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Math 8301: Topology & Manifolds 1
Homework 3    18/20

① Using the classification of closed, connected surfaces according to orientability and Euler characteristic, describe the two surfaces obtained by using the following strings to identify edges:

(i) $acca^{-1}bdbe^{-1}d^{-1}e^{-1}$

(ii) $abcb^{-1}defghg^{-1}f^{-1}a^{-1}h^{-1}d^{-1}c^{-1}e^{-1}$

Proof

(i) We will apply a handy homeomorphism $aaw \sim waa \sim awa^{-1}$ where $w$ can be a string of symbols, not just a single symbol. We have

$$S = acca^{-1}bdbe^{-1}d^{-1}e^{-1}$$
$$= \underbrace{acca^{-1}}_{S_1} \# \underbrace{bdbe^{-1}d^{-1}e^{-1}}_{S_2}$$

We have

$$S_1 = a\underbrace{cca^{-1}} \sim a\underbrace{a^{-1}cc} \sim cc \sim \mathbb{R}P^2$$

$$S_2 = \underbrace{bdb}e^{-1}d^{-1}e^{-1} \sim bbd^{-1}e^{-1}d^{-1}e^{-1} \sim \mathbb{R}P^2 \# d^{-1}e^{-1}d^{-1}e^{-1}$$
$$\sim \mathbb{R}P^2 \# dede \text{ (relabeling)}$$
$$\sim \mathbb{R}P^2 \# dd^{-1}ee$$
$$\sim \mathbb{R}P^2 \# ee \sim \mathbb{R}P^2 \# \mathbb{R}P^2$$

Thus $S \sim S_1 \# S_2 \sim \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$, which is the connected sum of three projective planes. We can write $S \sim (\mathbb{R}P^2)^{\#3}$.

Recall that the Euler's characteristic of $S \# T$ is

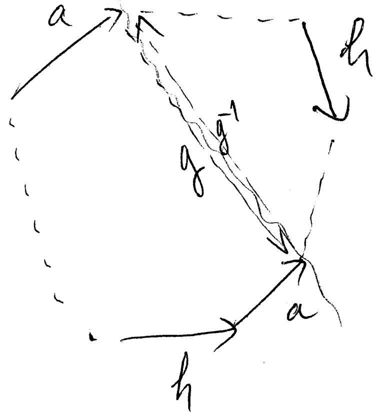
$$\chi(S \# T) = \chi(S) + \chi(T) - 2 \quad \forall \text{ closed surfaces } S \text{ and } T$$

Thus, with $\chi(\mathbb{R}P^2) = 1$, we have $\chi(\mathbb{R}P^2)^{\#n} = 2 - n$. Thus in our case $\chi(S) = 2 - 3 = -1$. Moreover, $S$ is not orientable since it contains $\mathbb{R}P^2$, which in turn contains a Möbius strip.

(ii) $S = abcb^{-1}de\underline{fg}h\underline{g^{-1}f^{-1}}a^{-1}h^{-1}d^{-1}c^{-1}e^{-1}$

We see that $f$ and $g$ always go together and $g^{-1}f^{-1} = (fg)^{-1}$. We can consolidate them as a single symbol $f$. Thus,

$$S \sim abcb^{-1}defhf^{-1}a^{-1}h^{-1}d^{-1}c^{-1}e^{-1}$$



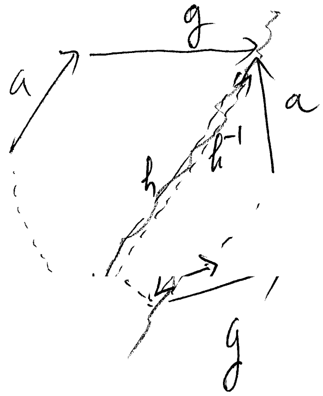

By cutting along (new) edge $g$ like in the figure, we have

$$S \sim aga^{-1}h^{-1}d^{-1}c^{-1}e^{-1} \# g^{-1}bcb^{-1}defhf^{-1}$$

Then we rotate two parts in order to glue $h, b^{-1}$ together

$$S \sim d^{-1}c^{-1}e^{-1}aga^{-1}h^{-1} \# hf^{-1}g^{-1}bcb^{-1}def$$

$\overset{\text{glue } h}{\sim}\ d^{-1}c^{-1}e^{-1}aga^{-1}f^{-1}g^{-1}bcb^{-1}def$




Cutting along (new) edge h like in the figure, we have

$$S \sim aghbcb^{-1}defd^{-1}c^{-1}e^{-1} \# h^{-1}a^{-1}f^{-1}g^{-1}$$

Rotate two parts in order to glue a together, we get

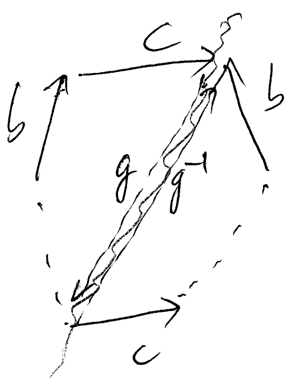
$$S \sim ghbcb^{-1}defd^{-1}c^{-1}e^{-1}a \# a^{-1}f^{-1}g^{-1}h^{-1}$$

$$\overset{\text{glue } a}{\sim} ghbcb^{-1}defd^{-1}c^{-1}e^{-1}f^{-1}g^{-1}h^{-1}$$

$$\overset{\text{rotate word}}{\sim} \underbrace{g^{-1}h^{-1}gh}\,\underbrace{bcb^{-1}defd^{-1}c^{-1}e^{-1}f^{-1}}$$

$$\sim \mathbb{T}_2 \# S_1$$

where $\mathbb{T}_2 = S^1 \times S^1$ is the torus.




Cutting $S_1$ along new edge g like in the figure, we have

$$S_1 \sim bcge^{-1}f^{-1} \# g^{-1}b^{-1}defd^{-1}c^{-1}$$

Rotate two parts in order to glue b together, we get

$$\cancel{S_1 \sim ge^{-1}f^{-1}bc \# c^{-1}g^{-1}b^{-1}defd^{-1}}$$

$$\overset{\text{glue } c}{\sim} \cancel{ge^{-1}f^{-1}bg^{-1}b}$$

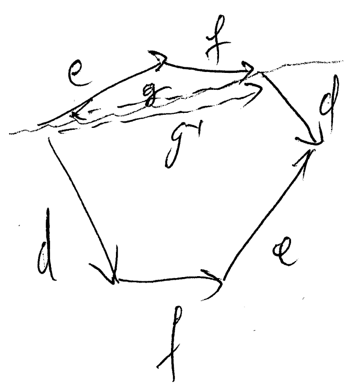
$$S_1 \sim cge^{-1}f^{-1}b \# b^{-1}defd^{-1}c^{-1}g^{-1}$$

4

$\overset{\text{glue } b}{\sim} c g e^{-1} f^{-1} d e f d^{-1} c^{-1} g^{-1}$

$\overset{\text{rotate}}{\sim} \underbrace{c^{-1} g^{-1} c g}_{T_2} \underbrace{e^{-1} f^{-1} d e f d^{-1}}_{S_2}$

Thus $S \sim T_2 \# T_2 \# S_2$ where $S_2 = e^{-1} f^{-1} d e f d^{-1}$.




Replacing $e^{-1}$ by $e$ and $f^{-1}$ by $f$, we have

$S_2 \sim e f d e^{-1} f^{-1} d^{-1}$

Cutting $S_2$ along new edge $g$ as in the figure, we have

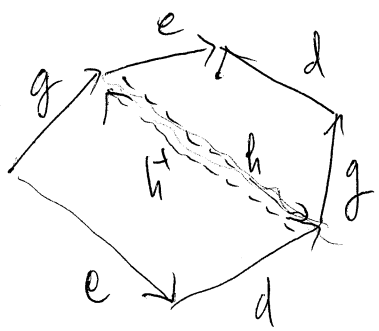
$S_2 \sim e f g \# g^{-1} d e^{-1} f^{-1} d^{-1}$

Rotate two parts to glue $f$ together, we have

$S_2 \sim g e f \# f^{-1} d^{-1} g^{-1} d e^{-1}$

$\overset{\text{glue } f}{\sim} g e d^{-1} g^{-1} d e^{-1}$

Cutting it into two parts along $h$ as in the figure, we obtain




$S_2 \sim g h d e^{-1} \# h^{-1} e d^{-1} g^{-1}$

$\overset{\text{rotate}}{\sim} g h d e^{-1} \# e d^{-1} g^{-1} h^{-1}$

$\overset{\text{glue } e}{\sim} g h d d^{-1} g^{-1} h^{-1}$.

collapsed $ghg^{-1}h^{-1} \sim T_2$

Thus $S \sim T_2 \# T_2 \# T_2 \sim (T_2)^{\#3}$, which is the connected sum of three tori. Recall the Euler's characteristic of $(T_2)^{\#n}$ is $2-2n$. Then

$$\chi(S) = 2 - 3.2 = -4$$

You identified both surfaces correctly, but you're supposed to use orientability and Euler char. to determine the surface, not the other way around!

Moreover, $S$ is orientable.

3/4

② An $n$-dimensional manifold with boundary is a topological space $M$ such that every point in $M$ has an open neighborhood $U$ which is homeomorphic to either $\mathbb{R}^n$ or $[0,\infty) \times \mathbb{R}^{n-1}$.

Assume without proof that no point of $\mathbb{R}^n$ has an open neighborhood homeomorphic to $[0,\infty) \times \mathbb{R}^{n-1}$. Define the boundary $\partial M$ of an $n$-dimensional manifold with boundary. Show that it is an $(n-1)$-manifold. And show that if $M$ is compact then $\partial M$ is closed.

Proof We will treat this problem part by part.

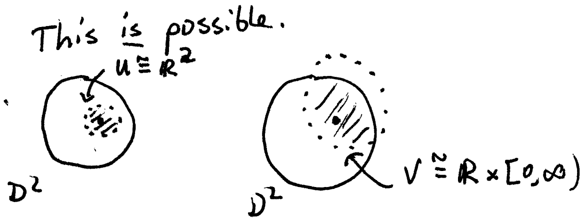
1) The definition of $n$-manifold with boundary makes it clear what should be ~~the~~ called the boundary $\partial M$ of an $n$-manifold with boundary $M$. We will define: the boundary $\partial M$ of an $n$-manifold with boundary $M$ is the

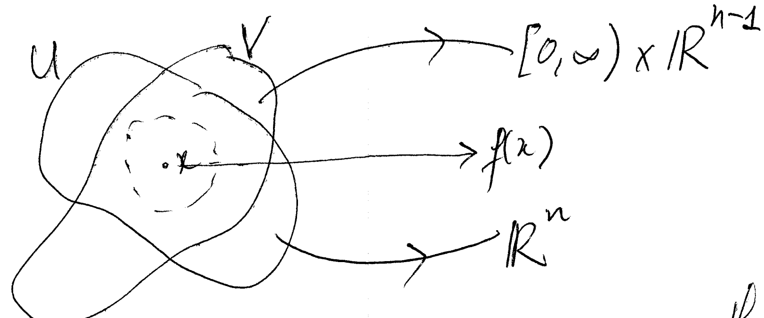
subset of all points in M which <u>has an</u> <u>open neighborhood</u> homeomorphic to $[0,\infty)\times\mathbb{R}^{n-1}$.

It's not enough to just have <u>one</u> such nbd. You need <u>all</u> nbs of a point in $\partial M$ to be homeomorphic to $[0,\infty)\times\mathbb{R}^{n-1}$

Sure, now comes the question: is it a well-made definition? What if a point in M has an open neighborhood homeomorphic to $[0,\infty)\times\mathbb{R}^{n-1}$ an another open neighborhood homeomorphic to $\mathbb{R}^n$? Strictly speaking, that does not affect the validity of the definition, ~~However, the defini~~ because it requires only <u>one</u> neighborhood homeomorphic to $[0,\infty)\times\mathbb{R}^{n-1}$. That definition will be, however, meaningless if we do not make the previous question clear. ~~Moreover,~~ It would be of no use and vague in sense that: in order to check if a point $x\in M$ belongs to the boundary or not, we usually examine <u>one</u> special neighborhood $U$ of $x$; ~~and~~ if $U\simeq\mathbb{R}^n$, we couldn't tell that $x\notin\partial M$; in other words, we have to check if <u>all</u> neighborhoods of $x$ in M are homeomorphic to $\mathbb{R}^n$. That's probably hopeless!

Suppose that a point $x\in M$ has an open neighborhood $U\simeq\mathbb{R}^n$ .. and another open neighborhood $V\simeq$ ~~$\mathbb{R}$~~ $[0,\infty)\times\mathbb{R}^{n-1}$! Now we'll try our best to show that this is impossible.






Let $f: U \to \mathbb{R}^n$, $g: V \to [0,\infty)\times\mathbb{R}^{n-1}$ be homeomorphisms. Since $f(x)\in\mathbb{R}^n$, there exists an open ball $B(f(x),\delta)$ in $\mathbb{R}^n$ such that $f^{-1}(B(f(x),\delta)) \subset (U\cap V)$.

Now $f^{-1}(B(f(x),\delta))$ is a subset of $V$ which is ~~too~~ homeomorphic to $\mathbb{R}^n$.

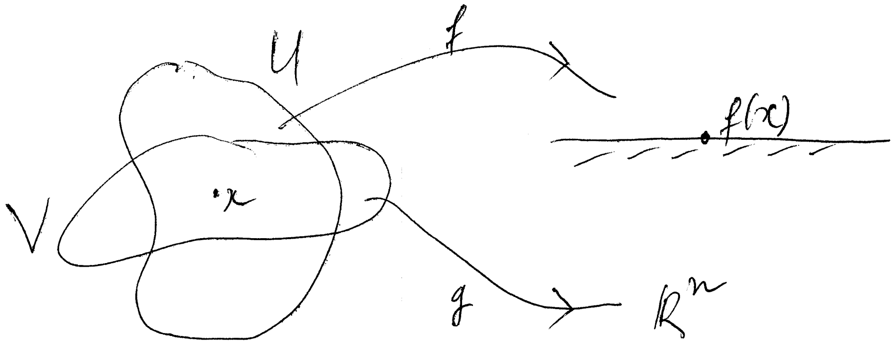
We really need that $f(x)\in\partial([0,\infty)\times\mathbb{R}^{n-1})$ to find a contradiction. Otherwise, if $f(x)$ lies in the interior of $[0,\infty)\times\mathbb{R}^{n-1}$, there will be no contradiction. For example any $x$ in the interior of $[0,\infty)\times\mathbb{R}^{n-1}$ has a neighborhood ~~to~~ homeomorphic to $[0,\infty)\times\mathbb{R}^{n-1}$, ~~whi~~ that is $[0,\infty)\times\mathbb{R}^{n-1}$ itself!


Thus the definition was not well-made. The hole is that we did not have the corresponding from $x$ to some point on the boundary of $[0,\infty)\times\mathbb{R}^n$.

thus another definition must remedy this pathology:

> "The boundary $\partial M$ of $M$ is the set of all points $x\in M$ such that there exists a neighborhood $U$ of $x$ in $M$ and a homeomorphism $f: U \to [0,\infty)\times\mathbb{R}^{n-1}$ that maps $x$ ~~onto~~ onto the boundary of $[0,\infty)\times\mathbb{R}^{n-1}$.
>
> This definition is good.

Now we'll verify that this is indeed a good definition. In other words, we shall show that if $x \in \partial M$ then there is no open neighborhood $V$ of $x$ in $M$ such that $V \simeq \mathbb{R}^n$. To speak another way, a boundary point of $M$ cannot be an interior point in $M$. Suppose that there ~~is~~ exists such a $V$ and a homeomorphism $g: V \to \mathbb{R}^n$.

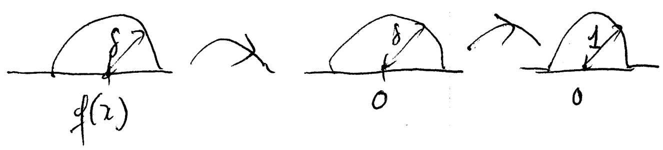



~~Since $g$ is continuous, there exist an open ball $B(g(x), \delta)$ in $\mathbb{R}^n$ such that $g^{-1}(B(g(x),\delta)) \subset (U \cap V)$. Put $W = g^{-1}(B(g(x),\delta))$. Then $W$ contains $x$ and open in $V$. Thus $W$ is also open in $U$. Thus $f(W)$ is open in $[0,\infty) \times \mathbb{R}^n$, containing $f(x)$; and $f(W) \simeq W \simeq B(g(x),\delta) \simeq \mathbb{R}^n$. Thus $f(x)$ is a point in $[0,\infty) \times \mathbb{R}^n$ that has~~ Since $f$ is continuous, there exists an open ball $W$ in $[0,\infty) \times \mathbb{R}^{n-1}$ such that $f^{-1}(W) \subset U \cap V$. We write

$$W = B(f(x), \delta) \cap \left([0,\infty) \times \mathbb{R}^{n-1}\right)$$

, where $B(f(x),\delta)$ is an open ball in $\mathbb{R}^n$.

We have
$$W \xrightarrow[\cong]{u \mapsto u - f(x)} B(0,\delta) \cap \left([0,\infty) \times \mathbb{R}^{n-1}\right)$$




$$\wr\wr \; u \mapsto \frac{u}{\delta}$$
$$B(0,1) \cap \left([0,\infty) \times \mathbb{R}^{n-1}\right)$$
$$\wr\wr \; u \mapsto \frac{u}{1-|u|}$$
$$[0,\infty) \times \mathbb{R}^{n-1}$$

Thus $W \simeq [0,\infty) \times \mathbb{R}^{n-1}$. Since $f^{-1}(W)$ is open in $V$ and $g$ is ~~a~~ homeomorphism, $g(f^{-1}(W))$ is open in $\mathbb{R}^n$. ~~Thus $g($~~ However,
$$g(f^{-1}(W)) \simeq f^{-1}(W) \simeq W \simeq [0,\infty) \times \mathbb{R}^{n-1}$$
This contradicts what we have assumed in the hypothesis. In short, our (second) definition might be a good one.

2) Now with the definition in part 1, we'll show that $\partial M$ is actually an $(n-1)$-manifold. This is absolutely mysterious. In our definition of $\partial M$, we thought * being-on-boundary is a "pointwise" property. We can say whether a point in $M$ is on boundary by seeking for a suitable open neighborhood $U$ and a homeomorphism $f: U \to [0,\infty) \times \mathbb{R}^{n-1}$ that maps $x$ to ~~the~~ some

point on the boundary of $[0,\infty)\times\mathbb{R}^{n-1}$. We didn't think about any relation between one point on the boundary ~~to~~ and other points on the boundary. We, in fact, has at least one clue ~~some clues~~ of the connection. Since the map $f: U \to [0,\infty)\times\mathbb{R}^{n-1}$ concerns not only $x \in U$ and $f(x) \in \partial([0,\infty)\times\mathbb{R}^{n-1})$, it concerns other point $y \in U$ such that $f(y) \in \partial([0,\infty)\times\mathbb{R}^{n-1})$. And ~~thus~~ these $y$'s are by definition also boundary points of $M$. Thus,

$$U_0 = f^{-1}\left(\partial([0,\infty)\times\mathbb{R}^{n-1})\right) \subset \partial M$$

If we can prove that $U_0 = U \cap (\partial M)$ then $U_0$ will be open in $\partial M$ and contains $x$ and $U_0 \simeq \partial([0,\infty)\times\mathbb{R}^{n-1})$; and thus our proof would finish if we can show

$$\partial([0,\infty)\times\mathbb{R}^{n-1}) \simeq \mathbb{R}^{n-1}$$

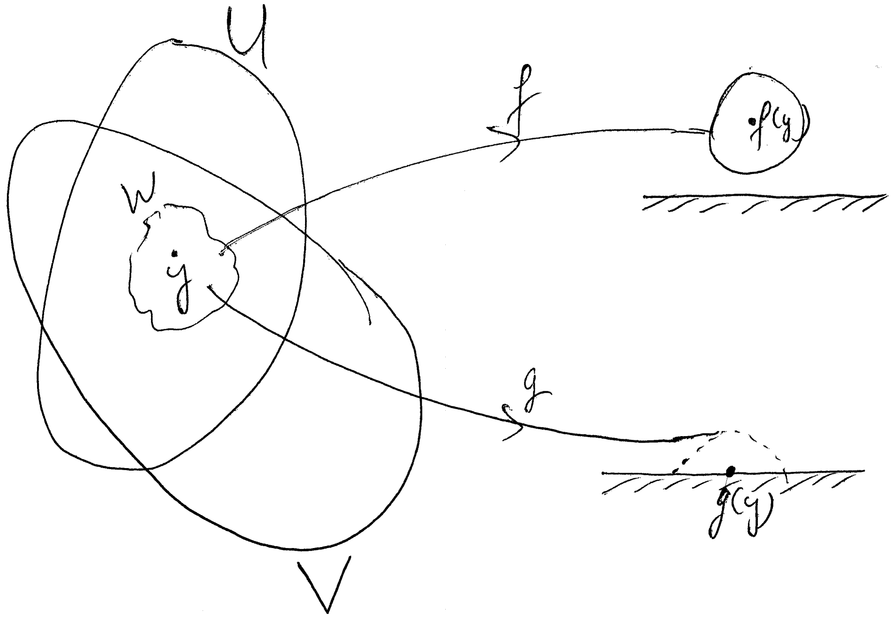
Therefore, there are two things we need to clarify:

1) $U_0 \overset{?}{=} U \cap (\partial M)$

2) $\partial([0,\infty)\times\mathbb{R}^{n-1}) \overset{?}{\simeq} \mathbb{R}^{n-1}$

Let's prove 1)

By the definition of $U_0$, we already had $U_0 \subset (U \cap \partial M)$. We need to show that $(U \cap \partial M) \subset U_0$. Take $y \in U \cap \partial M$, we'll show that $y \in U_0$, or $f(y) \in \partial([0,\infty) \times \mathbb{R}^{n-1})$. Suppose by contradiction that $f(y)$ lies in the interior of $[0,\infty) \times \mathbb{R}^{n-1}$.




Since $y \in \partial M$, there exists an open neighborhood $V$ of $y$ in $M$ and a homeomorphism $g : V \longrightarrow [0,\infty) \times \mathbb{R}^{n-1}$ such that $g(y) \in \partial([0,\infty) \times \mathbb{R}^{n-1})$.

Since $f(y) \in \mathrm{Int}([0,\infty) \times \mathbb{R}^{n-1})$, there exist $\delta > 0$ such that

$$W = f^{-1}(B(f(y), \delta)) \subset (U \cap V)$$

Since $y \in W$ open and $g$ is a homeomorphism, there exists $\delta' > 0$ such that

$\cancel{g^{-1}(B(g(y), \delta')) \subset W.}$ ~~W~~ $g^{-1}(B(g(y), \delta') \cap ([0,\infty) \times \mathbb{R}^{n-1})) \subset W$

12

As the first part has proved, we have

$$
\begin{aligned}
[0,\infty)\times\mathbb{R}^{n-1} &\simeq B(g(y),\delta')\cap([0,\infty)\times\mathbb{R}^{n-1})\\
&\simeq g^{-1}(B(g(y),\delta')\cap([0,\infty)\times\mathbb{R}^{n-1}))\\
&\subset W = f^{-1}(B(f(x),\delta))\\
&\simeq B(f(x),\delta)\\
&\simeq \mathbb{R}^{n},
\end{aligned}
$$

which says that $\mathbb{R}^{n}$ has a open subset homeomorphic to $[0,\infty)\times\mathbb{R}^{n-1}$.

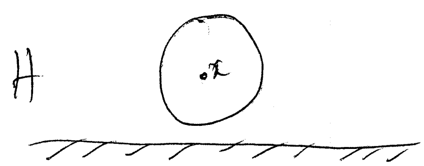
This is a contradiction.

Let's prove 2) Let's make it clear that while proving 1), we've already used the result of 2). The notation we used $\partial([0,\infty)\times\mathbb{R}^{n-1})$ refered to both the set $\{(x_1,\dots,x_n)\in\mathbb{R}^n : x_1=0\}$ (while showing $B(0,1)\cap\partial([0,\infty)\times\mathbb{R}^{n-1}) \simeq \partial([0,\infty)\times\mathbb{R}^{n-1})$) and the newly defined boundary of an n-manifold. Thus, it's very important that we make clear that the two ways of representation are equivalent. In notation, what we need ~~to~~ prove is

$$\partial([0,\infty)\times\mathbb{R}^{n-1}) = \{(x_1,\dots,x_n)\in\mathbb{R}^n : x_1=0\}$$

Denote by ~~H~~ $H$ the set on the right hand side. Then $H \subset \partial([0,\infty)\times\mathbb{R}^{n-1})$ because the identity map ~~on~~ on $[0,\infty)\times\mathbb{R}^{n-1}$ is a homeomorphism from $[0,\infty)\times\mathbb{R}^{n-1}$ to $[0,\infty)\times\mathbb{R}^{n-1}$ which maps each point in $H$ to itself (therefore in $H$). Now we'll prove that $\partial([0,\infty)\times\mathbb{R}^{n-1}) \subset H$. Let $x \in \partial([0,\infty)\times\mathbb{R}^{n-1})$ and suppose that $x \notin H$. Then there exists an open ball in $\mathbb{R}^n$ called ~~$B(x,\delta)\subset H$~~.

$$B(x,\delta) \subset [0,\infty)\times\mathbb{R}^{n-1}.$$




Since $x \in \partial([0,\infty)\times\mathbb{R}^{n-1})$, there exists a homeomorphism $f: U \to [0,\infty)\times\mathbb{R}^{n-1}$ where $U \ni x$ is open in $[0,\infty)\times\mathbb{R}^{n-1}$, such that $f(x) \in H$. Put

$$V = B(x,\delta) \cap U$$

then $V \ni x$ is open in $\mathbb{R}^n$. Since $f$ is continuous, there exists $\delta' > 0$ such that

$$\underbrace{f^{-1}\big(B(f(x),\delta') \cap ([0,\infty)\times\mathbb{R}^{n-1})\big)}_{\cong [0,\infty)\times\mathbb{R}^{n-1}} \subset V \subset \mathbb{R}^n$$

thus $\mathbb{R}^n$ has an open subset homeomorphic to $[0,\infty)\times\mathbb{R}^{n-1}$. This is a contradiction!

Then the part $\partial([0,\infty)\times\mathbb{R}^{n-1}) \overset{?}{\cong} \mathbb{R}^{n-1}$ becomes very simple:

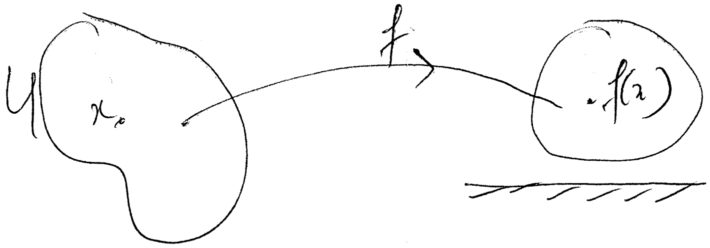
$$\phi : \mathbb{R}^{n-1} \to H$$

$$(x_2,\dots,x_n) \mapsto (0,x_2,\dots,x_n)$$

is a homeomorphism.

3) Let us be straight here that $\partial M$ is <u>always</u> closed in $M$, regardless of whether $M$ is compact or not. Indeed, if $\partial M=\phi$ or $\partial M = M$, then it is closed in $M$. Otherwise, we'll show that $M\setminus\partial M$ is open in $M$. Let $x\in M\setminus\partial M$. We'll show that there exists an open neighborhood $U\ni x$ in $M\setminus\partial M$. ~~Since x any there are two possibilities: first there exists $V\ni x$ open in $M$ and $V\cong\mathbb{R}^n$; second, there exist~~ As proved in part 1), a point in $M$ cannot has two open neighborhood such that one is homeomorphic to $\mathbb{R}^n$, the other is homeomorphic to $[0,\infty)\times\mathbb{R}^{n-1}$ in the way that it is mapped onto $\partial([0,\infty)\times\mathbb{R}^{n-1})$. Thus there are two possibilities of $x$: first there exists $V\ni x$ open in $M$ and $V\simeq\mathbb{R}^n$; second there exists $U\ni x$ open in $M$ and $U\simeq[0,\infty)\times\mathbb{R}^{n-1}$ and the homeomorphism $f: U\to[0,\infty)\times\mathbb{R}^{n-1}$ doesn't map $x$ to $\partial([0,\infty)\times\mathbb{R}^{n-1})$.

The second possibility is, in fact, coincide the first because we can




take an open ball $B(f(x),\delta) \subset [0,\infty)\times\mathbb{R}^{n-1}$, and set $V = f^{-1}(B(f(x),\delta))$. And then we'll return to the first possibility. Now every point in $V$ has an open neighborhood, which is $V$, homeomorphic to $\mathbb{R}^n$. Thus all of them are not on the boundary. Thus $V \subset M \setminus \partial M$.

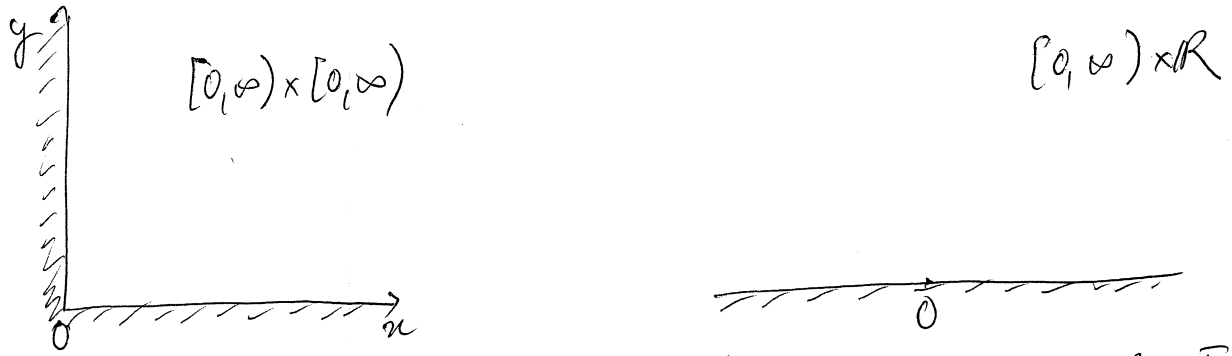
Now that we know that $\partial M$ is always closed in $M$. In case $M$ is compact, $\partial M$ will be compact. And since $\partial M$ is a compact $(n-1)$-dimensional manifold, it is closed (the terminology is itself the definition). 3/4

why is $\partial(\partial M) = \emptyset$?

(3) An $n$-dimensional manifold with corners in a topological space $M$ such that every point in $M$ has an open neighborhood $U$ which is homeomorphic to $[0,\infty)^p \times \mathbb{R}^{n-p}$ for some $0 \le p \le n$. Show that any $n$-dimensional manifold ~~Proof~~ with corners in an $n$-dimensional manifold with boundary.

Proof At the first look, the statement seems to be handy in sense that: to check if $M$ is an $n$-manifold with boundary, we don't need to know beforehand what boundary-points of $M$ are. Instead, what we need to know is that whether there exists a special open neighborhood (at each point) that is homeomorphic to some $[0,\infty)^p \times \mathbb{R}^{n-p}$. That might be helpful in case the boundary points of $M$ are hard to trace out. However, ~~do~~ we don't put so strong interest in this result since being able to find a homeomorphism from $U$ to $[0,\infty)^p \times \mathbb{R}^{n-p}$ somehow means we already know where the boundary points are.

The term $[0,\infty)^p \times \mathbb{R}^{n-p}$ may intimidate us because it has nothing to do with our definition of manifold with boundary. Thus, we* should try to relate it to $[0,\infty)^{1} \times \mathbb{R}^{n-1}$ (corresponding to the case $p=1$). The simplest case may be $[0,\infty)\times[0,\infty)$ and $[0,\infty)\times\mathbb{R}$.




These two spaces are indeed homeomorphic by rotation $z \mapsto z^2$. In other words, the map

$$f : [0,\infty)\times[0,\infty) \to [0,\infty)\times\mathbb{R}$$
$$(x,y) \mapsto \cancel{(x^2-y^2,\,2xy)}\ (2xy,\, x^2-y^2)$$

is a homeomorphism between them. And now we are holding the key to solve this problem. Suppose that we can conveniently use the association of product topological spaces $(A\times B)\times C \simeq A\times(B\times C)$ then

$$[0,\infty)\times \mathbb{R}^{n-1} \simeq \big([0,\infty)\times\mathbb{R}\big)\times\mathbb{R}^{n-2}$$
$$\simeq [0,\infty)^2\times\mathbb{R}^{n-2}$$
$$\simeq [0,\infty)\times\big([0,\infty)\times\mathbb{R}\big)\times\mathbb{R}^{n-3}$$
$$\simeq [0,\infty)\times[0,\infty)^2\times\mathbb{R}^{n-3} \simeq [0,\infty)^3\times\mathbb{R}^{n-3}$$
$$\simeq \dots$$
$$\simeq [0,\infty)^n$$

Thus for all $0<p\leq m$ we have $[0,\infty)\times\mathbb{R}^{n-1}\simeq[0,\infty)^p\times\mathbb{R}^{n-p}$. This is a very good news! because we have related the intimidating term $[0,\infty)^p\times\mathbb{R}^{n-p}$ to the term we need $[0,\infty)\times\mathbb{R}^{n-1}$. Now look back (return) to the problem. Each point in M has an open neighborhood either homeomorphic to $\mathbb{R}^n$ (for $p=0$) or $[0,\infty)\times\mathbb{R}^{n-1}$ (for $0<p\leq n$). And this is exactly the definition of n-manifold with boundary! It is worth adding one remark!

A point which has an open neighborhood $U$ homeomorphic to $[0,\infty)^p\times\mathbb{R}^{n-p}, p\geq 1$, and is mapped to $\partial([0,\infty)^p\times\mathbb{R}^{n-p}) = \{(x_1,\dots,x_p,x_{p+1},\dots,x_n)\in\mathbb{R}^n;$

$x_1,\dots,x_p\geq 0$ and $\exists\, 1\leq i\leq p: x_i=0\}$,

then $U$ is homeomorphic to $[0,\infty)\times\mathbb{R}^{n-1}$ in the way such that the point

$$\begin{tikzcd} U \arrow[r] \arrow[dr] & {[0,\infty)^p\times\mathbb{R}^{n-p}} \arrow[d] \\ & {[0,\infty)\times\mathbb{R}^{n-1}} \end{tikzcd}$$

is mapped ~~on~~ to $[0,\infty)\times\mathbb{R}^{n-1}$. (Recall that in Homework 1 we showed that a homeomorphism maps boundary points to boundary points, interior points to interior points (by using the Invariance of the Boundary theorem, which is ~~ex~~ equivalent to the assumption in Problem 2).

Now what is left is to show that $(A\times B)\times C\simeq A\times(B\times C)$. Let

$$f: (A\times B)\times C\longrightarrow A\times(B\times C)$$
$$((x,y),z)\longmapsto (x,(y,z))$$

$f$ is obviously a bijection. We'll show that $f$ is bi-continuous. Since the family of $\{O\times U : O \text{ open in } A, \text{ and } U \text{ open in } B\times C\}$ ~~f~~ is a basis for the product space $A\times(B\times C)$, it suffices to show that $f^{-1}(O_1\times U_1)$ is open in $(A\times B)\times C$, where $O_1$ is open in $A$, and $U_1$ is open in $(B\times C)$. Since the set family of $\{O\times U : O$ is open in $B$, and $U$ is open in $C\}$ is a basis for $B\times C$, it suffices to choose $U_1 = O_2\times O_3$ where $O_2$ is open in $B$ and $O_3$ is open in $C$. By the definition of $f$,

$$f^{-1}(O_1\times(O_2\times O_3)) = (O_1\times O_2)\times O_3,$$

which is open in $(A\times B)\times C$. The reverse is also true,

$$f((O_1\times O_2)\times O_3) = O_1\times(O_2\times O_3)$$

Thus $f$ is bi-continuous.

(4) Suppose that a topological space $X$ has a function $m: X\times X\to X$. Show that if $\alpha$ and $\beta$ are any paths in $X$, the definition

$$(\alpha*\beta)(t) = m(\alpha(t),\beta(t))$$

is homotopy invariant, in sense that $[\alpha]*[\beta]=[\alpha*\beta]$ is well-defined on homotopy classes of paths.

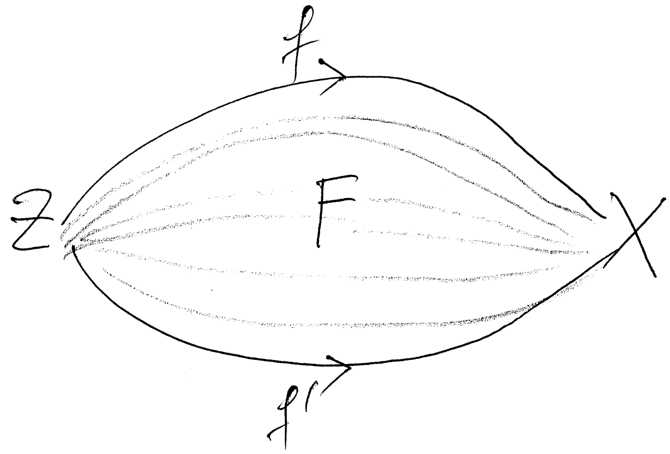
<u>Proof</u> To extract knowledge from this specific problem, let's put

put it in a more general set up. We know that path-homotopy is just a special case of ~~homo~~ relative homotopy. Let $Z$ and $X$ be topological spaces, and $A \subset Z$. Then two functions $f, g : Z \to X$ are said to be homotopic relatively $A$ if there exists a homotopy $H$ between them that is stationary on $A$. That is

$$H : Z \times [0,1] \to X$$

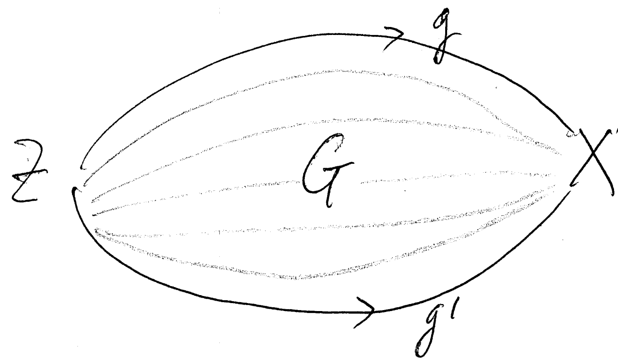
satisfying $H(\cdot, 0) = f$, $H(\cdot, 1) = g$ and $H(z, t) = f(z)$
$\forall z \in A, \forall t \in [0,1]$.

Path-homotopy is a special case of relative homotopy when $Z = [0,1]$ and $A = \{0,1\}$. Note that $Z$ is understood as a "parametrization" of the set $f(Z)$. We see that $f(Z)$ and $g(Z)$ are parametrized by $Z$ in different ways (via $f$ and $g$). And the homotopy $H$ is a process of reparametrizing $f(Z)$ so that it coincides $g(Z)$. In the following, all ~~homotopy~~ homotopies are understood as homotopies stationary on a set $A \subset Z$.



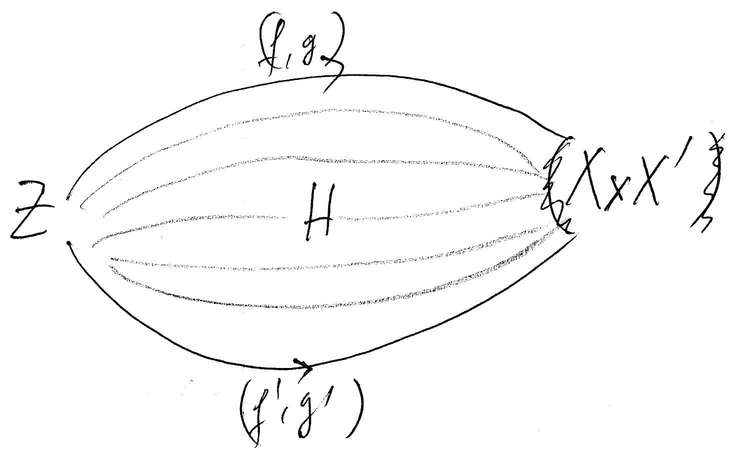

$F: Z \times [0,1] \to X$

$F(\cdot,0) = f \,;\ F(\cdot,1) = f'$



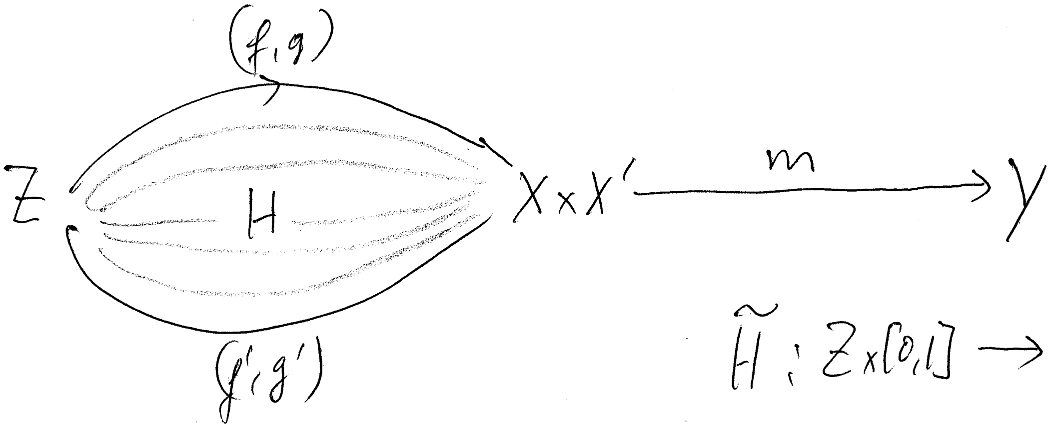

$G: Z \times [0,1] \to X'$

$G(\cdot,0) = g \,;\ G(\cdot,1) = g'$




$H: Z \times [0,1] \to X \times X'$

$H = (F, G)$




$\tilde{H}: Z \times [0,1] \to Y$

$\tilde{H} = m \circ H$

22

Thus $m \circ (f, g)$ and $m \circ (f', g')$ are homotopic relative to $A$ as long as $m$ is continuous $\Rightarrow$ $\tilde{H}$ is continuous since $H$ is. Now we consider $Z = [0,1]$, which is used to parametrize paths, $X' = X$, $f$ is path $\alpha$, $f'$ is path $\alpha'$, $g$ is path $\beta$, $g'$ is path $\beta'$. Then if $\sim$ denotes path-homotopic relation we have

$$\begin{cases} \alpha \sim \alpha' \\ \beta \sim \beta' \end{cases} \Longrightarrow m \circ (\alpha, \beta) \sim m \circ (\alpha', \beta')$$

Thus our definition is homotopy invariant. In other words, we can write

$$[\alpha] * [\beta] = m([\alpha], [\beta])$$

In case $X = \mathbb{R}^n$, the easiest nontrivial $m$ is probably the average function

$$m : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$(x, y) \longmapsto \frac{x + y}{2}$$

~~why is $\tilde{H}$ continuous?~~

4/4

(5) Show that the product of the previous problem satisfies an interchange law

$$(\alpha \cdot \beta) * (\gamma \cdot \delta) = (\alpha * \gamma) \cdot (\beta * \delta) \qquad (1)$$

whenever the left-hand side is defined.

Proof Let $\alpha$ be a path from $p_\alpha$ to $q_\alpha$

$\beta$ ———————— $p_\beta$ — $q_\beta$

$\gamma$ ———————— $p_\gamma$ — $q_\gamma$

$\delta$ ———————— $p_\delta$ — $q_\delta$

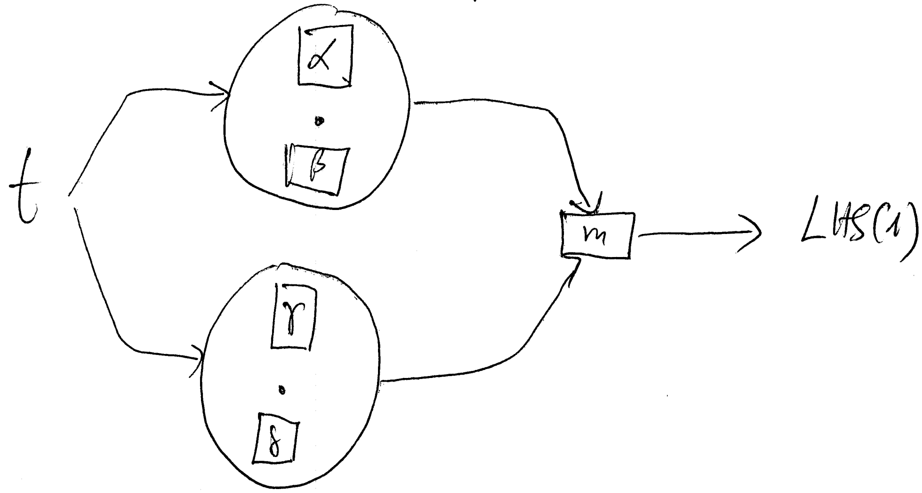
To make the LHS of (1) well-defined, we need $q_\alpha = p_\beta$ and $q_\gamma = p_\delta$.

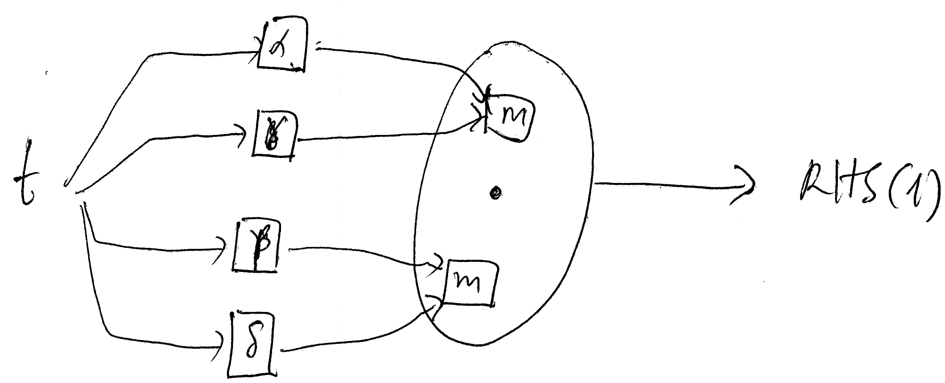
Then $(\alpha * \gamma)(t) = m(\alpha(t), \gamma(t))$ and $(\beta * \delta)(t) = m(\beta(t), \delta(t))$.

We have $(\alpha * \gamma)(1) = m(\alpha(1), \gamma(1)) = m(q_\alpha, q_\gamma) = m(p_\beta, p_\delta)$

$$= (\beta * \delta)(0)$$

Thus $\alpha * \gamma$ and $\beta * \delta$ are composable.




We have

$$LHS(1) = [(\alpha \cdot \beta) * (\gamma \cdot \delta)](t) = m((\alpha \cdot \beta)(t), (\gamma \cdot \delta)(t))$$

$$= \begin{cases} m(\alpha(2t), \gamma(2t)) & 0 \le t \le 1/2 \\ m(\beta(2t-1), \delta(2t-1)) & 1/2 \le t \le 1 \end{cases}$$

24

$$= \begin{cases} m(\alpha,\gamma)(2t) & 0 \le t \le 1/2 \\ m(\beta,\delta)(2t-1) & 1/2 \le t \le 1 \end{cases}$$

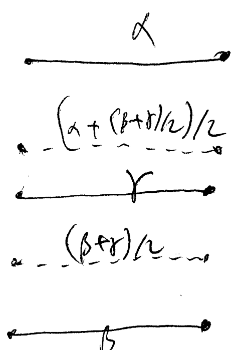
$$= \left(m(\alpha,\gamma) \cdot m(\beta,\delta)\right)(t)$$

$$= RHS(1). \qquad 4/4$$

It is worth to comment about the operation $*$. Problem 4) says that it can be defined as an ~~ho~~ operation on path-homotopy classes. Problem 5) gives us a correlation between $*$ and the path composition which together with all path-homotopy classes and all points forms a category. ~~We~~ Now we have three ingredients: all points $a \in X$, all path-homotopy class $Hom(a,b)$, and a new type of path composition $[\alpha]*[\beta]$ where $[\alpha] \in Hom(b,c)$, and $[\beta] \in Hom(a,b)$ (in fact $*$ can act on any pair $[\alpha]$, $[\beta]$ with arbitrary end points). We have the right to ask: are we having a new category? One criterion of a category is the ~~fun~~ associativity of morphisms composition. Thus we should check $([\alpha]*[\beta])*[\gamma] \overset{?}{=} [\alpha]*([\beta]*[\gamma])$. Since $*$ is not a normal function composition (just as path composition), the identity is not at all obvious and absolutely no ~~doubt~~ doubt. The ~~fo~~ above identity is equivalent to $m(\alpha,\beta) * \gamma \overset{?}{=} \alpha * m(\beta,\gamma)$ (up to path-homotopy)

or $m(m(\alpha,\beta),\gamma) \overset{?}{=} m(\alpha, m(\beta,\gamma))$. We can take $m(\alpha,\beta) = \frac{\alpha+\beta}{2}$

in $\mathbb{R}^2$ for instance.




$\alpha, \beta, \gamma$ are three segment equally spaced, and

$$\gamma = \frac{\alpha+\beta}{2} = m(\alpha,\beta)$$

Thus $m(m(\alpha,\beta),\gamma) = \gamma$. However

$$m(\alpha, m(\beta,\gamma)) = \frac{\alpha+(\beta+\gamma)/2}{2} \neq \gamma$$

Thus $*$ couldn't be the function (morphism) composition of any category because $*$ ~~not~~ is not associative.