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Math 8301: Topology & Manifolds

Homework 11

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(1) Let X be a topological space. We'll use the Mayer-Vietoris sequence to compute the homology groups of $X \times S^1$.

We put $U' = \{e^{i2\pi t} \mid \frac{1}{2} < t < 2\}$, $V = \{e^{i2\pi t} \mid -\frac{1}{2} < t < 1\}$.

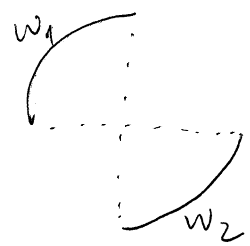
U' $\qquad\qquad$ V'

Then $U = X \times U'$ and $V = X \times V'$ satisfy: U and V are open in $X \times S^1$, and $U \cup V = X \times S^1$. Thus we have the Mayer-Vietoris sequence

$$\cdots \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(X \times S^1) \to$$
$$\to H_{n-1}(U \cap V) \to H_{n-1}(U) \oplus H_{n-1}(V) \to H_{n-1}(X \times S^1) \to$$
$$\to \cdots \qquad \cdots \to$$
$$\to H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(X \times S^1) \to$$
$$\to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(X \times S^1) \to 0$$

We see that $U \cap V = (X \times U') \cap (X \times V') = X \times (U' \cap V')$. Put

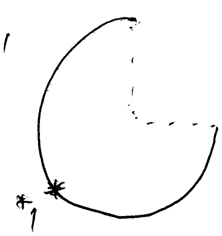
$W_1' = \{e^{i2\pi t} \mid \frac{1}{2} < t < 1\}$ and $W_2' = \{e^{i2\pi t} \mid \frac{3}{2} < t < 2\}$, we have $U' \cap V' = W_1' \cup W_2'$ and this is a disjoint union. Then



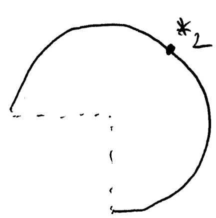
$$U \cap V = (X \times W_1') \amalg (X \times W_2')$$

Thus $H_n(U \cap V) = H_n(X \times W_1') \oplus H_n(X \times W_2')$ for all $n \geq 0$.

Since U' is contractible to a point, denoted by $*_1$, the following map is a homotopy equivalence:
$$i': U \longrightarrow X \times \{*_1\}$$
$$(x, \alpha) \longmapsto (x, *_1) \text{ for all } x \in X, \alpha \in U'$$



Similarly, we have a homotopy equivalence:
$$j': V \longrightarrow X \times \{*_2\}$$
$$(x, \beta) \longmapsto (x, *_2) \text{ for all } x \in X, \beta \in V'.$$



We can identify $X \times \{*_1\}$ and $X \times \{*_2\}$ with X and write $i': U \to X$, $i' = pr_X$, and
$$j': V \to X, \; j' = pr_X.$$

Then i' and j' induce the following homology homomorphisms

$$i'_*: H_n(U) \longrightarrow H_n(X), \qquad \text{and} \qquad j'_*: H_n(V) \longrightarrow H_n(X),$$
$$i'_*(z) = pr_X \circ z, \qquad\qquad j'_*(z) = pr_X \circ z.$$

In other words, for each singular simplex $\sigma: \Delta^n \to U = X \times U'$, we have $i'_*(\sigma) = pr_X \circ \sigma : \Delta^n \to X$. The similar explanation applies for j'_*.

The injections $U \cap V \xrightarrow{i} U$ and $U \cap V \xrightarrow{j} V$ also induce homology homomorphisms $i_*: H_n(U \cap V) \to H_n(U)$ and $j_*: H_n(U \cap V) \to H_n(V)$. For each $z \in H_n(U \cap V)$, z can be written uniquely as a sum $z = z_1 + z_2$ where $z_1 \in H_n(X \times W_1')$ and $z_2 \in H_n(X \times W_2')$. We define the map

$$\varphi: H_n(U \cap V) \longrightarrow H_n(X) \oplus H_n(X),$$

$$z = z_1 + z_2 \longmapsto \big(pr_X(z_1),\ pr_X(z_2)\big),$$

where the projection pr_X is understood as above.

$$\begin{tikzcd} \Delta^n \arrow[r, "\sigma"] \arrow[dr, "pr_X(\sigma)"'] & X \times W_1' \arrow[d, "pr_X"] \\ & X \end{tikzcd}, \qquad pr_X\Big(\sum n_\sigma \sigma\Big) := \sum n_\sigma\, pr_X(\sigma).$$

Then by definition, φ is an isomorphism. We have the commutative diagram

$$\begin{tikzcd} H_n(U) \arrow[r, "j'_*", "\sim"'] & H_n(X) \\ H_n(U \cap V) \arrow[u, "i_*"] \arrow[d, "j_*"'] \arrow[r, "\varphi", "\sim"'] \arrow[ur, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(X) \oplus H_n(X) \arrow[u, "p"'] \arrow[d, "p"] \\ H_n(V) \arrow[r, "j'_*", "\sim"'] & H_n(X) \end{tikzcd}$$

where $p(a,b) := a + b$ for all $a, b \in H_n(X)$.

Indeed, for each $z \in H_n(U \cap V)$, we can write $z = z_1 + z_2$ for $z_1 \in H_n(X \times W_1')$ and $z_2 \in H_n(X \times W_2')$. Then

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$$\begin{tikzcd}
z_1+z_2 \arrow[r, "i'_*"] & pr_X(z_1)+pr_X(z_2) \\
z=z_1+z_2 \arrow[u, "i_*"] \arrow[d, "j_*"'] \arrow[r, "\varphi"] & (pr_X(z_1), pr_X(z_2)) \arrow[u, "p"'] \arrow[d, "p"] \\
z_1+z_2 \arrow[r, "j'_*"] & pr_X(z_1)+pr_X(z_2)
\end{tikzcd}$$

Note that i'_* and j'_* are isomorphisms due to homotopy equivalence. Thus, the Mayer-Vietoris sequence can be replace by the following exact sequence:

$$\begin{aligned}
\cdots &\to H_n(X)\oplus H_n(X) \xrightarrow{f} H_n(X)\oplus H_n(X) \xrightarrow{g} H_n(X\times S^1) \\
&\to H_{n-1}(X)\oplus H_{n-1}(X) \to H_{n-1}(X)\oplus H_{n-1}(X) \to H_{n-1}(X\times S^1) \\
&\to \cdots \\
\cdots &\to H_1(X)\oplus H_1(X) \to H_1(X)\oplus H_1(X) \to H_1(X\times S^1) \\
&\to H_0(X)\oplus H_0(X) \to H_0(X)\oplus H_0(X) \to H_0(X\times S^1) \to 0.
\end{aligned}$$

The function f and g are given by $f: H_n(X)\oplus H_n(X) \to H_n(X)\oplus H_n(X)$,

$$(a,b) \mapsto (a+b, a+b),$$

and $g: H_n(X)\oplus H_n(X) \to H_n(X\times S^1)$,

$$(a,b) \mapsto a-b.$$

We have $Im f = \{(a+b, a+b) / a,b\in H_n(X)\}$

$$= \{(a+b)(1,+1) / a,b\in H_n(X)\}$$

$$= \{ a(1,+1) \,/\, a \in H_n(X) \} \simeq H_n(X)$$

This ~~set~~ subgroup of $H_n(X) \oplus H_n(X)$ has a direct summand $\{ a(1,0) \,/\, a \in H_n(X)\}$. Thus, $(H_n(X) \oplus H_n(X)) / Im f \simeq H_n(X)$. Since $Im f = \ker g$, we have

$$Im\, g \simeq (H_n(X) \oplus H_n(X)) / \ker g = (H_n(X) \oplus H_n(X)) / Im f \simeq H_n(X).$$

For $n \geq 1$, we have the sequence,

$$H_n(X \times S^1) \xrightarrow{h} H_{n-1}(X) \oplus H_{n-1}(X) \xrightarrow{k} H_{n-1}(X) \oplus H_{n-1}(X),$$

where k is defined like f, with $n-1$ instead of n. Specifically,

$$k(a,b) = (a+b, a+b).$$

Then

$$\ker(k) = \{ (a,b) \in H_{n-1}(X) \oplus H_{n-1}(X) \mid a+b = 0 \}$$
$$= \{ (a,-a) \mid a \in H_{n-1}(X) \}$$
$$= \{ a(1,-1) \mid a \in H_{n-1}(X) \} \simeq H_{n-1}(X).$$

~~This subgroup of $H_{n-1}(X) \oplus H_{n-1}(X)$ has a direct summand $\{a(1,0) / a \in H_{n-1}(X)\}$~~,

~~Thus H_{n-1}~~ By the exactness, $Im\, h = \ker(k) \simeq H_{n-1}(X)$. Thus, $H_n(X \times S^1) / \ker h \simeq Im\, h \simeq H_{n-1}(X)$. The exact sequence

$$H_n(X) \oplus H_n(X) \xrightarrow{g} H_n(X \times S^1) \xrightarrow{h} H_{n-1}(X) \oplus H_{n-1}(X)$$

gives us the following short exact sequence

$$0 \to Im\, g \xrightarrow{i} H_n(X \times S^1) \xrightarrow{proj.} H_n(X \times S^1) / \ker h \to 0$$

If we can show that this sequence is split, then

$$H_n(X\times S^1) \simeq \cancel{H_n}\, \mathrm{Im}\, g \oplus \left(H_n(X\times S^1)/\ker h\right) \simeq H_n(X)\oplus H_{n-1}(X).$$

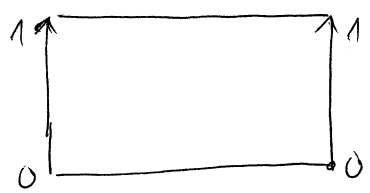
To show that the above short exact sequence is split, we should look for a map ψ such that $\psi\circ i = id_{\mathrm{Im}\, g}$.

$$0 \to \mathrm{Im}\, g \underset{\psi}{\overset{i}{\rightleftarrows}} H_n(X\times S^1) \longrightarrow H_n(X\times S^1)/\ker h \longrightarrow 0$$

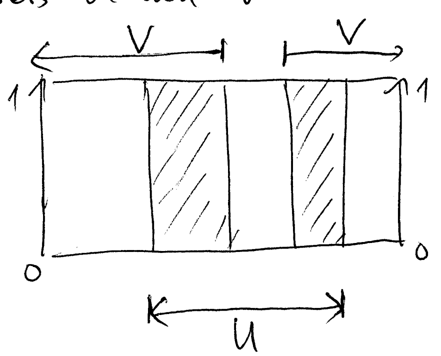
$\mathrm{Im}\, g \subset H_n(U)\oplus H_n(V)$

In other words, we should find a function $\psi: H_n(X\times S^1) \to \cancel{H_n(U)\oplus H_n(V)}$ such that $\psi(a-b) = (a,b)$ for all $a,b\in H_n(X)$. It is hard to construct such a function in general case. We will, however, construct it for a specifically simple case, $X=[0,1]\subset \mathbb{R}$. Then $X\times S^1$ is obtained by edge identification

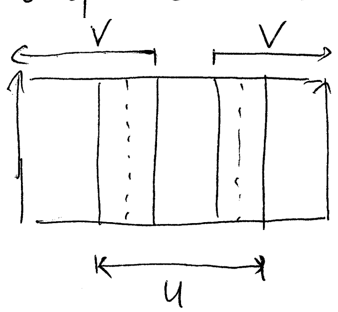
Just use the retraction $X\times S^1 \to X\times\{0\}$



The open sets U and V are denoted in the picture. The shaded region is $U\cap V$.

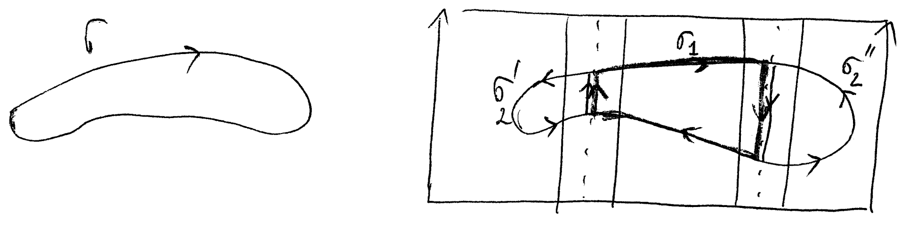


We take two special vertical lines in each connected component of $U\cap V$.



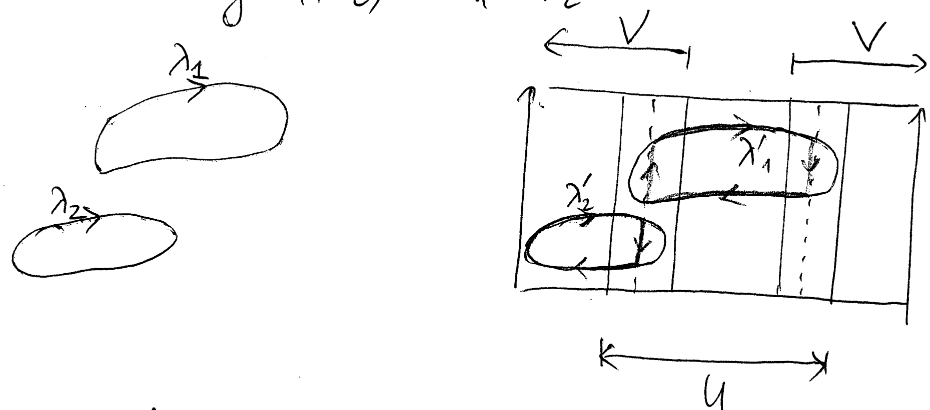
In general space X, such a line is $X\times\{*\}$, where $*$ lies in the intersection $U\cap V$.

For each cycle $\sigma \in H_n(X \times S^1)$, the two vertical lines will determine two cycles $\sigma_1 \in H_n(U)$ and $\sigma_2 \in H_n(V)$ such that $\sigma = \sigma_1 - \sigma_2$.



$\sigma_2 := \sigma_2' + \sigma_2''$.

The decomposition from σ into (σ_1, σ_2) is denoted by function $\psi : H_n(X \times S^1) \to H_n(U) \oplus H_n(V)$, $\psi(\sigma) = (\sigma_1, \sigma_2)$. Now we'll show that $\psi \circ i = id_{Im(g)}$. For $\lambda_1 \in H_n(U)$, $\lambda_2 \in H_n(V)$, we have $g(\lambda_1, \lambda_2) = \lambda_1 - \lambda_2$.



Up to the boundary of a high dimensional domain, we have $\lambda_1 = \lambda_1'$ and $\lambda_2 = \lambda_2'$.

Thus $\psi(\lambda_1 - \lambda_2) = \psi(\lambda_1' - \lambda_2') = (\lambda_1', \lambda_2') = (\lambda_1, \lambda_2)$. Thus $\psi \circ g(\lambda_1, \lambda_2) = (\lambda_1, \lambda_2)$.

Assuming that the sequence $0 \to Img \to H_n(X \times S^1) \to H_n(X \times S^1)/\ker h \to 0$ is also split in general cases, we conclude that $H_n(X \times S^1) \simeq H_n(X) \oplus H_{n-1}(X)$ for all $n \geq 1$. For $n = 0$, we have the exact sequence

$$H_0(X) \oplus H_0(X) \xrightarrow{f} H_0(X) \oplus H_0(X) \xrightarrow{r} H_0(X \times S^1) \longrightarrow 0$$

We still have $\operatorname{Im} f \simeq H_0(X)$ and $(H_0(X) \oplus H_0(X))/\operatorname{Im} f \simeq H_0(X)$. Since $\operatorname{Im} f = \ker r$, we have $(H_0(X) \oplus H_0(X))/\ker r \simeq H_0(X)$. Since r is surjective, $H_0(X \times S^1) \simeq (H_0(X) \oplus H_0(X))/\ker r \simeq H_0(X)$. In conclusion,

$$H_n(X \times S^1) \simeq \begin{cases} H_n(X) \oplus H_{n-1}(X) & \text{for } n \geq 1, \\ H_0(X) & \text{for } n = 0. \end{cases}$$

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② Let A, B, X be topological spaces such that $A \subset B \subset X$. First, we'll show that there is a short exact sequence of chain complexes:

$$0 \longrightarrow C_*(B,A) \longrightarrow C_*(X,A) \longrightarrow C_*(X,B) \longrightarrow 0$$

The inclusions $A \subset B \subset X$ induces the inclusions of chains $C_n(A) \subset C_n(B) \subset C_n(X)$. Consider the boundary operator $\partial : C_n(X) \longrightarrow C_{n-1}(X)$. We have $\partial(C_n(B)) = C_{n-1} B$ and $\partial(C_n(A)) = C_{n-1}(A)$. Therefore, we have a commutative diagram

$$\begin{tikzcd} C_n(B)/C_n(A) \arrow[r, "\tilde{i}"] \arrow[d, "\partial"'] & C_n(X)/C_n(A) \arrow[d, "\partial"] \\ C_n(B)/C_{n-1}(A) \arrow[r, "\tilde{i}"] & C_{n-1}(X)/C_{n-1}(A) \end{tikzcd}$$

where the maps are given by

$$\begin{tikzcd} z + C_n(A) \arrow[r, "\tilde{i}"] \arrow[d, "\partial"'] & z + C_n(A) \arrow[d, "\partial"] \\ \partial z + C_{n-1}(A) \arrow[r] & \partial z + C_{n-1}(A) \end{tikzcd}$$

for all $z \in C_n(B)$.

Likewise, we have the following commutative diagram

$$\begin{tikzcd} C_n(X)/C_nA \arrow[r, "\tilde{j}"] \arrow[d, "\partial"'] & C_nX/C_n(B) \arrow[d, "\partial"] \\ C_{n-1}X/C_{n-1}A \arrow[r, "\tilde{j}"] & C_{n-1}X/C_{n-1}B \end{tikzcd}$$

where the maps are given by

$$\begin{tikzcd} z + C_nA \arrow[r, "\tilde{j}"] \arrow[d, "\partial"'] & z + C_nB \arrow[d, "\partial"] \\ \partial z + C_{n-1}A \arrow[r, "\tilde{j}"] & \partial z + C_{n-1}B \end{tikzcd}$$

for all $z \in C_nX$.

Also, $\tilde{i}$ and $\tilde{j}$ defined above are respectively injective and surjective. Thus we have the commutative diagram

You should check exactness here

$$\begin{tikzcd} 0 \arrow[r] & C_n(B)/C_nA \arrow[r, "\tilde{i}"] \arrow[d, "\partial"'] & C_nX/C_nA \arrow[r, "\tilde{j}"] \arrow[d, "\partial"] & C_nX/C_nB \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & C_{n-1}B/C_{n-1}A \arrow[r, "\tilde{i}"] & C_{n-1}X/C_{n-1}A \arrow[r, "\tilde{j}"] & C_{n-1}X/C_{n-1}B \arrow[r] & 0 \end{tikzcd}$$

By definition, $C_nB/C_nA = C_n(B,A)$, $C_nX/C_nA = C_n(X,A)$ and $C_nX/C_nB = C_n(X,B)$.

Therefore, we obtain an exact sequence of chain complexes:

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 \arrow[r] & C_n(B,A) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & C_n(X,B) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_{n-1}(B,A) \arrow[r] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & C_{n-1}(X,B) \arrow[r] \arrow[d] & 0 \\ & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 \arrow[r] & C_0(B,A) \arrow[r] \arrow[d] & C_0(X,A) \arrow[r] \arrow[d] & C_0(X,B) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Next, whenever we have a short exact sequence of chain complexes $0 \to C_*(B,A) \to C_*(X,A) \to C_*(X,B) \to 0$, we have a long exact sequence of homology groups (by Zigzag Lemma).

$$
\begin{array}{l}
\cdots \to H_n(B,A) \to H_n(X,A) \to H_n(X,B) \to \\
\to H_{n-1}(B,A) \to H_{n-1}(X,A) \to H_{n-1}(X,B) \to \\
\to \cdots \qquad \cdots \to \\
\to H_1(B,A) \to H_1(X,A) \to H_1(X,B) \to \\
\to H_0(B,A) \to H_0(X,A) \to H_0(X,B) \to 0
\end{array}
$$

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(3) Let X be a topological space and $A, U, V \subset X$ such that $X = U \cup V$, $A \subset U \cap V$ and U, V are open in X. We'll show that there is a Mayer-Vietoris sequence relating $H_*(X,A)$, $H_*(U,A)$, $H_*(V,A)$ and $H_*(U \cap V, A)$.

~~Proof~~ We see that $A \subset U \cap V \subset U$. Thus, by the previous exercise, there is an exact sequence of chain complexes

$$0 \to C_*(U \cap V, A) \to C_*(U,A) \to C_*(U, U \cap V) \to 0.$$

Also, we see that $A \subset V \subset X$. Thus, there is an exact sequence:

$$0 \to C_*(V,A) \to C_*(X,A) \to C_*(X,V) \to 0.$$

Because $\begin{array}{ccccc} A & \subset & U \cap V & \subset & U \\ \cap & & \cap & & \cap \\ A & \subset & V & \subset & X \end{array}$, we have a commutative diagram

of chain complexes:

$$\begin{tikzcd}
0 \arrow[r] & C_*(U\cap V, A) \arrow[r] \arrow[d, "\tilde{i}"'] & C_*(U,A) \arrow[r] \arrow[d, "\tilde{i}"'] & C_*(U, U\cap V) \arrow[r] \arrow[d, "\tilde{i}"'] & 0 \\
0 \arrow[r] & C_*(V,A) \arrow[r] & C_*(X,A) \arrow[r] & C_*(X,V) \arrow[r] & 0
\end{tikzcd}$$

where $\tilde{i}$ is the chain map induced by $i : U \hookrightarrow X$. This diagram induces a diagram of homology groups:

$$\begin{tikzcd}
H_n(U\cap V, A) \arrow[r] \arrow[d, "i_*"'] \arrow[dr, phantom, "\circlearrowleft"] & H_n(U,A) \arrow[r] \arrow[d, "i_*"'] \arrow[dr, phantom, "\circlearrowleft"] & H_n(U, U\cap V) \arrow[d, "i_*"'] \\
H_n(V,A) \arrow[r] & H_n(X,A) \arrow[r] & H_n(X,V)
\end{tikzcd}$$

Moreover, the diagram of chain complexes induces a connecting homomorphism ∂_* such that

$$\begin{tikzcd}
H_n(U, U\cap V) \arrow[r, "\partial_*"] \arrow[d] & H_{n-1}(U\cap V, A) \arrow[d] \\
H_n(X,V) \arrow[r, "\partial_*"] & H_n(V,A)
\end{tikzcd}$$

commutes.

Therefore, we have a long diagram of homology groups.

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(U\cap V, A) \arrow[r] \arrow[d] & H_n(U,A) \arrow[r] \arrow[d] & H_n(U, U\cap V) \arrow[r, "\partial_*"] \arrow[d, "i_*"] & H_{n-1}(U\cap V, A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_n(V,A) \arrow[r] & H_n(X,A) \arrow[r] & H_n(X,V) \arrow[r, "\partial_*"] & H_{n-1}(V,A) \arrow[r] & \cdots
\end{tikzcd}$$

By the previous exercise, the upper and lower rows are exact. We see that U^c is closed in X, and $U^c = X \setminus U \in V$. Thus U^c is closed in V. Thus by excision, we have $H_n(X,V) \simeq H_n(X\setminus U^c, V\setminus U^c)$. Since $X \setminus U^c = X\setminus(X\setminus U) = U$ and $V \setminus U^c = V \cap U$, we have $H_n(X,V) \simeq H_n(U, U\cap V)$. Thus the map i_* in the previous diagram is an isomorphism for all $n \geq 0$. Thus that diagram gives us the following exact sequence.

$$
\begin{aligned}
&\cdots \to H_n(U\cap V, A) \to H_n(U,A) \oplus H_n(V,A) \to H_n(X,A) \to \\
&\to H_{n-1}(U\cap V, A) \to H_{n-1}(U,A) \oplus H_{n-1}(V,A) \to H_{n-1}(X,A) \to \\
&\to \cdots \qquad \cdots \to \\
&\to H_1(U\cap V, A) \to H_1(U,A) \oplus H_1(V,A) \to H_1(X,A) \to \\
&\to H_0(U\cap V, A) \to H_0(U,A) \oplus H_0(V,A) \to H_0(X,A) \to 0
\end{aligned}
$$

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This is the Mayer-Vietoris sequence that we need to find.

(5) Let M be a k-manifold, $k \geq 0$, and $p \in M$. We will compute the relative homology groups $H_*(M, M\setminus\{p\})$.

For $k=0$, M is just a single point. Thus $M\setminus\{p\} = \phi$. Thus $H_n(M,\phi) \simeq H_n(M) = H_n(*) = \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{if } n \geq 1 \end{cases}$.

Now we only consider the case $k \geqslant 1$, which implies $M \setminus \{p\} \neq \phi$. First, we'll prove the following lemma:

[Let $f: A \to B$ be a homeomorphism and C, D be subsets of A and B respectively such that $D = f(C)$. Then $H_n(A, C) \simeq H_n(B, D)$ for all $n \geqslant 0$.]

Proof The idea is to use the five-lemma. Since $C \subset A$, we have the exact sequence of homology groups:

$$\cdots \to H_n(C) \to H_n(A) \to H_n(A, C) \to H_{n-1}(C) \to H_{n-1}(A) \to \cdots$$

Since $D \subset B$, we have the exact sequence of homology groups:

$$\cdots \to H_n(D) \to H_n(B) \to H_n(B, D) \to H_{n-1}(D) \to H_{n-1}(B) \to \cdots$$

Since $A \xrightarrow{f} B$ and $C \xrightarrow{f} D$, we have the commutative diagram where the vertical maps are induced by f:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(C) \arrow[r] \arrow[d, "(1)"] & H_n(A) \arrow[r] \arrow[d, "(2)"] & H_n(A, C) \arrow[r] \arrow[d, "(3)"] & H_{n-1}(C) \arrow[r] \arrow[d, "(4)"] & H_{n-1}(A) \arrow[r] \arrow[d, "(5)"] & \cdots \\
\cdots \arrow[r] & H_n(D) \arrow[r] & H_n(B) \arrow[r] & H_n(B, D) \arrow[r] & H_{n-1}(D) \arrow[r] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

The homology groups of index $n-1$ are replaced by 0 in case $n = 0$. The group morphisms (1) and (5) are isomorphisms because $C \xrightarrow{f} D$. The morphisms (2) and (4) are also isomorphisms because $A \xrightarrow{f} B$. Thus (3) is an isomorphism by five-lemma. QED //

Return to the problem. Since M is a k-manifold, p has a Euclidean neighborhood U, i.e. $U \simeq \mathbb{R}^k$ and U is open in M. We have $U^c = M \setminus U \subset M \setminus \{p\}$. Thus we can apply the excision property here. We have

$$H_n(M, M\setminus\{p\}) \simeq H_n(M\setminus U^c, (M\setminus\{p\})\setminus U^c) = H_n(U, U\setminus\{p\}).$$

Let $f: U \to \mathbb{R}^k$ be an homeomorphism with $f(p) = q$. Then we can apply the above lemma for $A = U$, $B = \{p\}$, $C = \mathbb{R}^k$ and $D = \{q\}$. We have

$$H_n(U, U\setminus\{p\}) \simeq H_n(\mathbb{R}^k, \mathbb{R}^k\setminus\{q\}).$$

We see that any translation map in $\mathbb{R}^k$ is a homeomorphism. Thus q can be taken zero. Moreover, the map $g: D_k = \{x \in \mathbb{R}^k \mid \|x\| < 1\} \to \mathbb{R}^k$ with

$g(z) = \dfrac{z}{1-\|z\|}$ is a homeomorphism sending 0 to 0. By the above lemma, we have $H_n(\mathbb{R}^k, \mathbb{R}^k\setminus\{0\}) \simeq H_n(D_k^*, D_k^*\setminus\{0\})$. Next we'll show that $H_n(D_k, D_k\setminus\{0\}) \simeq H_n(D_k, S_{k-1})$ by homotopy invariance of homology groups and the five-lemma.

The map $r: D_k\setminus\{0\} \to S_{k-1}$, $r(z) = \dfrac{z}{\|z\|}$ is a deformation retraction. Thus, the inclusion $i: S_{k-1} \to D_k\setminus\{0\}$ is a homotopy equivalence. Since we have the inclusions $D_k\setminus\{0\} \xrightarrow{i} D_k$ and $S_{k-1} \xrightarrow{i} D_k$, there is a commutative diagram where the vertical maps are induced by the inclusion maps.

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(S_{k-1}) \arrow[r] \arrow[d, "(1)"'] & H_n(D_k) \arrow[r] \arrow[d, "(2)"'] & H_n(D_k, S_{k-1}) \arrow[r] \arrow[d, "(3)"'] & H_{n-1}(S_{k-1}) \arrow[r] \arrow[d, "(4)"'] & H_{n-1}(D_{k-1}) \arrow[r] \arrow[d, "(5)"'] & \cdots \\
\cdots \arrow[r] & H_n(D_k \setminus \{0\}) \arrow[r] & H_n(D_k) \arrow[r] & H_n(D_k, D_k \setminus \{0\}) \arrow[r] & H_{n-1}(D_k \setminus \{0\}) \arrow[r] & H_n(D_{k-1}) \arrow[r] & \cdots
\end{tikzcd}$$

The homology groups of index $n-1$ will be replaced by 0 in case $n=0$. (2) and (4) are identity maps. (1) and (3) are isomorphisms because $i: S_{k-1} \to D_k \setminus \{0\}$ is a homotopy equivalence. Thus, by the five-lemma, (3) is an isomorphism. Thus $H_n(D_k, D_k \setminus \{0\}) \simeq H_n(D_k, S_{k-1})$. Up to now we have,

$$H_n(M, M \setminus \{p\}) \simeq H_n(D_k, S_{k-1}) = \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

Combining this result with the case $k=0$, we have

$$H_n(M, M \setminus \{p\}) = \begin{cases} \mathbb{Z} & \text{if } n = k, \\ 0 & \text{if } n \neq k, \end{cases} \quad \text{for all } k \geq 0. \checkmark$$

Therefore, only H_k is nontrivial where k was given as a "dimension" of M. Suppose that M is also an m-manifold with $k \neq m$. Then $H_m(M, M \setminus \{p\})$ is nontrivial. This is a contradiction because we know that only H_k is nontrivial. Therefore, k is a well-defined dimension of a manifold with respect to a given point $p \in M$. To show that k is actually independent

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Continue the proof of the lemma of problem 4

$$V_n \cap A_m = (X \setminus \{x_p : p \geq n\}) \cap A_m = (X \setminus \{x_p : n \leq p < m\}) \cap A_m.$$

Since the set $\{x_p : n \leq p < m\}$ is finite, and X is a T_1 space, this set is closed in X. Thus $X \setminus \{x_p : n \leq p < m\}$ is open in X. Thus $V_n \cap A_m$ is open in A_m.

Because this is true for all $m \in \mathbb{N}$, we conclude V_n is open in X. Since

$$A \subset X = \bigcup_{k=1}^{\infty} V_k.$$

and A is compact, there exists $i_1 < i_2 < \dots < i_m$ such that $A \subset \bigcup_{k=1}^{m} V_{i_k}$. Since the sequence (V_k) is ascending, we have $A \subset V_{i_m}$. This is a contradiction because we knew that $x_{i_m} \in A \setminus V_{i_m}$.

of the choice of points p, we need the connectivity of M.

~~Nons~~ Now assume that M is connected and $p, p' \in M$. We'll show that $H_n(M, M\setminus\{p\}) \simeq H_n(M, M\setminus\{p'\})$ for all $n \geq 0$. Since M is a manifold, it is locally path-connected. Together with ~~the~~ its connectedness, we conclude that M is path-connected. Thus there exists a path $\gamma: [0,1] \to M$ connecting p and p'. In problem 3, HW4, we showed that there exists a number $m \in \mathbb{N}$ and open subsets $U_0, U_1, \dots, U_{m-1}$ of M such that $U_j \simeq \mathbb{R}^k$ and $\gamma([\frac{j}{m}, \frac{j+1}{m}]) \subset U_j$ for all $0 \leq j < m$, where k is the dimension of M corresponding to point p. Put $p_j = \gamma(\frac{j}{m})$ for all $0 \leq j \leq m$. Then $p = \gamma(0) = p_0$ and $p' = \gamma(1) = p_m$. To show that $H_n(M, M\setminus\{p\}) \simeq H_n(M, M\setminus\{p'\})$, we will show that $H_n(M, M\setminus\{p_j\}) \simeq H_n(M, M\setminus\{p_{j+1}\})$ for all $0 \leq j < m$. Since $p_j, p_{j+1} \in U_j$ and $U_j \simeq \mathbb{R}^k$, what we need to show is equivalent to show that $H_n(M, M\setminus\{p\}) \simeq H_n(M, M\setminus\{p'\})$ provided that there exists an open subset U of M such that $p, p' \in U$ and $U \simeq \mathbb{R}^k$ (homeomorphic). Like before, by excision property, we have

$$H_n(M, M\setminus\{p\}) \simeq H_n(U, U\setminus\{p\})$$

$$H_n(M, M\setminus\{p'\}) \simeq H_n(U, U\setminus\{p'\})$$

Since U and $\mathbb{R}^k$ are homeomorphic, there are corresponding q and q' in $\mathbb{R}^k$ such that $H_n(U, U\setminus\{p\}) \simeq H_n(\mathbb{R}^k, \mathbb{R}^k\setminus\{q\})$ and $H_n(U, U\setminus\{p'\}) \simeq H_n(\mathbb{R}^k, \mathbb{R}^k\setminus\{q'\})$. The translation $\phi: x \mapsto x+q'-q$ is a homeomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$ mapping q to q'. Thus,

$$H_n(\mathbb{R}^k, \mathbb{R}^k\setminus\{q\}) \simeq H_n(\mathbb{R}^k, \mathbb{R}^k\setminus\{\phi(q)\}) = H_n(\mathbb{R}^k, \mathbb{R}^k\setminus\{q'\}).$$ 4/4

(4) We assume the following lemma for a moment:

<u>Lemma</u> Let X be a T_1-space and subspaces $A_1 \subset A_2 \subset A_3 \cdots$ be such that $\bigcup_{k=1}^{\infty} A_k = X$ and a subspace U is closed in X if and only if $U \cap A_n$ is closed in A_n for all $n \geq 1$. Then for every compact set $A \subset X$, there exists $j \in \mathbb{N}$ such that $A \subset A_j$.

Let X be a space satisfying the conditions in the above lemma. First, we will show that each element in $H_k(X)$ is the image of an element in $H_k(A_j)$ for some $j \in \mathbb{N}$. Let $[z] \in H_k(X)$, where z is a ~~cycle~~ ~~in chain~~ cycle in $C_k(X)$. We have

$$z = \sum_{l=1}^{m} m_l \sigma_l \quad \text{with } \partial z = 0 \text{ in } C_{k-1}(X), \text{ where}$$

$\sigma_l : \Delta^k \to X$ is a singular k-simplex for all $l = 1, \dots, m$. Since σ_l is continuous and ~~$\sigma_l(\Delta^k)$~~ Δ^k is compact, $\sigma_l(\Delta^k)$ is a compact subspace of X. Using the lemma, we conclude that there exists $j_\ell \in \mathbb{N}$ such that $\sigma_\ell(\Delta^k) \subset A_{j_\ell}$.

Put $j = \max\{j_1, j_2, \dots, j_m\}$. Then $\sigma_\ell(\Delta^k) \subset A_j$ for all $\ell = 1, \dots, m$. Thus $\sigma_\ell \in C_k(A_j)$. Thus,
$$z = \sum_{\ell=1}^{m} n_\ell \sigma_\ell \in C_k(A_j).$$

Since $\partial z = 0$ in $C_{k-1}(X)$ and $C_{k-1}(A_j) \subset C_{k-1}(X)$, we have $\partial z = 0$ in $C_{k-1}(A_j)$.

Therefore, $[z]$ with respect to X is the image of $[z]$ with respect to A_j. In otherwords, the inclusion $A_j \xhookrightarrow{i} X$ induces a homomorphism $i_*: H_k(A_j) \to H_k(X)$ such that $i_*([z]) = [z]$.

Secondly, we'll show that two elements in $H_k(A_j)$ become the same in $H_k(X)$ iff there exists $j' \geq j$ such that their images in $H_k(A_{j'})$ coincide.

($\Leftarrow$) Let $[z_1], [z_2] \in H_k(A_j)$, where z_1 and z_2 are cycles in $C_k(A_j)$, such that there exists $j' \geq j$ with $i_*(z_1) = i_*(z_2)$, where i is the inclusion $A_j \xhookrightarrow{i} A_{j'}$. Then $z_1 - z_2 = \partial \tau$ where $\tau \in C_{k+1}(A_{j'})$. Then $\tau \in C_{k+1}(X)$. Then $z_1 - z_2$ is equal to a boundary of a $(k+1)$-chain in X. Thus $[z_1]$ with respect to X is the same as $[z_2]$ with respect to X.

($\Rightarrow$) Let $[z_1], [z_2] \in H_k(A_j)$ where z_1 and z_2 are two cycles in $C_k(A_j)$ such that $i_*(z_1) = i_*(z_2)$ where i is the inclusion $A_j \hookrightarrow X$. Then there exists $\tau \in C_{k+1}(X)$

such that $z_1 - z_2 = \partial\tau$. We write $\tau = \sum_{l=1}^{m} n_l \sigma_l$ where each σ_l is a continuous function from Δ^{k+1} to X. Like before, there exists an index $j' \geq j$ such that $\sigma_l(\Delta^{k+1}) \subset A_{j'}$. Thus $\sigma_l \in C_{k+1}(A_{j'})$. Thus $\tau \in C_{k+1}(A_{j'})$. Therefore, $z_1 - z_2$ is the boundary of a $(k+1)$-chain in $A_{j'}$. Thus, $[z_1]$ with respect to $C_k(A_{j'})$ is the same as $[z_2]$ with respect to $C_k(A_{j'})$.

Proof of the lemma

Let X be a space as in the hypothesis. Suppose that a subspace V of X satisfies $V \cap A_n$ is open in A_n for all $n \in \mathbb{N}$. Then, if we write $V^c := X \setminus V$ then $V^c \cap A_n = A_n \setminus (A_n \cap V)$, which is ~~open~~ closed in A_n for all $n \in \mathbb{N}$. Thus V^c is closed in X. Thus V is open in X. This means any subspace of X whose intersection with A_n is open in A_n is also open in X.

Let A be compact subspace of X. Suppose by contradiction that $A \not\subset A_n$ for any n. then there exists $x_n \in A \setminus A_n$. We define $V_n = X \setminus \{x_m : m \geq n\} = X \setminus \{x_n, x_{n+1}, \ldots\}$. Then $V_1 \subset V_2 \subset V_3 \subset \cdots$ and $\bigcup_{k=1}^{\infty} V_k = X$. We'll show that $V_n \cap A_m$ is open in A_m for every $m, n \in \mathbb{N}$. Since $x_p \notin A_p$ and $A_m \subset A_p$ for all $p \geq m$, we have $x_p \notin A_m$ for all $p \geq m$. Therefore,

(proof continues after page 15 in this Problem set)

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