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Topology and Manifolds

Homework 2 20/20

① Show graphically that the simplicial complex with 7 vertices, generated by the triangles below, gives rise to a space homeomorphic to the torus.

123 127 134 145 156 167 236

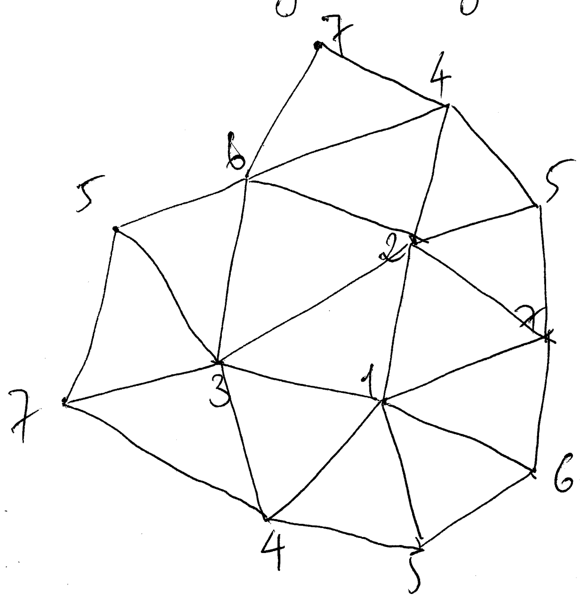
245 246 257 347 356 357 467

Proof

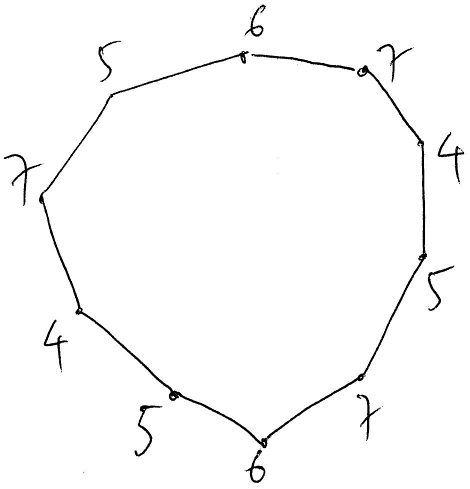
We will follow the following steps:

Step 1: Find a description of the simplicial complex as a polygon with edge identification.

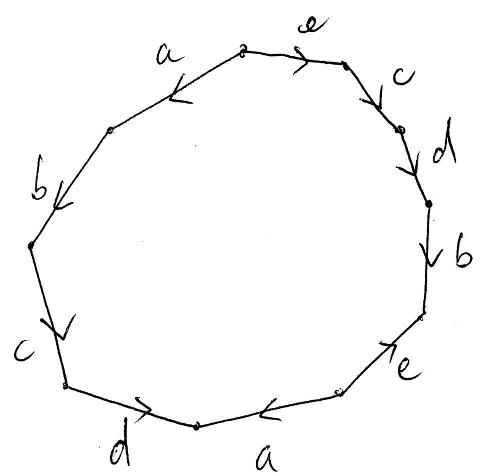
We glue the above triangles along their edges as follow



Now the points 1, 2 and 3 become interior point. Thus only 7-3=4 vertices are left. We redraw the graph as follow

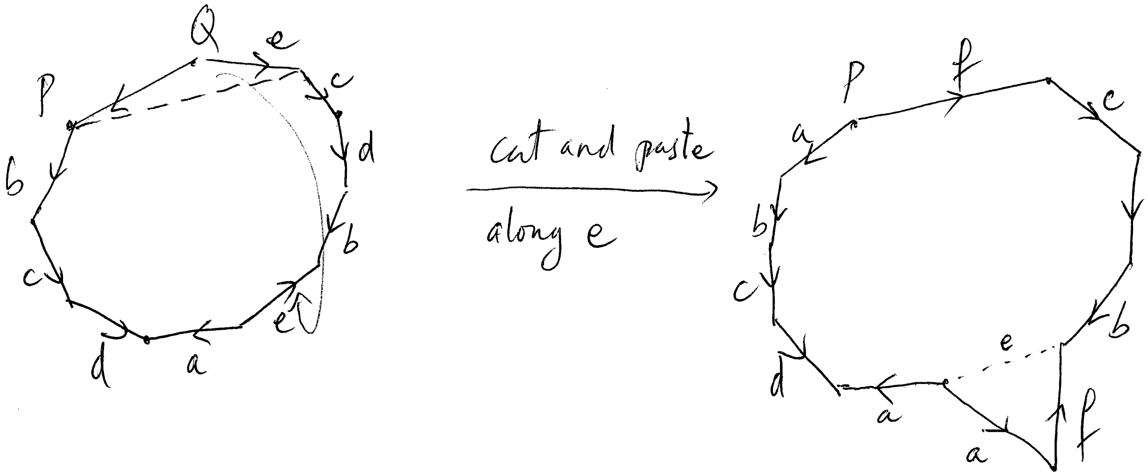


And now we obtain ~~the~~ a description of the given simplicial complex as a polygon with edge identifications as follow

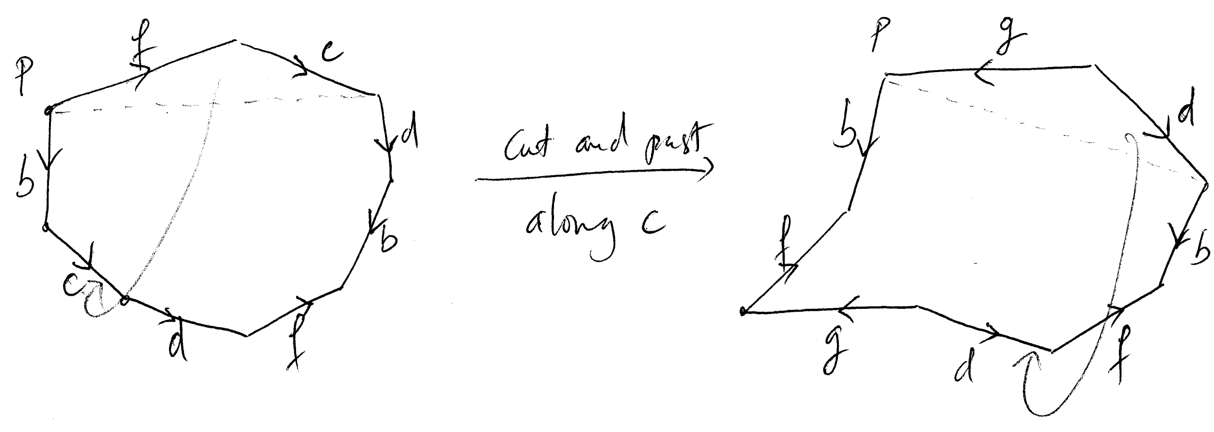


<u>Step 2</u> : Try to reduce the polygon by either ~~reduce~~ cutting, pasting or folding. Our goal is to reduce ~~it~~ to the case in which ~~eve~~ all vertices are the same.

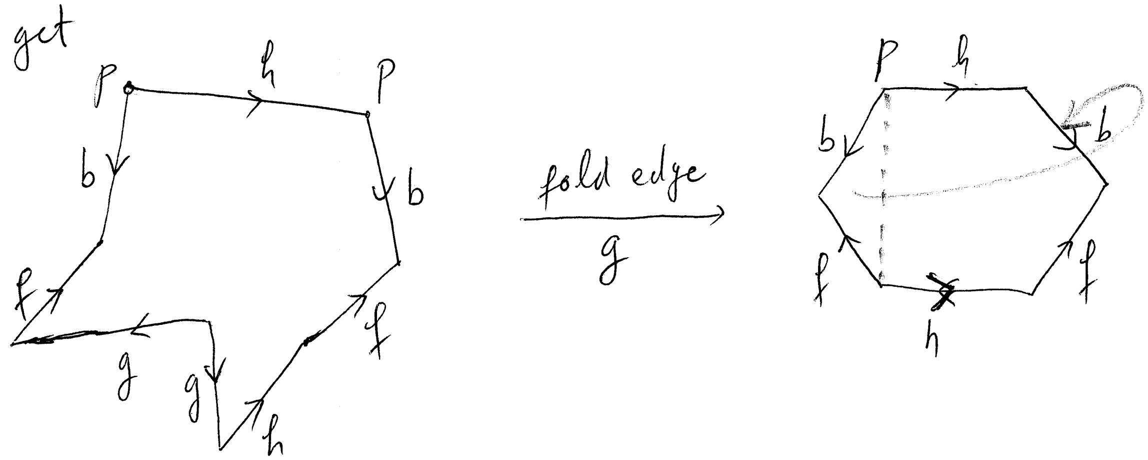
We see that the point P in the left figure below has the maximal number of vertices identified with P. Thus we cut the triangle and paste it along edge e, as shown in the right figure



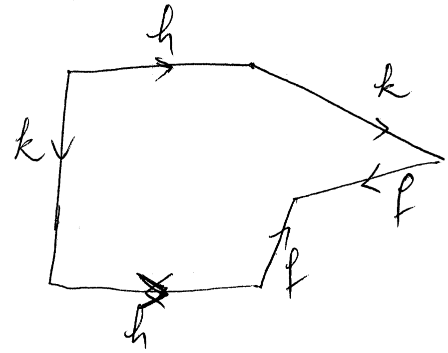
Then we fold edges a and a^{-1} to get



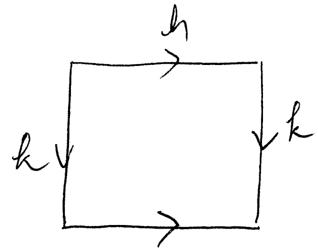
We continue to cut the triangle on the right figure and paste it along edge d to get



We continue to cut the triangle in the above figure and paste along b to get

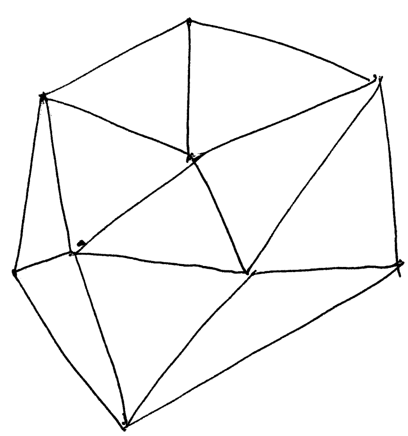


We fold edges l and l^{-1} to get



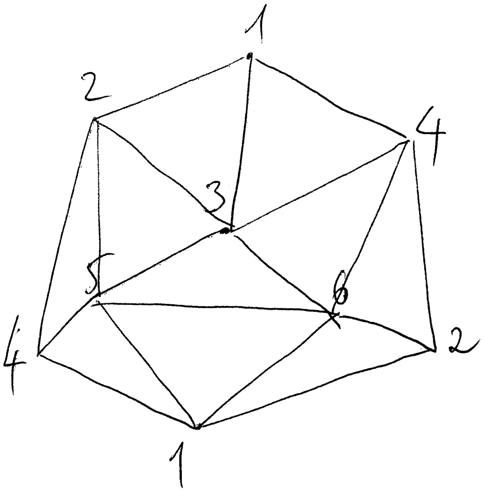
And this is the space homeomorphic to the torus. 4/4

(4) With an appropriate viewpoint, we will see an icosahedron, which is a equilateral 20-faced solid, as follow



This projection is exactly a half of the real icosahedron.

In case of real projective plane, we will identify the antipodal vertices of this icosahedron and label them as follow



We obtain a triangulation, which we will prove to be a triangulation of $\mathbb{RP}^2$, as follow:

123	~~356~~	123	235
134		134	246
235		126	245
246		145	356
		156	346

this is a simplicial complex with

$$V = \{1,2,3,4,5,6\} \text{ - the set of vertices}$$

$$F = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{1,5\},\{1,6\},$$
$$\{2,3\},\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\},$$
$$\{1,2,3\},\{1,3,4\},\{1,2,6\},\{1,4,5\},\{1,5,6\},\{2,3,5\},\{2,4,6\},\{2,4,5\},$$
$$\{3,5,6\},\{3,4,6\}\} \text{ - the set of all faces.}$$

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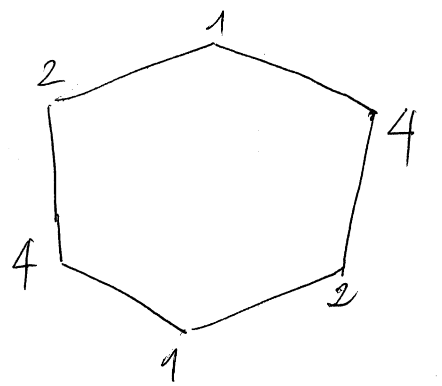
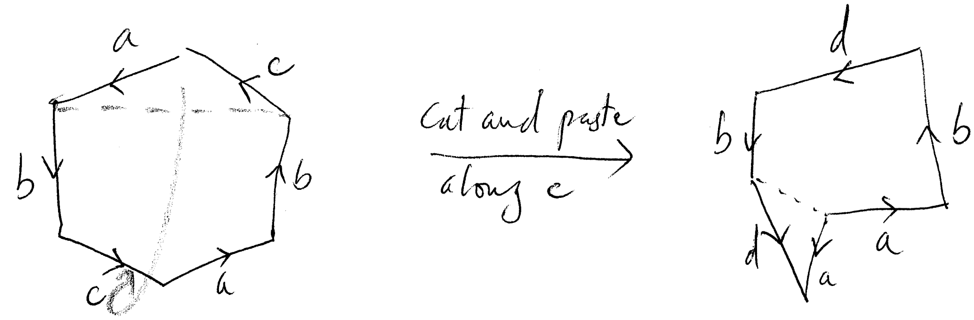
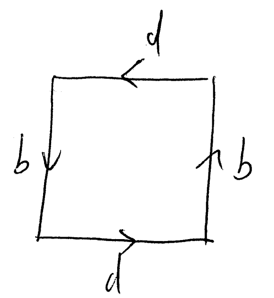
We will show that $(\mathcal{O}, \mathcal{F})$ is homeomorphic to $\mathbb{RP}^2$. We see that 3, 5 and 6 are already interior points. Thus the realization of the simplicial complexes remains:



We obtain the description as a polygon with edge identification



We try to reduce to the case in which only one vertex is left. First we cut the triangle and paste it along edge c as shown above. Then we fold edges a and a^{-1} to get



which is well-known as $\mathbb{RP}^2$.

4/4

(2) For a 2-dimensional simplicial complex with v vertices, e edges and f faces, the Euler characteristic χ is defined to be $v-e+f$. If this simplicial complex gives rise to a compact surface, give formulas for e and f in terms of χ and v which are nondecreasing in v.

Proof We have $\chi = v-e+f$. Thus $e - f = v - \chi$.

Moreover, if the simplicial complex ~~ha~~ gives rise to a closed surface, then each edge belongs to exactly two faces. ~~and~~ Moreover, we know that each face has exactly three edges. We put

$$S = \{(a,b) : a \text{ is an edge of face } b \text{ in the simplicial complex}\}$$

Then $|S| = 2e$ because each edge a appears in exactly ~~three~~ two pairs. Moreover, $|S| = 3f$ because each face b appears in exactly three pairs. Thus we must have $2e = 3f$. Then we get

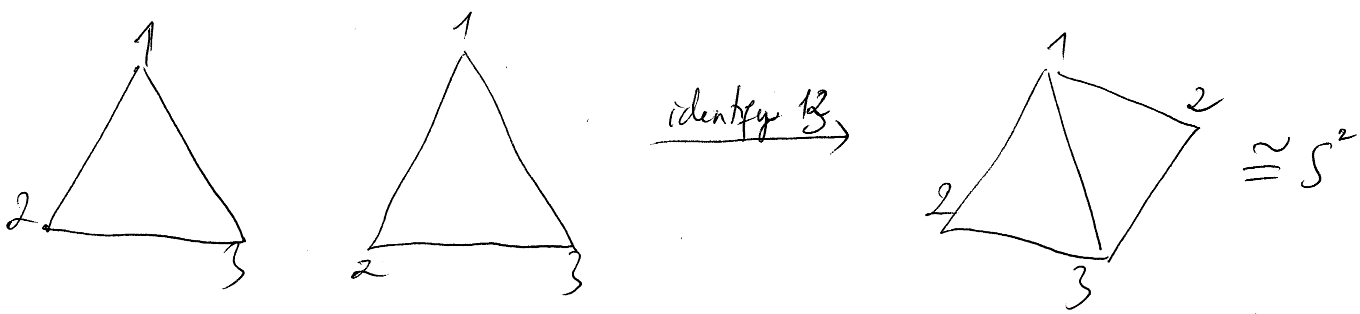
$$\begin{cases} e-f = v-\chi \\ 2e-3f = 0 \end{cases}$$

which gives $e = 3(v-\chi)$ and $f = 2(v-\chi)$.

If χ is fixed, then e and f are increasing in v. 4/4

(3) Using the formulas from the previous problem, show that any triangulation of a compact surface of Euler characteristic 0 requires at least 7 vertices, and any of Euler characteristic 1 requires at least 6 vertices.

Proof Let ~~S be the generating set of the simplicial complex, i.e. S contains all triangles in the triangulation~~. If there are two triangles in the triangulation that are identical, ~~then~~ say 123 and 123, then after identifying edges, these



two triangles give a sphere, which is closed. Thus the triangulation has no other triangles than these two ones (Otherwise, the third triangle has no place to join the sphere without destroying the "2-manifold" property).

In that case $v=3$, $f=2$ and $e=3$. Then $\chi = v-e+f=2$, not in our concern.

Now we consider the case in which no face can duplicate. Then the number of faces is at most C_v^3, that is the number of all sets of three vertices. Thus $f \le C_v^3$.

The number of edges is at most C_v^2, that is the number of all combinations of two vertices. Thus $e \leq C_v^2$. Then we have

$$\begin{cases} f \leq C_v^3 \\ e \leq C_v^2 \end{cases}$$

By the previous problem, in case of closed surface $f = 2(v-\chi)$ and $e = 3(v-\chi)$. We have

$$\begin{cases} 2(v-\chi) \leq C_v^3 \\ 3(v-\chi) \leq C_v^2 \end{cases}$$

<u>If $\chi = 0$:</u>

Then $\begin{cases} 2v \leq C_v^3 \\ 3v \leq C_v^2 \end{cases}$

We test each $v = 1, 2, 3, \dots$ and see that $v = 7$ is the first value that makes these inequality true. Thus $v \geq 7$ and $f = 2v \geq 14$, $e = 3v \geq 21$.

<u>If $\chi = 1$:</u>

Then $\begin{cases} 2(v-1) \leq C_v^3 \\ 3(v-1) \leq C_v^2 \end{cases}$

We test each $v = 1, 2, 3, \dots$ and see that $v = 6$ is the first value that makes these inequality true. Thus $v \geq 6$ and $f = 2(v-1) \geq 10$, $e = 3(v-1) = 15$.

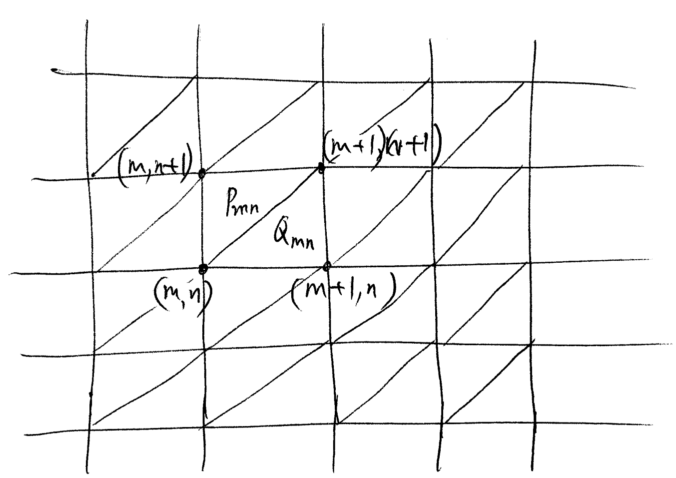
(5) Suppose that you are given a simplicial complex with set $\mathcal{V}$ of vertices and set $\mathcal{F} \subset \mathcal{P}(\mathcal{V})$ of faces, satisfying two properties.

(a) Any face $U \in \mathcal{F}$ satisfies $|U| \leqslant 3$.

(b) For any vertices $a \neq b$ such that $\{a,b\} \in \mathcal{F}$, there are precisely two other vertices c such that the three-element set $\{a,b,c\}$ is in $\mathcal{F}$.

Does this simplicial complex necessarily give rise to a compact (closed) surface? Either give a proof or counterexample. (Be careful)

Proof We notice that $\mathcal{V}$ is not necessarily finite, and that is the source of our idea! We will try to construct a counterexample in which the surface derived from the simplicial complex $(\mathcal{V}, \mathcal{F})$ is unbounded and thus not compact.



For each pair of integer (m,n), we denote v_{mn} to be the point in $\mathbb{R}^2$ which has coordinate (m,n). We also denote the closed triangle with three vertices $v_{mn}, v_{m,n+1}, v_{m+1,n+1}$ to be P_{mn}, and that with three vertices $v_{mn}, v_{m+1,n}, v_{m+1,n+1}$

to be Q_{mn}. We put

$$\mathcal{V} = \{v_{mn} : m, n \in \mathbb{Z}\}$$

$$\mathcal{F} = \bigcup_{m,n\in\mathbb{Z}} \Big\{ \{v_{mn}, v_{m,n+1}, v_{m+1,n+1}\}, \{v_{mn}, v_{m,n+1}\}, \{v_{mn}, v_{m+1,n+1}\}, \{v_{m,n+1}, v_{m+1,n+1}\}, \{v_{mn}\}, \{v_{m,n+1}\}, \{v_{m+1,n+1}\} \Big\}$$

$$\cup \left(\bigcup_{m,n\in\mathbb{Z}} \Big\{ \{v_{mn}, v_{m+1,n}, v_{m+1,n+1}\}, \{v_{mn}, v_{m+1,n}\}, \{v_{mn}, v_{m+1,n+1}\}, \{v_{m+1,n}, v_{m+1,n+1}\}, \{v_{mn}\}, \{v_{m+1,n}\}, \{v_{m+1,n+1}\} \Big\} \right)$$

Because of either the requirement for $(\mathcal{V}, \mathcal{F})$ to be an (abstract) simplicial complex or the requirement for its realization in $\mathbb{R}^2$ (which is the set of well ordered triangles in the figure) to be a Euclidean simplicial complex is satisfied, we conclude that $(\mathcal{V}, \mathcal{F})$ is a simplicial complex. The requirements for $(\mathcal{V}, \mathcal{F})$ to be a simplicial complex are

1) Each element in $\mathcal{F}$ is a finite subset of $\mathcal{V}$.
2) If $s \in \mathcal{F}$ the every nonempty subset of s also belongs to $\mathcal{F}$
3) $(\mathcal{V}, \mathcal{F})$ is locally finite, i.e. each vertex of $\mathcal{V}$ appears in only finitely many elements of $\mathcal{F}$

Requirements 1) and 2) automatically holds because of the definition of $\mathcal{F}$. Requirement 3) is also satisfied ~~by~~ because each vertex appears in exactly 6 elements in $\mathcal{F}$. A realization of $(\mathcal{V}, \mathcal{F})$ is the union of all triangles P_{mn}, Q_{mn} in $\mathbb{R}^2$, which is equal to $\mathbb{R}^2$ itself. And we know that any possible realizations of $(\mathcal{V}, \mathcal{F})$ are homeomorphic. Thus we can safely say that $(\mathcal{V}, \mathcal{F})$ gives rise to $\mathbb{R}^2$ with regular topology. Interestingly, ~~each~~ edge in $\mathcal{F}$ appears in exactly 2 triangles. Specifically,

- Edge $\{v_{mn}, v_{m,n+1}\}$ appears in only triangles P_{mn} and $Q_{m-1,n}$.
- Edge $\{v_{m,n+1}, v_{m+1,n+1}\}$ appears in only P_{mn} and $Q_{m,n+1}$.
- Edge $\{v_{mn}, v_{m+1,n+1}\}$ appears in only P_{mn} and Q_{mn}.

Thus the simplicial complex $(\mathcal{V}, \mathcal{F})$ satisfies the conditions in the problem. The surface it gives rise to, however, is $\mathbb{R}^2$, which is not a compact topological space. 4/4

We can continue to show that even if $(\mathcal{V}, \mathcal{F})$ is given one more constraint

(c) $(\mathcal{V}, \mathcal{F})$ is finite, i.e. $\mathcal{V}$ is a finite set

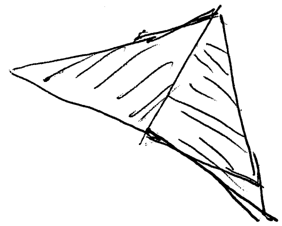
This actually isn't true: consider a "pinched torus" or two tetrahedra joined at one vertex (or even a single point)

Then $(\mathcal{V}, \mathcal{F})$ gives rise to a ~~closed~~ compact surface. (By struggling with the following proof)

<u>Proof</u> Put $\mathcal{V} = \{v_1, \ldots, v_n\}$. Let $\{e_1, \ldots, e_n\}$ be the normal (standard) basis of $\mathbb{R}^n$. We consider the natural map $f_0 : \mathcal{V} \to \{e_1, \ldots, e_n\}$
$$v_i \mapsto e_i$$

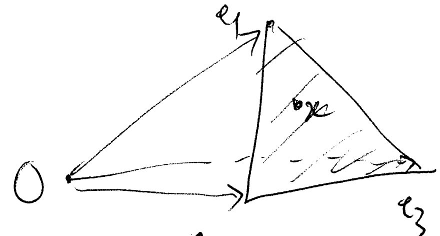
Each face ~~$s \in \mathcal{F}$~~ $\{v_{i_1}, \ldots, v_{i_k}\}$ in $\mathcal{F}$ corresponds to the convex hull of $\{f_0(v_{i_1}), \ldots, f_0(v_{i_k})\}$ in $\mathbb{R}^n$. Thus we get a realization of $(\mathcal{V}, \mathcal{F})$ in $\mathbb{R}^n$, which is denoted by K.

Now we'll show that K is actually a closed surface.



~~Tak~~ Firstt, Since K is contained in the cubic (polyhedron) made by $e_1,\dots,e_n$ on $\mathbb{R}^n$, and K is closed, it must be compact. To show that K is a 2-manifold, we show that each $x \in K$ has a neighborhood in K that is ~~isomor~~ homeomorphic to $\mathbb{R}^2$ or $\mathbb{B}^2$. We consider 3 cases:

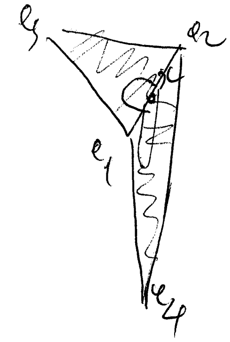
<u>Case 1</u>: x is in the interior of a triangle



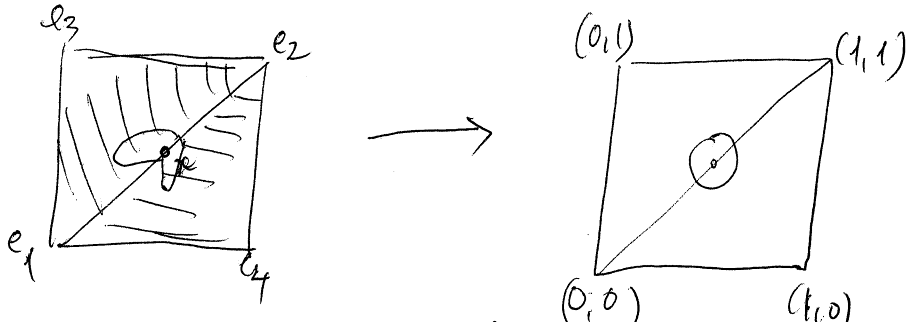
WLOG, we can assume $x \in [e_1, e_2, e_3]$ (the ~~affined space generate~~ 2-simplex spanned by e_1, e_2, e_3)

Then x is not on any ~~segment~~ segment $[e_i, e_j]$ and faces ~~$=[f(v_i), f(v_j)]$~~ because these e_i's are affinely independent. Thus the distance between x to any edges $[e_i, e_j]$ and faces is positive; and because ~~are~~ there are only finitely many edges and faces, there ~~min~~ exists the ~~minimum~~ minimum distance from x to all edges and faces, say $\delta > 0$. Then the ball $B(x, \frac{\delta}{2})$ in $\mathbb{R}^n$ intersects K only in the interior of $[e_1, e_2, e_3]$. Thus x has a neighborhood in K that is ~~to~~ homeomorphic to $\mathbb{R}^2$ (it takes some effort to write explicitly the homeomorphism).

Case 2: x lies on an edge, but not coincides any vertex.



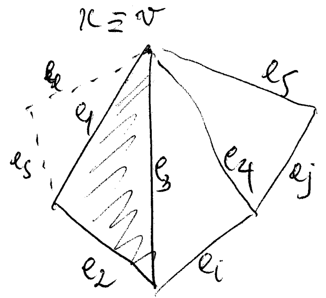
WLOG, suppose that $x \in [e_1, e_2] \setminus \{e_1, e_2\}$. By the problem's hypothesis (a), there exist exactly two vertice, say e_3 and e_4 for example, such that $[e_1, e_2, e_3]$ and $[e_1, e_2, e_4]$ are contained in K. By the same reasoning as the previous case, there exists $\delta > 0$ such that the ball $B(x, \delta)$ in $\mathbb{R}^n$ intersects K at $[e_1, e_2, e_3]$ and $[e_1, e_2, e_4]$ only. Then we can define the map from $[e_1, e_2, e_3] \cup [e_1, e_2, e_4]$ to the square $[0,1] \times [0,1]$ in $\mathbb{R}^2$



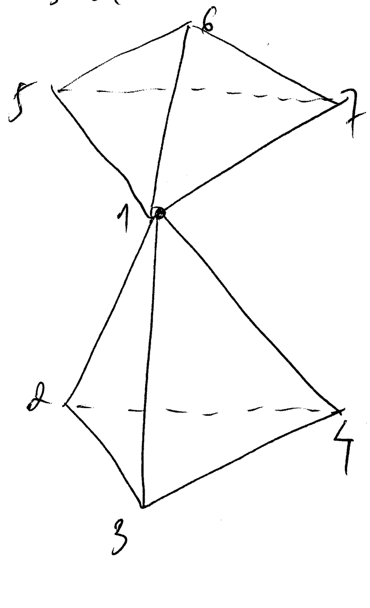
which is affine and the neighborhood of x in K becomes ~~a ne~~ something in $[0,1] \times [0,1]$ that is homeomorphic to $\mathbb{R}^2$. Again, it takes some effort to write everything down explicitly.

Case 3: x coincides one vertex

Suppose that x ~~coided~~ coincides vertex v of ~~the~~ the triangle $[e_1, e_2, e_3]$



Since e_3 has to appear in exactly 2 2-simplices, there is some e_4 such that $v \in e_4$. Again, since e_4 must appear in exactly 2 2-simplices, the must be some e_5 such that $v \in e_5$, ... The process must finish because we have only finitely many triangles $[e_i, e_j, e_k]$, and because each side (or edge) must belong to exactly 2 ~~tri~~2-simplices, the sequence $e_3, e_4, e_5, \ldots$ must go back to e_1. Then everything will be fine if there exist $\delta > 0$ such that the ball $B(v, \delta)$ in $\mathbb{R}^n$ only intersect K at faces $[e_1, e_2, e_3]$, $[e_3, e_4, e_i]$, $[e_4, e_5, e_j]$, ..., $[e_q, e_1, e_s]$. After some effort, we can show that this intersection is homeomorphic to $\mathbb{R}^2$. However, what if there is no such δ. Look at the following picture:



There are two tetrahedra which intersect at only vertex 1. And we see that any Ball $B(1, \delta)$ in $\mathbb{R}^3$ cut the simplicial compact at very weird ~~domain~~ domain. This domain is in fact not isomorphic to $\mathbb{R}^2$ because if we remove ~~one~~ point 1, it becomes disconnected, while $\mathbb{R}^2$ removed one point is still connected.

It demonstrates that now we have one more Counterexample in case $\mathcal{V}$ is finite, and it is surprising!!

$$\mathcal{V} = \{1,2,3,4,5,6,7\}$$

$$\mathcal{F} = \{ \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{1,2\},\{1,3\},\{2,3\},\{1,5\},\{1,6\},$$
$$\{1,7\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},$$
$$\{1,5,6\},\{1,5,7\},\{1,6,7\},\{5,6,7\} \}$$

The final point we would like to make here is that if $\mathcal{V}$ is infinite, ~~the~~ any Simplicial complex $(\mathcal{V},\mathcal{F})$ ~~satisfying~~ does not give rise to a compact surface. (or more generally compact Euclidean simplicial complex)

Proof Suppose that $(\mathcal{V},\mathcal{F})$ has a geometric realization that is compact in some $\mathbb{R}^n$. ~~WLOG~~ we can assume that $(\mathcal{V},\mathcal{F})$ itself is an Euclidean simplicial complex whose polyhedron form is a compact topological subspace in $\mathbb{R}^n$. $(\mathcal{V},\mathcal{F})$ by definition must satisfy 3 conditions (Lee, p. 149):

(i) If $\sigma \in \mathcal{F}$ then every face of σ is in $\mathcal{F}$.

(ii) the intersection of any two simplices in $\mathcal{F}$ is either empty or a face of each.

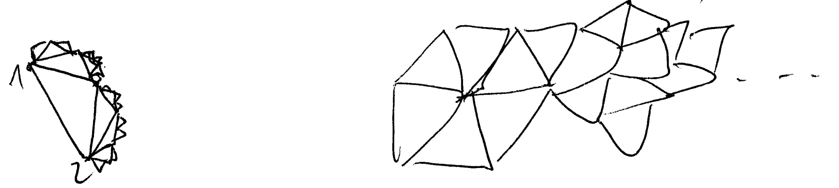
(iii) $\mathcal{F}$ is a locally finite collection, i.e. every point $x \in \mathbb{R}^n$ has a neighborhood which intersects only finitely many elements in $\mathcal{F}$.

Let K be an ~~Euclidean simplicial comp~~ an polyhedron form of $(\mathcal{V},\mathcal{F})$.

Put $V = \{v_1, v_2, v_3, \dots\}$. Then $v_1, v_2, \dots \in K$. Since K is compact, ~~it must~~ the sequence (v_n) must has a limit point $v \in K$. Now we see that any neighborhood of v in $\mathbb{R}^n$ must contain infinitely many v_n's, and thus it intersects infinitely many elements of $\mathcal{F}$. This is a contradiction!

In conclusion, a simplicial complex $(V, \mathcal{F})$ whose V is infinite cannot give rise to a compact surface (or compact polyhedron form) due to two risks; either the polyhedron form is unbounded, or, if bounded, will break the locall finiteness property.

The last thing we want to comment is that there is no way to ~~have an equivalent condition~~ impose a condition on an abstract simplicial complex $(V, \mathcal{F})$ to make sure that its polyhedron form satisfies the local finiteness property. The reason is that $(V, \mathcal{F})$ only describes the list of vertices, faces, edges, ... and how they structurally connect to each other; it does not indicate the distance between two vertices. For example, the below simplicial complexes are the same, but one



is locally finite, the other is not.