r(\partial M) = S^1$ gives us the following commutative diagram:

$$\begin{tikzcd}
H_n \partial M \arrow[r] \arrow[d] & H_n M \arrow[r] \arrow[d, "\cong"] & H_n(M,\partial M) \arrow[r] \arrow[d] & H_{n-1}\partial M \arrow[r] \arrow[d] & H_{n-1} M \arrow[r] \arrow[d, "\cong"] & H_{n-1}(M,\partial M) \arrow[d] \\
H_n S^1 \arrow[r] & H_n S^1 \arrow[r] & H_n(S^1,S^1) \arrow[r] & H_{n-1} S^1 \arrow[r] & H_{n-1} S^1 \arrow[r] & H_{n-1}(S^1,S^1)
\end{tikzcd}$$

The rows are exact.

* For $n \geq 3$, we have $H_n M \simeq H_n S^1 \simeq 0$ and $H_{n-1}\partial M \simeq H_{n-1} S^1 \simeq 0$. We have the exact sequence $0 \to H_n(M,\partial M) \to 0$. Thus $H_n(M,\partial M) = 0$.
* For $n=2$, we have $H_n M \simeq H_n S^1 \simeq 0$ and $H_{n-1}\partial M \simeq H_{n-1} S^1 \simeq H_{n-1} M \simeq H_{n-1} S \simeq \mathbb{Z}$.

Thus, we have

This is actually multiplication by 2

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H_2(M,\partial M) \arrow[r, "\varphi_4"] \arrow[d] & \mathbb{Z} \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & H_1(M,\partial M) \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\varphi_3"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Suppose that $H_1(M,\partial M) = 0$ (we'll show this below). Then φ_1 is a surjection from $\mathbb{Z}$ to $\mathbb{Z}$. Thus φ_1 is an isomorphism. By commutativity, $\varphi_3 \circ \varphi_2$ is an isomorphism. Thus φ_2 is injective. By commutativity, $\varphi_2 \circ \varphi_4 = 0$. Thus $\varphi_4 = 0$. Then $0 \to H_2(M,\partial M) \to 0$ is exact. Thus $H_2(M,\partial M) = 0$.

* For $n = 1$, we have

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, "\cong"] & H_1(M,\partial M) \arrow[r, "\varphi_4'"] \arrow[d] & \mathbb{Z} \arrow[r, "\varphi_1'"] \arrow[d, "\varphi_2'"] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & H_0(M,\partial M) \arrow[d] \\
\mathbb{Z} \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\varphi_3'"] & \mathbb{Z} \arrow[r] & \cancel{H_0}\ 0
\end{tikzcd}$$

We use the same approach as in the case $n=2$. Granting for a moment that

$H_0(M,\partial M)=0$, we'll show that $H_1(M,\partial M)$. By exactness on the first row, φ_1' is surjective. Thus φ_1' is bijective. By commutativity, $\varphi_3'\circ\varphi_2'$ is bijective. Thus φ_2' is injective. By commutativity, $\varphi_2'\circ\varphi_4' = 0$. Thus $\varphi_4'=0$. Then we get the exact sequence $0 \longrightarrow H_1(M,\partial M) \longrightarrow \cancel{0}$. Thus $H_1(M,\partial M)=0$,

* For $n=0$: we'll show that $H_0(M,\partial M)=0$.

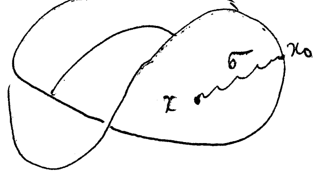
The commutative diagram only gives us

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\varphi"] \arrow[d, "\wr"'] & H_0(M,\partial M) \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

We couldn't distinguish the cases $\varphi=0$ and φ is an isomorphism. Thus, we need to refer directly to the geometry of the Möbius to find $H_0(M,\partial M)$. Therefore, we return to the proof for case $n=0$ of the first approach. Namely,

$$Z_0(M,\partial M) = C_0(M,\partial M),$$

$$B_0(M,\partial M) = \left\{ \tau = \sum_{\text{path } \sigma} m_\sigma \partial\sigma + C_0(\partial M) \right\}$$



For each $x\in M$, we take some $x_0\in\partial M$ and a path σ in M connecting x_0 to x. Then
$$\partial\sigma + C_0(\partial M) = \sigma(1)-\sigma(0)+C_0(\partial M) = \{x\}-\{x_0\}+C_0(\partial M) = \{x\}+C_0(\partial M)$$

Thus $C_0(M,\partial M)\subset B_0(M,\partial M)$. Thus $Z_0(M,\partial M) = B_0(M,\partial M)$. Therefore,

$$H_0(M,\partial M) = Z_0(M,\partial M)\,/\,B_0(M,\partial M) = 0. \qquad 1/4$$

You should really be using the long exact sequence for the pair $(M,\partial M)$ in this problem.