Name: Tuan Pham | Math 8301: Topology and Manifolds | 1

ID: 4652218 | Homework 4 | 20/20

(1) Suppose you are given a simplicial complex with set $\mathcal{V}$ of vertices and $\mathcal{F}$ of faces. Let X be the space you get by realizing this simplicial complex. For definiteness, we'll let $\mathbb{V}$ be the vector space with basis $\mathcal{V}$, and define

$$X = \bigcup_{U \in \mathcal{F}} \left\{ \sum_{v \in U} t_v \cdot v \;\middle|\; t_v \geq 0, \sum t_v = 1 \right\} \subset \mathbb{V}.$$

(a) Given a edge $\{a,b\} \in \mathcal{F}$, define a path $p_{a,b}$ from a to b in X.

(b) Given a triangle $\{a,b,c\} \in \mathcal{F}$, show that there is a homotopy of paths from $p_{a,c}$ to $p_{a,b} \cdot p_{b,c}$.

Proof Let us make it clear that $\mathcal{V}$ is a set of points in some $\mathbb{R}^n$, so that the definition of X is meaningful. Since $\mathbb{V}$ is the vector space with basis $\mathcal{V}$, it is actually a Euclidean space $\mathbb{V} = \mathbb{R}^n$. Each face $U \in \mathcal{F}$ is now realized as a simplex in $\mathbb{R}^n$:

$$\tilde{U} = \left\{ \sum_{v \in U} t_v \cdot v; \quad \text{for } t_v \geq 0, \sum t_v = 1 \right\}$$

$\uparrow$ realization of U

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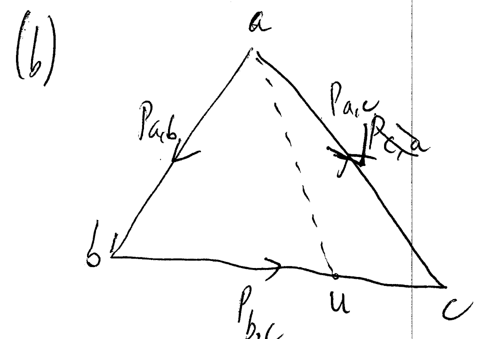
For an edge $\{a,b\}$ in F, $U=\{a,b\}$ has the following realization $\tilde{U}=\{t_a \cdot a + t_b \cdot b \,;\, t_a, t_b \geq 0,\, t_a+t_b=1\}$. This realization can be parametrized as a path from a to b in X:

$$p_{a,b} : [0,1] \longrightarrow X$$

$$p_{a,b}(t) = (1-t)a + tb \checkmark$$

Check? $p_{a,b}$ is continuous because it is affine in t. Moreover $p_{a,b}(0)=a$ and $p_{a,b}(1)=b$.

(b)



Since $\{a,b,c\} \in U$, the simplex generated by them, which is a triangle, belongs to X. To ~~show~~ find a homotopy from $p_{a,c}$ to $p_{a,b} \cdot p_{b,c}$, we only need to find a way to deform $p_{a,c}$ in the triangle to make it $p_{a,b} \cdot p_{b,c}$. The idea is to replace $p_{a,c}$ by a pathe from a to u in the figure and let u move from c to b along $p_{c,b}$.

At stage $s=0$, u is standing at c. At stage $s=1$, u will be at b. Thus $u=(1-s)c+sb$. Now we want to go from a to c by first going from a to u, then from u to c. We need to choose appropriate speeds on $(a \to u)$ and $(u \to c)$ so that the time $(a \to u)$ is 1 at stage $s=0$, and it decreases gradually to $\frac{1}{2}$ at stage $s=1$. Equivalently, we should determine the ~~speed~~ time on $(u\to c)$ such that it is 0 at stage $s=0$ and $\frac{1}{2}$ at stage $s=1$. The easiest choice is $\frac{s}{2}$. Thus the time from a to u at stage s is $1-\frac{s}{2}$. Then the velocity of travelling from a to u is

$$v_1 = \frac{\text{distance}}{\text{time}} = \frac{u-a}{1-\frac{s}{2}},$$

and that of travelling from u to c is $v_2 = \frac{\text{distance}}{\text{time}} = \frac{c-u}{\frac{s}{2}}$

Thus our position at time $0\le t\le 1-\frac{s}{2}$ is $a+v_1t = a+\frac{u-a}{1-\frac{s}{2}}t$.

And our position at time $1-\frac{s}{2}\le t\le 1$ is $u+\frac{c-u}{\frac{s}{2}}\left(t-\left(1-\frac{s}{2}\right)\right)$.

In conclusion, our travelling plan is

$$H(t,s)=\begin{cases} a+\dfrac{u-a}{1-\frac{s}{2}}t, & 0\le t\le 1-\frac{s}{2} \\ u+\dfrac{c-u}{\frac{s}{2}}\left(t-\left(1-\frac{s}{2}\right)\right), & 1-\frac{s}{2}\le t\le 1\end{cases}$$

~~Of co~~ Let's double check: H is continuous because the two functions agree at $t=1-\frac{s}{2}$; and $H(t,0)= a+(u-a)t = a+(c-a)t = f_{a,c}(t)$.

$$H(t,1)=\begin{cases} a+2(b-a)t, & 0\le t\le \frac{1}{2} \\ a+2(c-b)(t-\frac{1}{2}), & \frac{1}{2}\le t\le 1\end{cases} = \begin{cases} f_{a,b}(2t), & 0\le t\le \frac{1}{2} \\ f_{b,c}(2t-1), & \frac{1}{2}\le t\le 1\end{cases} = f_{a,b}\cdot f_{b,c}(t)$$

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We can write H in term of s and t by replacing $u = (1-s)c + sb$:

$$H(t,s) = \begin{cases} a + \dfrac{(1-s)c+sb-a}{1-\frac{s}{2}}t, & 0 \le t \le 1-\frac{s}{2} \\ (1-s)c+sb + \dfrac{c-(1-s)c-sb}{\frac{s}{2}}\left(t-\left(1-\frac{s}{2}\right)\right), & 1-\frac{s}{2} \le t \le 1 \end{cases}$$

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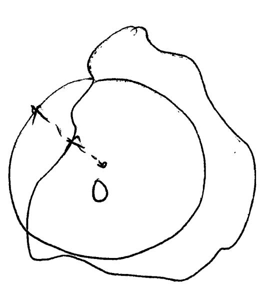
② Suppose $f(z)$ is a monic polynomial $z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$ whose coefficients are complex numbers. Recall $S^1 = \{w \in \mathbb{C} \mid |w| = 1\}$. Show that there is a sufficiently large real number $R > 0$ such that

(a) $f(z) \neq 0$ when $|z| = R$, and

(b) the resulting function $S^1 \to \mathbb{C}\setminus\{0\}$, given by $w \mapsto f(Rw)$ is homotopic to the map $w \mapsto (Rw)^n$.

Proof It's worth examining what happens if R is not large enough. We want to deform the circle $w \mapsto (Rw)^n$ to $w \mapsto f(Rw)$ without crossing the origin. We can only do so in the left figure by using



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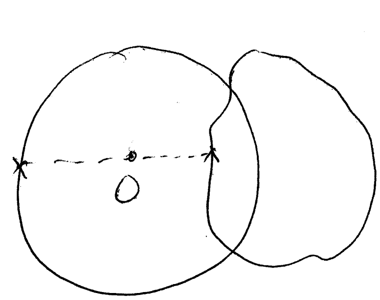


undesired

straight-line homotopy. We couldn't do so for the right figure. Thus we need very large R such that any straight segment connecting $f(Rw)$ to $(Rw)^n$

does not pass through 0. Put $g, h : S^1 \to \mathbb{C}\setminus\{0\}$ such that $g(w) = f(Rw)$ and $h(w) = (Rw)^n$ (as if we already had R such that $g(w) \neq 0 \ \forall w \in S^1$). The straight-line connecting $g(w)$ and $h(w)$ is $(1-t)g(w) + th(w)$. Put $z = Rw$. We don't want $(1-t)f(z) + tz^n$ to be 0 for any $t \in [0,1)$ and any z sufficiently far away from 0.

We have

$$
\begin{aligned}
\phi(z,t) := (1-t)f(z) + tz^n &= (1-t)(z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0) + tz^n \\
&= z^n + (1-t)(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0) \\
&= z^n\left[1 + (1-t)\left(\frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)\right]
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|\phi(z,t)| &= |z|^n \left|1 + (1-t)\left(\frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)\right| \\
&\geq |z|^n \left(1 - (1-t)\left|\frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right|\right) \\
&\geq |z|^n \left(1 - \underbrace{\left|\frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right|}_{\to 0 \text{ as } z \to \infty}\right)
\end{aligned}
$$

Thus there exists $R > 0$ such that $\phi(z,t) \neq 0 \ \forall |z| = R$ and $\forall t \in [0,1]$

With this R, now we know that $(1-t)g(w) + th(w) \neq 0 \ \forall w \in S^1, t \in [0,1]$.

Thus, we can define

$$H(\omega,t) = (1-t)g(\omega) + t\,h(\omega) = (1-t)f(R\omega) + t(R\omega)^n$$

we see that $H : S^1 \times [0,1] \to \mathbb{C}\setminus\{0\}$ and is a continuous map. Moreover,

$H(\omega, 0) = f(R\omega)$ and $H(\omega, 1) = (R\omega)^n$. Thus H is a homotopy from $\omega \mapsto f(R\omega)$ to $\omega \mapsto (R\omega)^n$. In this process, the maps are from S^1 to $\mathbb{C}\setminus\{0\}$. 4/4

(3) Suppose M is an m-manifold and that $\gamma : [0,1] \to M$ is a path in M. Show that there is a homotopic path $\gamma' \sim \gamma$ and an integer n satisfying the following: for all $0 \le k < n$, there is an open set $U_k \subset M$ and a homeomorphism $\phi_k : U_k \to \mathbb{R}^m$ such that

- $\gamma'\left(\left[\frac{k}{n}, \frac{k+1}{n}\right]\right) \subset U_k$, and
- the composite function $\phi_k \circ \gamma' : \left[\frac{k}{n}, \frac{k+1}{n}\right] \to \mathbb{R}^m$ is affine

Proof The ~~con~~ statement is very intuitive. First, we will prove the following lemma to avoid any possible annoyance in making intuition rigorous:

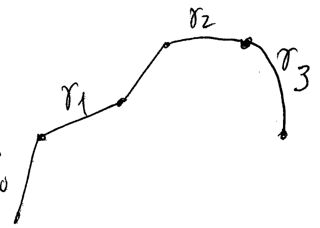
Let $\gamma : [0,1] \to X$ be a path, and $n \in \mathbb{N}$. For each $0 \le k < n$, we define a path $\gamma_k : [0,1] \to X$ such that

$$\begin{tikzcd} {[0,1]} \arrow[r, "T_{nk}"] \arrow[rr, bend right, "\gamma_k"'] & {\left[\frac{k}{n}, \frac{k+1}{n}\right]} \arrow[r, "\gamma"] & \gamma\left(\left[\frac{k}{n}, \frac{k+1}{n}\right]\right) \end{tikzcd}$$

$\gamma_k = \gamma \circ T_{nk}$, where $T_{nk}(t) = \frac{t}{n} + \frac{k}{n}$.

Then $\gamma \sim \gamma_0 \cdot \gamma_1 \cdot \ldots \cdot \gamma_{n-1}$

The lemma means that we can compose n paths into one by subdividing the t-domain. <u>Proof of the lemma.</u>



We have $\gamma_0\cdot\gamma_1\cdots\gamma_{n-1} = (\dots((\gamma_0\cdot\gamma_1)\cdot\gamma_2)\cdot\dots)\cdot\gamma_{n-1}$

Thus the time budget on γ_0 is $\frac{1}{2^n}$, on $\gamma_1 = \frac{1}{2^n}$, on γ_2 is $\frac{1}{2^{n-1}}$, ..., on γ_{n-1} is $\frac{1}{2}$. That's the ~~sch~~ schedule at stage $s=0$. At stage $s=1$, the time budget should be $\frac{1}{n}$ on every path. Thus, the time budget at any stage s

on γ_0 is $\frac{1}{2^{n-1}}(1-s)+\frac{1}{n}s$

on γ_1 is $\frac{1}{2^{n-1}}(1-s)+\frac{1}{n}s$

on γ_2 is $\frac{1}{2^{n-2}}(1-s)+\frac{1}{n}s$

$\vdots$

on γ_{n-1} is $\frac{1}{2}(1-s)+\frac{1}{n}s$

The corresponding speeds are just the reciprocal of the alloted time on each interval. Thus we get the ~~hom~~ path-homotopy from $\gamma_0\cdot\gamma_1\cdots\gamma_n$ to γ as follow

$$H(t,s)=\begin{cases}\gamma_0\left(\dfrac{t}{\frac{1}{2^{n-1}}(1-s)+\frac{1}{n}s}\right), & 0\le t\le \frac{1}{2^{n-1}}(1-s)+\frac{1}{n}s\\[2ex] \gamma_1\left(\dfrac{t-\left(\frac{1}{2^{n-1}}(1-s)+\frac{1}{n}s\right)}{\frac{1}{2^{n-1}}(1-s)+\frac{1}{n}s}\right), & \cancel{0\le}\ \frac{1}{2^{n-1}}(1-s)+\frac{1}{n}s\le t \le\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}\right)(1-s)+\frac{2}{n}s\\ \quad\vdots & \\ \gamma_k\left(\dfrac{t-\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}+\frac{1}{2^{n-2}}+\cdots+\frac{1}{2^{n-k+1}}\right)(1-s)+\frac{k}{n}s}{\frac{1}{2^{n-k}}(1-s)+\frac{1}{n}s}\right) & \\ \quad\text{for }\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}+\cdots+\frac{1}{2^{n-k+1}}\right)(1-s)+\frac{k+1}{n}s\le t\le\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}+\cdots+\frac{1}{2^{n-k}}\right)(1-s)+\frac{k+1}{n}s & \\ \quad\vdots & \\ \gamma_{n-1}(\cdots) & \end{cases}$$

Therefore $\gamma_0\cdot\gamma_1\cdot\ldots\cdot\gamma_{n-1}\sim\gamma$

Return to the problem. First, assume that we can find a homotopic path $\tilde\gamma\sim\gamma$ and an integer n such that for all $0\le k<n$, there is an open set $U_k\subset M$ such that $U_k\overset{\phi_k}{\cong}\mathbb{R}^m$ and $\tilde\gamma\left(\left[\frac{k}{n},\frac{k+1}{n}\right]\right)\subset U_k$.

We'll find a ~~homeomorphism $\phi_k: U_k\to\mathbb{R}^n$ such that $\phi_k\circ\tilde\gamma|_{[\frac{k}{n},\frac{k+1}{n}]}\to\mathbb{R}^m$ is affine.~~ homotopic $\gamma'\sim\tilde\gamma$ such that $\gamma'\left(\left[\frac{k}{n},\frac{k+1}{n}\right]\right)\subset U_k$ and

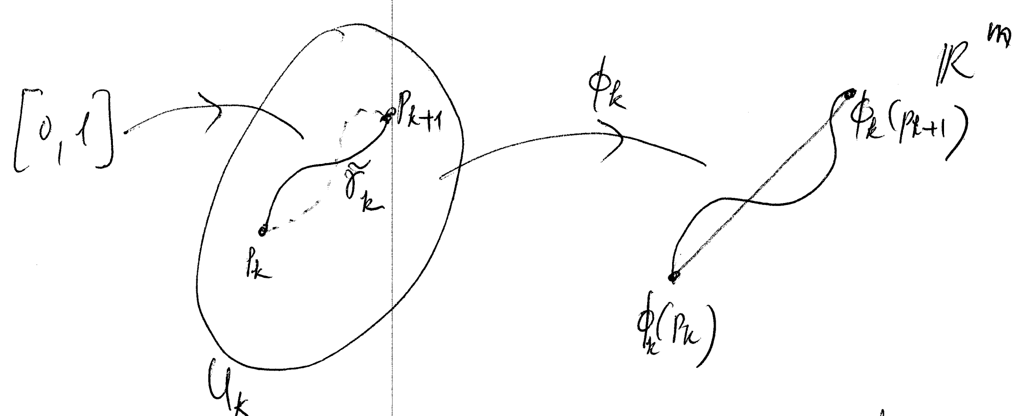
~~We decompose $\tilde{\gamma}$ into n paths like in the lemma~~:

~~$\tilde{\gamma}$~~ ~~γ_0~~.

$\phi_k \circ \gamma' : \left[\frac{k}{n}, \frac{k+1}{n}\right] \longrightarrow \mathbb{R}^m$ is affine. We decompose $\tilde{\gamma}$ into n sub-paths like in the lemma: $\tilde{\gamma} \sim \tilde{\gamma}_0 . \tilde{\gamma}_1 . \cdots . \tilde{\gamma}_{n-1}$ where

$$\tilde{\gamma}_k = \tilde{\gamma} \circ \pi_{nk}$$

$$\begin{tikzcd}
{[0,1]} \arrow[r, "\pi_{nk}"] \arrow[rr, bend right, "\tilde{\gamma}_k"'] & \left[\frac{k}{n}, \frac{k+1}{n}\right] \arrow[r, "\tilde{\gamma}"] & \tilde{\gamma}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right]\right)
\end{tikzcd}$$



Since ϕ_k is continuous, $\phi_k \circ \tilde{\gamma}_k$ is also a path in $\mathbb{R}^m$, which is ~~home~~ homotopic to the straight-path from $\phi_k(p_k)$ to $\phi_k(p_{k+1})$ (the homotopy is simply the straight-line homotopy). Since ϕ_k^{-1} is also continuous, the preimage of the straight path in $\mathbb{R}^m$ is homotopic to $\tilde{\gamma}_k$, which we call $\gamma_k' : [0,1] \to U_k$. Thus $\phi_k \circ \gamma_k' : [0,1] \to \mathbb{R}^m$ is affine. Let $\gamma' : [0,1] \to M$ be the map such that $\gamma_k' = \gamma' \circ \pi_{nk}$, or equivalently $\gamma' = \gamma_k' \circ \pi_{nk}^{-1}$ on $\left[\frac{k}{n}, \frac{k+1}{n}\right]$.

Then by the lemma that we've just prove, $\gamma' \sim \gamma_1' \cdot \gamma_2' \cdots \gamma_{n-1}'$. Since $\gamma_k' \sim \tilde{\gamma}_k$, we have $\gamma' \sim \tilde{\gamma}_0 \cdot \tilde{\gamma}_1 \cdots \tilde{\gamma}_{n-1} \sim \tilde{\gamma}$. Then the proof completes. Now what we have to prove is to find a homotopic path $\tilde{\gamma} \sim \gamma$ and an integer n such that for all $0 \leq k < n$, there exists an open set $U_k \subset M$ such that $U_k \simeq \mathbb{R}^m$ and $\tilde{\gamma}([\frac{k}{n}, \frac{k+1}{n}]) \subset U_k$.

We shall prove that in fact we can choose $\tilde{\gamma} = \gamma$, i.e. we shall prove that [there exists an integer n such that for all $0 \leq k < n$, there exists U_k open in M such that $U_k \simeq \mathbb{R}^m$ and $\gamma([\frac{k}{n}, \frac{k+1}{n}]) \subset U_k$]

Since M is an m-manifold, for each $t \in [0,1]$ there exists an open neighborhood V_t of $\gamma(t)$ such that $V_t \simeq \mathbb{R}^m$. Since $\gamma([0,1])$ is compact and $\gamma([0,1]) \subset \bigcup_{t \in [0,1]} V_t$, there exists a finite subcovering, i.e. there exists open sets $V_1, V_2, \dots, V_p$, each of which is homeomorphic to $\mathbb{R}^m$ and $\gamma([0,1]) \subset (V_1 \cup V_2 \cup \dots \cup V_p)$. We have $[0,1] = \gamma^{-1}(V_1 \cup \dots \cup V_p)$

$$[0,1] = \underbrace{\gamma^{-1}(V_1)}_{\text{open}} \cup \dots \underbrace{\gamma^{-1}(V_p)}_{\text{open}}$$

Since $[0,1]$ is compact, the cover $\{\gamma^{-1}(V_1), \dots, \gamma^{-1}(V_p)\}$ has a Lebesgue number $\varepsilon > 0$. ✓ Let $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. Then every subset of $[0,1]$ of diameter $\frac{1}{n}$ will be contained in one of the open set $\gamma^{-1}(V_i)$'s.

Thus $[\frac{0}{n}, \frac{1}{n}]$ is contained in some $\gamma^{-1}(V_i)$ which we call ~~U_0~~ that V_i U_0

$[\frac{1}{n}, \frac{2}{n}]$ ———————————— ~~U_1~~ that V_i U_1

$\vdots$ $\vdots$

$[\frac{n-1}{n}, \frac{n}{n}]$ ———————————— ~~U_{n-1}~~ that V_i U_{n-1}.

Thus $\gamma([\frac{k}{n}, \frac{k+1}{n}]) \subset U_k \simeq \mathbb{R}^m$ for every $0 \leq k < n$. 4/4

(4) Let $\mathcal{C}$ be a category, and for any $a, b \in \mathcal{C}$ let $Iso_{\mathcal{C}}(a,b) \subset Hom_{\mathcal{C}}(a,b)$ be the set of maps $a \to b$ which are isomorphisms in $\mathcal{C}$. Show that there is a groupoid $\mathcal{C}^w$ with the same collection of objects as $\mathcal{C}$, but with $Hom_{\mathcal{C}^w}(a,b) = Iso_{\mathcal{C}}(a,b)$.

Proof We write down what it means by $\mathcal{C}$ is a category:

- $\mathcal{C}$ has a collection of objects called $Ob(\mathcal{C})$.
- $\mathcal{C}$ has morphisms between two objects called $Hom_{\mathcal{C}}(a,b)$.
- $\mathcal{C}$ has composition operations $\circ : Hom_{\mathcal{C}}(b,c) \times Hom_{\mathcal{C}}(a,b) \longrightarrow Hom_{\mathcal{C}}(a,c)$

which satisfies two conditions

* Associativity: For any $f \in Hom_{\mathcal{C}}(a,b)$, $g \in Hom_{\mathcal{C}}(b,c)$ and $h \in Hom_{\mathcal{C}}(c,d)$

we have $(h \circ g) \circ f = h \circ (g \circ f)$.

$*$ Identity: for any object $x \in \mathcal{C}$, there exists $1_x \in \mathrm{Hom}_{\mathcal{C}}(x,x)$ such that for any $f \in \mathrm{Hom}_{\mathcal{C}}(a,b)$, we have $f \circ 1_a = f = 1_b \circ f$.

A groupoid is a category, with one additional condition:

$*$ Each $f \in \mathrm{Hom}_{\mathcal{C}}(a,b)$ has an inverse $g \in \mathrm{Hom}_{\mathcal{C}}(b,a)$ in sense that

$$f \circ g = 1_b \quad \text{and} \quad g \circ f = 1_a$$

Now we'll check that $\mathcal{C}^w$ with the same objects as $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}^w}(a,b) = \mathrm{Iso}_{\mathcal{C}}(a,b)$ and the same ~~op~~ composition as $\mathcal{C}$, is a groupoid. First we need to check if $\mathcal{C}^w$ is a category. Since $\mathcal{C}$ is already a category, we need to check only two things:

- Let $f \in \mathrm{Iso}_{\mathcal{C}}(a,b)$ and $g \in \mathrm{Iso}_{\mathcal{C}}(b,c)$. Then $g \circ f \in \mathrm{Iso}_{\mathcal{C}}(a,c)$?
- $1_a \in \mathrm{Iso}_{\mathcal{C}}(a,a)$?

<u>Check the first statement</u>

$f \in \mathrm{Iso}_{\mathcal{C}}(a,b)$ implies there exists $f' \in \mathrm{Iso}_{\mathcal{C}}(b,a)$ such that

$$f \circ f' = 1_b \quad \text{and} \quad f' \circ f = 1_a$$

$g \in \mathrm{Iso}_{\mathcal{C}}(b,c)$ implies there exists $g' \in \mathrm{Iso}_{\mathcal{C}}(c,b)$ such that

$$g \circ g' = 1_c \quad \text{and} \quad g' \circ g = 1_b$$

Put $h = f' \circ g' \in \mathrm{Hom}_{\mathcal{C}}(c, a)$. Then

$$
\begin{aligned}
(g \circ f) \circ h = (g \circ f) \circ (f' \circ g') &= g \circ (f \circ f') \circ g' \\
&= g \circ 1_b \circ g' \\
&= g \circ g' \\
&= 1_c
\end{aligned}
$$

and $h \circ (g \circ f) = (f' \circ g') \circ (g \circ f) = f' \circ (g' \circ g) \circ f = f' \circ 1_b \circ f = f' f = 1_a$

Thus $h \in \mathrm{Iso}_{\mathcal{C}}(c, a)$ and $g \circ f \in \mathrm{Iso}_{\mathcal{C}}(a, c)$ and they are inverses of each other.

Check the second statement

$$1_a \circ 1_a = 1_a \circ 1_a = 1_a$$

Thus $1_a \in \mathrm{Iso}_{\mathcal{C}}(a, a)$ and its inverse is itself.

Therefore, $\mathcal{C}^w$ is a category. Then it is straightforward that $\mathcal{C}^w$ is a groupoid since each $f \in \mathrm{Hom}_{\mathcal{C}^w}(a, b) = \mathrm{Iso}_{\mathcal{C}}(a, b)$ has an inverse $f' \in \mathrm{Iso}_{\mathcal{C}}(b, a) = \mathrm{Hom}_{\mathcal{C}^w}(b, a)$.

<u>Example</u> The category of sets with an additional condition

- Objects are sets,
- morphisms are bijections,
- ~~Compos~~ morphism compositions are map compositions.

Category of topological spaces with an additional condition:

- Objects are topological spaces,
- Morphisms are homeomorphisms,
- morphism compositions are map compositions

Category of vector spaces with an additional condition:

- Objects are vector spaces,
- morphisms are linear isomorphisms,
- morphism compositions are map compositions.

Another example

Let G be a group whose every element is of order 2. Such a group exists, for example $(\mathbb{Z}_2, +)$, $(\mathbb{Z}_2 \times \mathbb{Z}_2, +), \ldots$. Moreover, such a group G is always abelian because ~~$e = (a-b)^2 = a^2$~~

$$\begin{cases} e = abba = (ab)(ba) \\ e = (ab)^2 = (ab)(ab) \end{cases}$$ Thus $ab = ba$.

Then we define an entity $\mathcal{C}$ including

- objects: each object is an element of G,
- morphism $H_{\mathcal{C}}(a,b)$ is defined as ab, which is also an object,
- morphism composition $\mathrm{Hom}_{\mathcal{C}}(b,c) \times \mathrm{Hom}_{\mathcal{C}}(a,b) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(a,c)$ is determined by $(bc, ab) \mapsto bcab = ac$.

Then $\mathcal{C}$ is a category, and also a ~~mono~~ groupoid.

(5) Show that a category with one object is equivalent data to a monoid, and that a groupoid with one object is equivalent data to a group.

Proof

First, let's consider a category $\mathcal{C}$ with only one object a. Put $M = \mathrm{Hom}_{\mathcal{C}}(a,a)$. Then $M \neq \phi$ because $1_a \in M$. For any $f, g \in M$, we have $f \circ g \in M$. Thus the morphism composition is a binary operator on M. By the associativity and unitality of $\mathcal{C}$, we have

$$(f \circ g) \circ h = f \circ (g \circ h) \quad \forall f, g, h \in M$$

$$1_a \circ f = f \circ 1_a = f \quad \forall f \in M$$

Thus $(M, \circ)$ is a monoid. Thus the category $\mathcal{C}$ is just an element "a" together with a monoid. Therefore, in term of data, $\mathcal{C}$ is just a monoid. ✓

Second, let's consider a groupoid $\mathcal{G}$ containing only one object a. Put $G = \mathrm{Hom}_{\mathcal{C}}(a,a)$. Since $\mathcal{G}$ is also a category, G is a monoid. Moreover, each $f \in G$ has an inversion $g \in G$ such that $f \circ g = 1_a = g \circ f$. Thus, $(G, \circ)$ is a group. Since $\mathcal{G}$ contains a single object "a" together with a group $(G, \circ)$, $\mathcal{G}$ is equivalent data to a group.

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