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Math 8301: Topology & manifolds

Homework 5

19/20

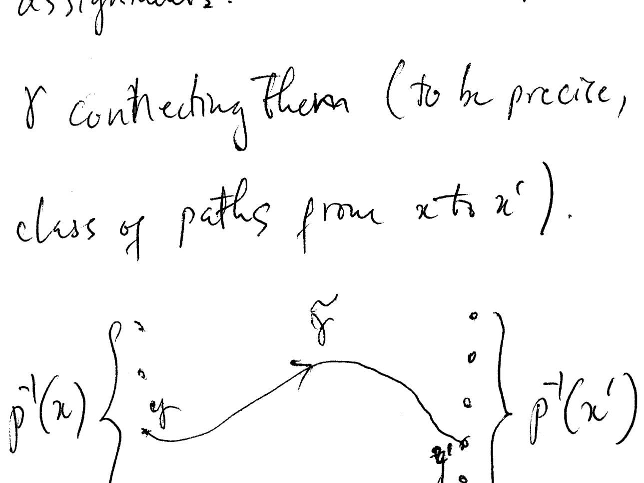
① For a space X, we'd like to define a groupoid $\Pi_1(X)$ as follows. The object of $\Pi_1(X)$ are the points of X. The morphisms $Hom_{\Pi_1(X)}(x,y)$ are homotopy classes of paths γ starting at x and ending at y. The composition of morphisms is given by the path composition $\gamma \cdot \gamma'$.

Explain why this, strictly speaking, does not give a definition of a category, and explain how to alter the definition of composition to fix it.

Proof $\Pi_1(X)$ is a collection of objects, which are points in X; and for two objects x and y a set $Hom(x,y)$ called morphisms; and for three objects x, y, z a law of composition $Hom(x,y) \times Hom(y,z) \longrightarrow Hom(x,z)$ which is defined by path composition. The standard notation for composition of morphisms in a category is, however, $Hom(y,z) \times Hom(x,y) \longrightarrow Hom(x,z)$. Thus we only need the exchange $[\gamma]$ and $[\gamma']$ when taking their composition. Particularly, we define $[\gamma'] \circ [\gamma] := [\gamma \cdot \gamma']$, where $\circ$ is morphism's composition and $\cdot$ is the usual path-composition. This is just the matter of notation. All of the properties of path-composition (associativity, unitality and inverse) are induced straightly to this morphism composition. Then $\Pi_1(X)$ will be a category. 4/4

(2) If $p: Y \to X$ is a covering space, generalize the action of fundamental group $\pi_1(X,x)$ on $p^{-1}(x)$ to show that the assignment $x \mapsto p^{-1}(x)$ extends to a functor p^{-1} from $\Pi_1(X)$ to the category of sets.

<u>Proof</u> We already have object assignments. Now we need ~~hom~~ morphism assignments. Consider two points x and x' in the base-space X, and a path γ connecting them (to be precise, γ is a representative of a ~~class of~~ homotopy class of paths from x to x'). We need to assign to γ a map ~~between~~ from $p^{-1}(x)$ ~~and~~ to $p^{-1}(x')$. we see that



at each $y \in p^{-1}(x)$, there exists a unique lift of γ in Y, say $\tilde{\gamma}$, such that $\tilde{\gamma}(0) = y$ and $p \circ \tilde{\gamma} = \gamma$. Then

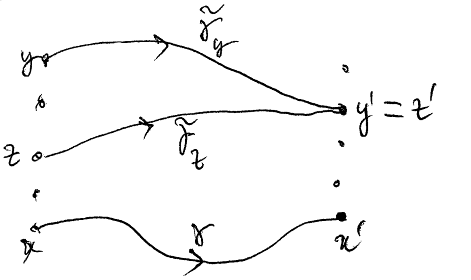
$$x' = \gamma(1) = (p \circ \tilde{\gamma})(1) = p(\tilde{\gamma}(1))$$

Thus $\tilde{\gamma}(1) \in p^{-1}(x')$. Then we assign $y \mapsto \tilde{\gamma}(1)$. Since $\tilde{\gamma}$ depends on y, we should use subscript to denote the dependency $\tilde{\gamma} = \tilde{\gamma}_y$. Then we have a map $\cancel{p^{-1}(X)}$ $P_\gamma : p^{-1}(x) \to p^{-1}(x')$

$$y \mapsto \tilde{\gamma}_y(1) \quad ✓$$

We will show, moreover, that p_γ^{-1} is in fact a bijection.

Check injectivity:



Suppose that $p_\gamma^{-1}(y) = p_\gamma^{-1}(z)$ and $y \neq z$. Then

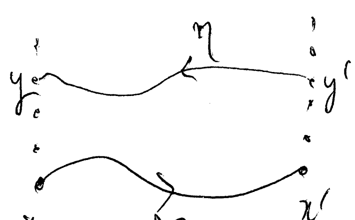
$$\tilde\gamma_y(1) = \tilde\gamma_z(1)$$

The inverse of γ is $\gamma^{-1}: t \mapsto \gamma(1-t)$. Then γ^{-1} is a path from x' to x. We have

$$\tilde\gamma_y^{-1}: t \mapsto \tilde\gamma_y(1-t) \text{ and } \tilde\gamma_z^{-1}: t \mapsto \tilde\gamma_z(1-t)$$

Then $\tilde\gamma_y^{-1}(0) = \tilde\gamma_z^{-1}(0) = y'$. Thus $\tilde\gamma_y^{-1}$ and $\tilde\gamma_z^{-1}$ are two different lifts of γ^{-1} into the covering space at y'. This is a contradiction.

Check surjectivity:



For each $y' \in p^{-1}(x')$, we'll find $y \in p^{-1}(x)$ such that $y' = p_\gamma^{-1}(y)$. There exists a lift of γ^{-1} into the covering space, called η, such that $\eta(0) = y'$. ~~Then η^{-1}~~ Put $\eta(1) = y$. Then η^{-1} is a lift of γ at y. Thus $p_\gamma^{-1}(y) = \eta^{-1}(1) = \eta(0) = y'$.

Now we have p_γ^{-1} is a bijection. Let $\mathcal{S}$ be the category of sets; the morphisms are bijections; the morphism composition is the usual map composition. Then $\mathcal{S}$ is actually a groupoid (see Problem 4, Homework #1). We define a functor p^{-1} from $\Pi_1(X)$, the groupoid of points in X, to

~~the~~ $\mathcal{S}$, the groupoid of sets as follow:

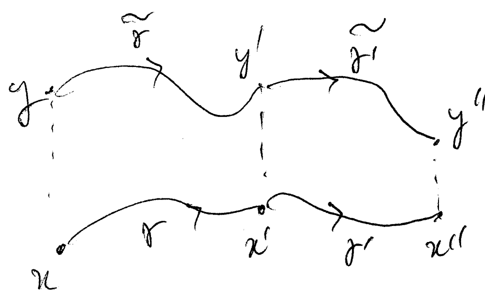
- Object assignment: $x \mapsto p^{-1}(x)$, the fiber over X.
- Morphism assignment: $\gamma \mapsto p_\gamma^{-1}$

Then ~~what~~ we need to check is just the compatibility of morphism composition in $\Pi_1(X)$ and that in $\mathcal{S}$. Let γ be a path from x to x', and γ' be a ~~map~~ path from x' to x''. We know that $[\gamma']\circ[\gamma] = [\gamma\cdot\gamma']$. We have

$$p^{-1}([\gamma']\circ[\gamma]) = p^{-1}(\gamma\cdot\gamma') = p^{-1}_{\gamma\cdot\gamma'}$$

which is a map from $p^{-1}(x)$ to $p^{-1}(x'')$ such that

$$p^{-1}_{\gamma\cdot\gamma'}(y) = \widetilde{\gamma\cdot\gamma'}_y(1) \qquad (*)$$



~~Since $\tilde\gamma\cdot\tilde\gamma'$ is a lift of $\gamma\cdot\gamma'$ such that (up to a homotopy) such that $\tilde\gamma\cdot\tilde\gamma'(0)=y$,~~ ~~it is equal to $\widetilde{\gamma\cdot\gamma'}$ (up to a homotopy)~~), ~~which is the lift~~ we have

$$p\circ(\tilde\gamma\cdot\tilde\gamma')(t) = p(\tilde\gamma\cdot\tilde\gamma'(t)) = \begin{cases} p(\tilde\gamma(2t)) & 0\le t\le 1/2 \\ p(\tilde\gamma'(2t-1)) & 1/2\le t\le 1 \end{cases}$$

$$= \begin{cases} \gamma(2t) & 0\le t\le 1/2 \\ \gamma'(2t-1) & 1/2\le t\le 1 \end{cases}$$

$$= (\gamma\cdot\gamma')(t)$$

Thus $\tilde\gamma\cdot\tilde\gamma'$ is a lift of $\gamma\cdot\gamma'$ such that $\tilde\gamma\cdot\tilde\gamma'(0)=y$. By the uniqueness of lifts, we have $\tilde{\gamma}.\tilde{\gamma}' = \widetilde{\gamma.\gamma'}$. Thus

$$(*) = \widetilde{\gamma.\gamma'}(1) = \tilde{\gamma}_y \cdot \tilde{\gamma}'_{y'}(1) = \tilde{\gamma}'_{y'}(1) = y''$$

Thus $p^{-1}_{\gamma.\gamma'}(y) = y''$. We have $p^{-1}_{\gamma'} \circ p^{-1}_{\gamma}(y) = p^{-1}_{\gamma'}(\tilde{\gamma}_y(1)) = p^{-1}_{\gamma'}(y') = y''$.

Thus $p^{-1}_{\gamma.\gamma'}(y) = p^{-1}_{\gamma'} \circ p^{-1}_{\gamma}(y)$ $\forall y \in p^{-1}(x)$. Thus $p^{-1}_{\gamma.\gamma'} = p^{-1}_{\gamma} \circ p^{-1}_{\gamma'}$. ✓

You also need to check that $p^{-1} : \Pi_1(X) \to \mathrm{Set}$ sends the identity to the identity. 3/4

Remark

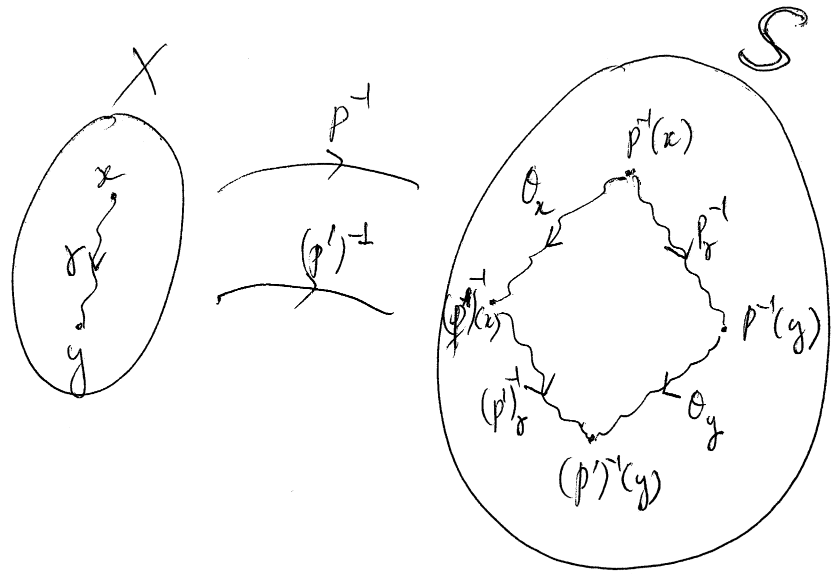
We have $p^{-1}(x) \simeq p^{-1}(y)$ in sense of set isomorphism. This ~~means~~ implies if there exists $x_0 \in X$ such that $|p^{-1}(x_0)| = n$ finite then $|p^{-1}(y)| = n$ for every $y \in X$ which is path-connected to x.

(3) Suppose $p : Y \to X$ and $p' : Y' \to X$ are covering maps, and $\phi : Y \to Y'$ is a homeomorphism such that $p'\phi = p$. Show that the functors p^{-1} and $(p')^{-1}$, from $\Pi_1(X)$ to the category of sets, are naturally isomorphic.

Proof Let $\mathcal{S}$ be the groupoid of sets as the previous problem. The existence of a homeomorphism $\phi : Y \to Y'$ such that $p'\phi = p$ says that p and p' are the same in some sense. We need to ~~specify~~ clarify that sense under the light of "natural isomorphism". We have $p, p' : \Pi_1(X) \to \mathcal{S}$. We need, for every $x \in X$, a choice of set-isomorphism from $p^{-1}(x)$ to $(p')^{-1}(x)$, called θ_x such that

$$\begin{tikzcd} p^{-1}(x) \arrow[r, "p^{-1}_{\gamma}"] \arrow[d, "\theta_x"'] & p^{-1}(y) \arrow[d, "\theta_y"] \\ (p')^{-1}(x) \arrow[r, "(p')^{-1}_{\gamma}"] & (p')^{-1}(y) \end{tikzcd}$$

commutes for every $x, y \in X$ and paths γ from x to y.



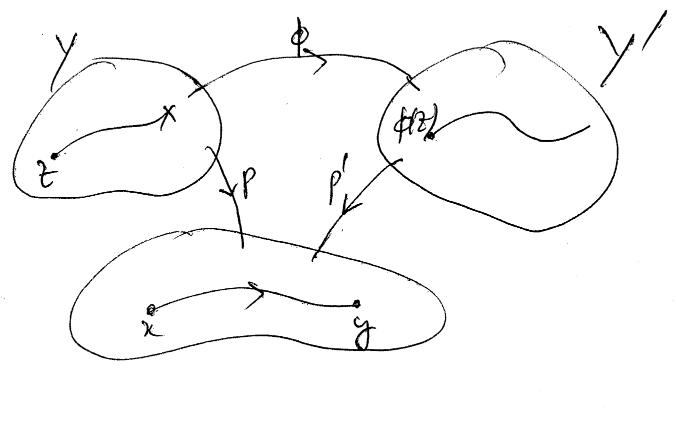
Since $p'\phi = p$, we have $p^{-1}(x) = \phi^{-1}((p')^{-1}(x))$. thus $\phi(p^{-1}(x)) = (p')^{-1}(x)$.
Thus ϕ is a ~~mor~~ isomorphism from $p^{-1}(x)$ to $(p')^{-1}(x)$. Thus we choose $O_x = \phi$ for all $x \in X$. Strictly speaking, it should be $O_x = \phi|_{p^{-1}(x)}$. We have

$$(p')^{-1}_\gamma \circ O_x(z) = (p')^{-1}_\gamma(\phi(z)) \overset{def}{=} \tilde{\gamma}'_{\phi(z)}(1) \qquad \forall z \in p^{-1}(x)$$

where $\tilde{\gamma}'_{\phi(z)}$ is the lift of γ into Y' such that $\tilde{\gamma}'_{\phi(z)}(0) = \phi(z)$. Similarly,

$$O_y \circ P_\gamma^{-1}(z) = \phi(P_\gamma^{-1}(z)) = \phi(\tilde{\gamma}_z(1)) \qquad \forall z \in p^{-1}(x)$$

where $\tilde{\gamma}$ is the lift of γ into Y such that $\tilde{\gamma}_z(0) = z$. Thus we need to check $\phi(\tilde{\gamma}_z(1)) = \tilde{\gamma}'_{\phi(z)}(1)$.



we have

$\gamma = p \circ \tilde{\gamma}_z = p' \circ \phi \circ \tilde{\gamma}_z$

$\gamma = p' \circ \tilde{\gamma}'_{\phi(z)}$

Since $\phi \circ \tilde{\gamma}_z(0) = \phi(z) = \tilde{\gamma}'_{\phi(z)}(0)$, both $\phi \circ \tilde{\gamma}_z$ and $\tilde{\gamma}'_{\phi(z)}$ are lifts

✓

of γ onto Y' at $\phi(z)$. By the uniqueness of the lifting path, we have $\phi \circ \tilde{\gamma}_z = \tilde{\gamma}'_{\phi(z)}$. In particular, $\phi \circ \tilde{\gamma}_z(1) = \tilde{\gamma}'_{\phi(z)}(1)$. 4/4

(4) Suppose $f(z)$ is a monic polynomial $z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$ whose coefficients are complex numbers. Define

$$S = \{z \in \mathbb{C} \mid f'(z) = 0\} \text{ and } T = f(S)$$

Show that the restricted map $f : \mathbb{C} \setminus f^{-1}(T) \to \mathbb{C} \setminus T$ is a covering map.

Proof First we'll prove the following lemma:

Let $g : \Omega \to \mathbb{C}$ analytic where Ω is open in $\mathbb{C}$; and $w \in \Omega$ such that $g'(w) \neq 0$. Then

Not needed. Just use the inverse function theorem.

(i) There exists $r_1 > 0$ such that $g|_{B(w,r_1)}$ is injective

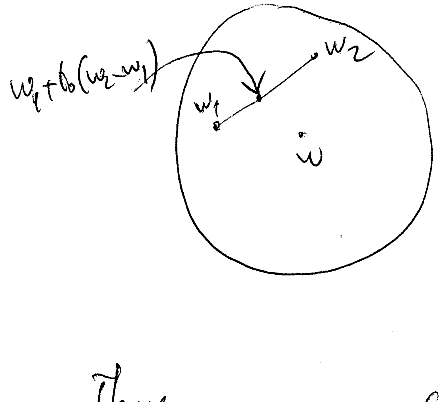
(ii) there exists $r_2 > 0$ such that $g|_{B(w,r_2)} : B(w,r_2) \to g(B(w,r_2))$ is a homeomorphism. Consequently, $g(B(w,r_2))$ is open in $\mathbb{C}$.

Proof of the lemma.

(i) There is a disk $B(w,r)$ containing in Ω. Since g is analytic on this disk, ~~it is infinitely~~ it's derivative is also continuous on this disk. Since $g'(w) \neq 0$, there exists $0 < r_1 < r$ such that $g'(z) \neq 0$ for every

$z \in B(w, r_1)$. We'll show that g is injective on $B(w, r_1)$. Suppose by contradiction that there exist w_1 and w_2 in $B(w, r_1)$ such that $w_1 \neq w_2$ and $g(w_1) = g(w_2)$. We define $h : [0,1] \to B(w, r_1)$ given by

$$h(t) = g(w_1 + t(w_2 - w_1)) \quad \forall t \in [0,1]$$



Then h is continuous on $[0,1]$, and differentiable on $(0,1)$. Thus by Mean-Value-Theorem, there exists $t_0 \in (0,1)$ such that $h'(t_0) = \dfrac{h(1) - h(0)}{1 - 0}$

Thus

$$\lim_{t \to t_0} \frac{g(w_1 + t(w_2 - w_1)) - g(w_1 + t_0(w_2 - w_1))}{t - t_0} = \frac{g(w_2) - g(w_1)}{1 - 0} = 0$$

Thus

$$\lim_{t \to t_0} \frac{g(w_1 + t(w_2 - w_1)) - g(w_1 + t_0(w_2 - w_1))}{t(w_2 - w_1) - t_0(w_2 - w_1)} = 0$$

Thus $g'(w_1 + t_0(w_2 - w_1)) = 0$. Because $B(w, r_1)$ is convex, and w_1, w_2 are in $B(w, r_1)$, the point $w_1 + t_0(w_2 - w_1)$ is also in $B(w, r_1)$. This contradicts the fact that $g'(z) \neq 0$ for every z in $B(w, r_1)$.

(ii) Let r_2 be such that $0 < r_2 < r_1$ where r_1 was found in (i). Then $\overline{B(w, r_2)} \subset B(w, r_1)$. Thus $g|_{\overline{B(w, r_2)}} : \overline{B(w, r_2)} \to g(\overline{B(w, r_2)})$ is injective, surjective and continuous map from a compact space to a Hausdorff space. Thus g is a homeomorphism from $\overline{B(w, r_2)}$ to $g(\overline{B(w, r_2)})$. Thus g is a homeomorphism from $B(w,r_2)$ to $g(B(w,r_2))$. Since they are both subsets in $\mathbb{C}$ and we know that $B(w,r_2)$ is open in $\mathbb{C}$, by the Invariance of the Domain theorem, $g(B(w,r_2))$ is open in $\mathbb{C}$. ∎

Back to the problem: take $z_0 \in \mathbb{C}\setminus T$. We'll find an open neighborhood U of z_0 in $\mathbb{C}\setminus T$ such that $f^{-1}(U)$ is homeomophic to $U \times F$ where F is discrete and

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[d, "\phi"'] & U \\ U \times F \arrow[ur, "\mathrm{proj}"'] & \end{tikzcd}$$

commutes.

By the Fundamental theorem of Algebra, there exists ~~no~~ n roots of the equation $f(z)=z_0$ called $z_1, z_2, \ldots, z_n$. They are pairwise distinct because otherwise, if $z_i = z_j$ is a multiple root then $f'(z_i)=0$, and $z_0 = f(z_i) \in T$, which is not allowed. Now, since $z_0 \in T$, $f'(z_k) \neq 0$ $\forall k=1,\ldots,n$. Applying the above lemma, for each $k=1,\ldots,n$, there exists $r_k>0$ such that the map

$$f_k : B(z_k, r_k) \longrightarrow f(B(z_k, r_k)) = U_k \quad \text{defined as } f_k(x)=f(x),$$

is a homeomorphism. Since z_k's are distinct, we can ~~take~~ assume that r_k's are already small enough so that $B(z_k, r_k)$'s are pairwise disjoint. Put $B_k = B(z_k, r_k)$. Since $z_1, \ldots, z_n$ are all root of the equation $f(z)-z_0=0$, there exists $\varepsilon > 0$ such that $|f(z)-z_0| > \varepsilon > 0$ $\forall z \in \tilde{\mathbb{C}} \setminus \left(\bigcup_{k=1}^{n} B_k\right)$.
where $\tilde{\mathbb{C}} = \mathbb{C} \setminus f^{-1}(T)$.

~~Put $B = B(z_0, \varepsilon')$ where $0 < \varepsilon' < \varepsilon$ small enou~~

By the above Lemma, U_k's are open. Since $z_0 \in U_k$ fore every k, $z_0 \in \bigcap_{k=1}^{n} U_k$. Thus there exist $0 < \varepsilon' < \varepsilon$ such that $B \subset \left(\bigcap_{k=1}^{n} U_k\right)$.

For each $x \in f^{-1}(B)$, we have $f(x) \in B$; thus $|f(x) - z_0| < \varepsilon' < \varepsilon$; thus $x \in \tilde{\mathbb{C}} \setminus \left(\bigcup_{k=1}^{n} B_k\right)$. Then $f^{-1}(B) \subset \tilde{\mathbb{C}} \setminus \left(\bigcup_{k=1}^{n} B_k\right)$

$$f^{-1}(B) = f^{-1}(B) \cap \left(\bigcup B_k\right) = \bigcup \left(f^{-1}(B) \cap B_k\right)$$
$$= \bigcup f_k^{-1}(B) \qquad (\text{because } B \subset U_k)$$

Each $f_k^{-1}(B)$ is homeomorphic to B, and $f_k^{-1}(B)$'s are disjoint. Thus

$$\begin{tikzcd} f_k^{-1}(B) \arrow[r, "f_k"] & B \\ f_{k+1}^{-1}(B) \arrow[ur, "f_{k+1}"] & \\ \cdots & \end{tikzcd}$$

$$\phi : f^{-1}(B) \longrightarrow B \times \{1, \dots, n\}$$
$$x \in f_k^{-1}(B) \longmapsto (f_k(x), k) = (f(x), k)$$

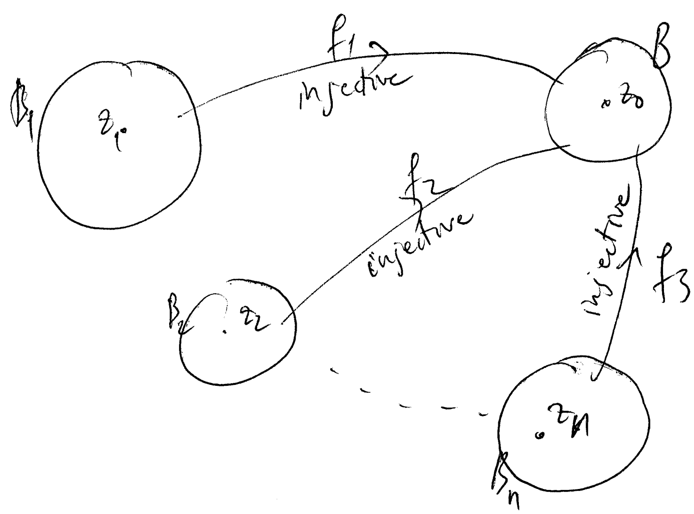
is a homeomorphism. Moreover

$$\begin{tikzcd} f^{-1}(B) \arrow[r, "f"] \arrow[d, "\phi"'] & B \\ B \times \{1, \dots, n\} \arrow[ur, "\mathrm{proj}"'] & \end{tikzcd}$$

Commutes by defintion of ϕ.

Thus f is a covering map from $\mathbb{C} \setminus f^{-1}(T)$ to $\mathbb{C} \setminus T$. Below is the picture illustrating the idea.

4/4



(5) Let X and Y be two topological spaces. Assume that there exists a covering map $p: Y \to X$. We'll examine whether the "locally homeomorphic to $\mathbb{R}^n$" property, Hausdorff property and second countable properties can ~~be~~ ~~succeeded~~ succeed to be transfered along p.

Examine the "locally homeomorphic to $\mathbb{R}^n$" property

Let's call a topological space to satisfy property $(*)$ if it is locally homeomorphic to $\mathbb{R}^n$. We'll show that this property can be transfer along p from Y to X and viceversa.

Suppose Y has prop. $(*)$

Recall that we have the property: any open set of a topological space satisfying $(*)$ also satisfying $(*)$. This was a part of a proof that every open set of an n-manifold is also an n-manifold. For each $x \in X$, we'll find an open neighborhood of x in X that is homeomorphic to $\mathbb{R}^n$.

Since p is a covering map, there exists an open neighborhood V of x in X, a discrete set F and a homeomorphism $\phi: p^{-1}(V) \to V\times F$ such that

$$\begin{tikzcd} p^{-1}(V) \arrow[r, "p"] \arrow[d, "\phi"'] & V \\ V\times F \arrow[ur, "\mathrm{proj}"'] & \end{tikzcd}$$

commutes

Since $p^{-1}(V)$ is open in Y, it also has prop. $(*)$. Since $V\times F \cong p^{-1}(V)$, $V\times F$ has prop. $(*)$. Take $\alpha \in F$ arbitrarily. Since $(x, \alpha) \in V\times F$, it has an open neighborhood W in $V\times F$ such that $\cong \mathbb{R}^n$. In particular, W has prop. $(*)$. Since $V\times\{\alpha\}$ is a product of an open set in X and an open set in F, it is open in $V\times F$. Thus $(V\times\{\alpha\})\cap W$ is open in W. Thus $(V\times\{\alpha\})\cap W$ has prop. $(*)$, we have

$$(V\times\{\alpha\})\cap W = U\times\{\alpha\}$$

where U is some ~~set~~ subset of X containing x. Since $U = \mathrm{proj}\left((V\times\{\alpha\})\cap W\right)$, and that $(V\times\{\alpha\})\cap W$ is open in $V\times F$, and that proj is an open map, U is also open in V. Thus U is open in X. Since $U \cong U\times\{\alpha\} = (V\times\{\alpha\})\cap W$, U has prop. $(*)$. Therefore we can find an open neighborhood of x in U (thus also in X) that is homeomorphic to $\mathbb{R}^n$.

Suppose X has prop. $(*)$

For each $y \in Y$, we'll find an open neighborhood of y in Y that has prop. $(*)$. Put $x = f(y)$. There exists U - open neighborhood of x in X -, a discrete space F, and a homeomorphism ϕ such that

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "p"] \arrow[d, "\phi"'] & U \\ U \times F \arrow[ur, "proj"'] & \end{tikzcd}$$

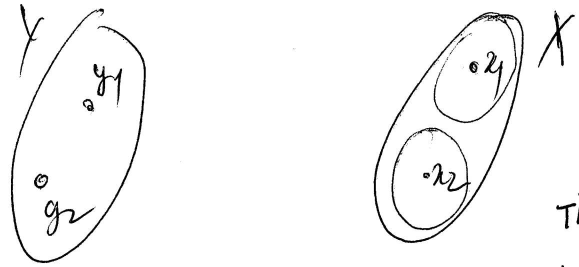
commutes

Since U is open in X, U also has prop. $(*)$. For each ~~$z \in$~~ $(z,\alpha) \in U \times F$, $(z,\alpha) \in U \times \{\alpha\}$ - an open set in $U \times F$. Since $U \times \{\alpha\} \cong U$, it also has prop. $(*)$. Thus every $(z,\alpha) \in U \times F$ has an open neighborhood having prop. $(*)$. Thus $U \times F$ has prop. $(*)$. Since $p^{-1}(U) \simeq U \times F$, $p^{-1}(U)$ has prop. $(*)$. Moreover, $p^{-1}(U)$ is open, ~~containing y~~ and contains y. ✓

Examine the Hausdorff property

First we have one thing may not be true (questionable): "Let X be a topological space in which every point has an open neighborhood that is Hausdorff if viewed as a subspace. Is X necessarily Hausdorff? "

Now first assume that X is Hausdorff. Then we take y_1, y_2 distinct in Y and put $x_1 = p(y_1)$, $x_2 = p(y_2)$. If $x_1 \neq x_2$ Then ~~therefore~~ there exists $U_1 \ni x_1$, $U_2 \ni x_2$ open, disjoint. There exist F_1, F_2, ϕ_1, ϕ_2 such that



This is if $p(y_1) \neq p(y_2)$.
what if $p(y_1) = p(y_2)$?

$$\begin{tikzcd} y_1 \in p^{-1}(U_1) \arrow[r, "p"] \arrow[d, "\phi_1"'] & U_1 \\ U_1 \times F_1 \arrow[ur, "\text{proj}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y_2 \in p^{-1}(U_2) \arrow[r, "p"] \arrow[d, "\phi_2"'] & U_2 \\ U_2 \times F_2 \arrow[ur, "\text{proj}"'] & \end{tikzcd} \quad \text{commute.}$$

Since U_1 and U_2 are disjoint, $p^{-1}(U_1)$ and $p^{-1}(U_2)$ are disjoint and open. Thus Y is also Hausdorff provided that p is injective. If it's not, we cannot conclude Y is Hausdorff. ~~Next, assume that Y is Hausdorff.~~

Conversely, assume that Y is Hausdorff. Take x_1 and x_2 in X. There exists $U_1 \ni x_1$ and $U_2 \ni x_2$ open such that

$$\begin{tikzcd} p^{-1}(U_1) \arrow[r, "p"] \arrow[d, "\phi_1"'] & U_1 \\ U_1 \times F_1 \arrow[ur, "\text{proj}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} p^{-1}(U_2) \arrow[r] \arrow[d, "\phi_2"'] & U_2 \\ U_2 \times F_2 \arrow[ur, "\text{proj}"'] & \end{tikzcd} \quad \text{commute}$$

Since Y is Hausdorff, $p^{-1}(U_1)$ and $p^{-1}(U_2)$ are also Hausdorff. Thus $U_1 \times F_1$ and $U_2 \times F_2$ are Hausdorff. Thus U_1 and U_2 are Hausdorff. Thus, X is a

Space in which every point has an open neighborhood that is Hausdorff.
We ~~conte~~ are really not sure whether X is ~~Hausdo~~ Hausdorff or not. ✓

Examine the second countable property

If a topological space has the property that every point has an open neighborhood that is second countable if viewed as a subspace, it is not necessarily a second countable space (for example the long line).

First, assume X is second contable. then the continuity of p doesn't tell us anything about ~~the~~ second countability of Y. The covering property of p, on the other hand, only gives us information about locals of Y; thus we cannot make a "global" statement about contability of Y.

Next, assume Y is second contable. The same reason as above; and we cannot judge the countability of X.

A counterexample in which X is contable while Y is not contable is

$X = \{0\}$, $\tau_X = \{\phi, X\}$

$Y = X \times [0,1]$ where $[0,1]$ has discrete topology.

$p : Y \to X$ such that $p(\cancel{\{0\}}(0,x)) = 0 \quad \forall x \in [0,1]$. ✓

4/4