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Math 8301 : Topology & Manifolds $\boxed{1}$
Homework 6 15/20

(1) For an integer n and a real number $R>0$, find the effect of the map $w \mapsto (Rw)^n : S^1 \to \mathbb{C}\setminus\{0\}$ on π_1.

Proof We will follow the following steps:

Step 0:

- We know that $\pi_1(S^1,1)$ is a cyclic group generated by the path $\gamma : [0,1] \to S^1$, $\gamma(t) = e^{i2\pi t}$. If we denote $\gamma^k := \gamma.\gamma.\dots.\gamma$ (mod homotopy equivalente), then

$$\pi_1(S^1,1) = \{\gamma^k : k \in \mathbb{Z}\}$$

- put $f : S^1 \to \mathbb{C}\setminus\{0\}$, $f(w)=(Rw)^n$. Then $f(1)=R^n$.
- $f^* : \pi_1(S^1,1) \to \pi_1(\mathbb{C}\setminus\{0\}, R^n)$ is defined by

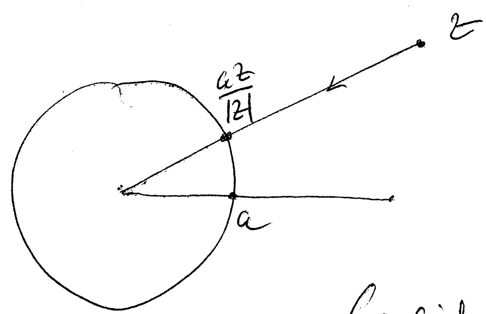
$$f^*(\alpha) = f\circ\alpha \quad \forall \alpha \in \pi_1(S^1,1).$$

Step 1 Find $\pi_1(\mathbb{C}\setminus\{0\}, R^n)$

Step 2 Find f^*.

Details: Step 1 Put $A = \{w\in\mathbb{C} : |w| = R^n\}$. We'll show that $\pi_1(\mathbb{C}\setminus\{0\}, R^n) = \pi_1(A, R^n)$.

Put ~~R^n $a = R^n \in \mathbb{C}\setminus\{0\}$.~~ $a = R^n > 0$.



We define $g : \mathbb{C}\setminus\{0\} \to A$

$$g(z) = a\frac{z}{|z|}$$

Then g is a retraction of $\mathbb{C}\setminus\{0\}$ on A.

Consider $H : (\mathbb{C}\setminus\{0\}) \times [0,1] \to (\mathbb{C}\setminus\{0\})$

$$H(z, s) = sz + (1-s)\, a\frac{z}{|z|}$$

Then H is a basepoint-preserving homotopy from $id_{(\mathbb{C}\setminus\{0\}, a)}$ to g. Therefore each loop $\tilde{\gamma}$ based at a in $\mathbb{C}\setminus\{0\}$ will be homotopic to $\tilde{\gamma}'$ based at a in A where ~~$\gamma'(t) = H(\gamma(t)$~~ $\gamma'(t) = H(\tilde{\gamma}(t), 1)$. Why? Put $H' : [0,1] \times [0,1] \to X$ defined by $H'(t, s) = H(\tilde{\gamma}(t), s)$. Then H' is continuous and

$$H'(t, 0) = H(\tilde{\gamma}(t), 0) = \tilde{\gamma}(t)$$

$$H'(t, 1) = H(\tilde{\gamma}(t), 1) = \tilde{\gamma}'(t)$$

$$H'(0, s) = H(\tilde{\gamma}(0), s) = H(a, s) = a$$ because H is basepoint-preserving.

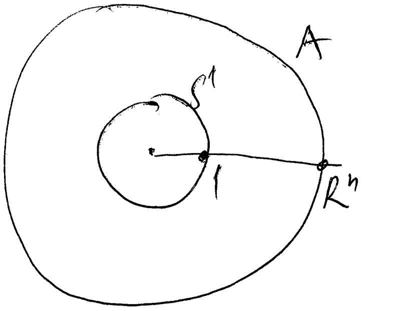
$$H'(1, s) = H(\tilde{\gamma}(1), s) = H(a, s) = a$$

therefore γ' is path-homotopic to γ. Thus $\gamma \in \pi_1(A, a)$. Thus $\pi_1(X, a) \subset \pi_1(A, a)$. Since $\pi_1(A, a) \subset \pi_1(\mathbb{C}\setminus\{0\}, a)$, we have

$X = \mathbb{C}\setminus\{0\}$

$$\pi_1(\mathbb{C}\setminus\{0\}, a) = \pi_1(A, a).$$

Therefore, $\pi_1(\mathbb{C}\setminus\{0\}, R^n) = \pi_1(A, R^n)$.



The map ~~g~~ $h: S^1 \to A$ is a homeomorphism.
$w \mapsto R^n w$

Thus $h_* : \pi_1(S^1, 1) \to \pi_1(A, R^n)$ is a group-isomorphism. Because $\pi_1(S^1,1) = \langle \gamma \rangle$ - cyclic group generated by γ, which is the loop $[0,1] \to S^1$, $\gamma(t) = e^{i2\pi t}$, we have $\pi_1(A, R^n) = \langle \alpha \rangle$ -cyclic group generated by α, which is the loop $[0,1] \to A$, $\alpha(t) = h \circ \gamma(t) = h(e^{i2\pi t}) = R^n e^{i2\pi t}$. Therefore,

$$\pi_1(\mathbb{C}\setminus\{0\}, R^n) = \langle \alpha \rangle = \{\alpha^k : k \in \mathbb{Z}\}$$

Step 2 By definition, $f_* : \pi_1(S^1, 1) \to \pi_1(\mathbb{C}\setminus\{0\}, R^n)$ such that $f_*(\beta) = f \circ \beta$. Thus $f_*(\gamma) = f \circ \gamma$. Particularly,

$$f_*(\gamma)(t) = f(\gamma(t)) = (R\gamma(t))^n = R^n \gamma(t)^n = R^n e^{2i\pi n t}$$

Put $\beta = f_*(\gamma)$. We have $\beta(t) = R^n e^{2i\pi n t}$. Thus

$$\beta|_{[0,\frac{1}{n}]} = \alpha,$$

$$\beta|_{[\frac{1}{n},\frac{2}{n}]} = \alpha,$$

$$\vdots$$

$$\beta|_{[\frac{n-1}{n},\frac{n}{n}]} = \alpha.$$

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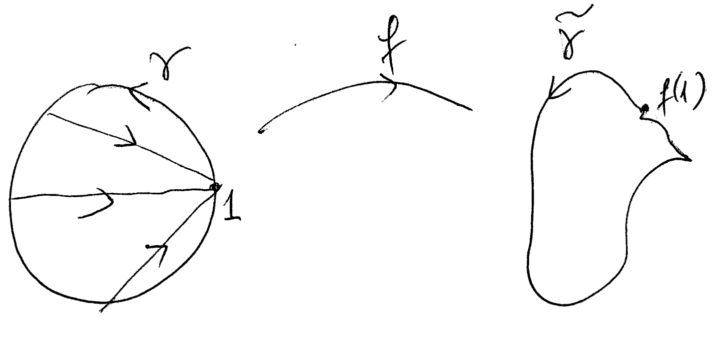
Therefore, as prove in Problem 3, Homework 4, β is path-homotopic to $\alpha.\alpha.\dots.\alpha = \alpha^n$. Thus, $f^*(\gamma) = \alpha^n$. Thus,

$$f^*(\gamma^k) = \alpha^{nk} \quad \forall k \in \mathbb{Z}$$

This is the effect of f on π_1 which we need to find. In the special case $n=0$, f^* is is the trivial homomorphism. Otherwise, $\operatorname{Im} f^* \simeq \pi_1(\mathbb{C}\setminus\{0\}, R)$ and f^* is injective.   4/4

(2) Show that if a polynomial $f(z)$ with complex coefficients has no zeros, then the induced map $\pi_1(S^1, 1) \to \pi_1(\mathbb{C}\setminus\{0\}, f(1))$ send all elements to the identity.

Proof   Because $\pi_1(S^1, 1)$ is a cyclic group generated by $\gamma: [0,1] \to S^1$, $\gamma(t) = e^{i2\pi t}$, it suffices to show that $f_*(\gamma)$ is path-homotopic to the constant path $t \in [0,1] \mapsto f(1)$. Put $\tilde{\gamma} = f_*(\gamma) = f \circ \gamma$.



We have $f: \mathbb{C} \to \mathbb{C}$ and γ is a ~~curve~~ loop based at 1 in $\mathbb{C}$, Thus there is a homotopy $H: [0,1] \times [0,1] \to \mathbb{C}$, $H(t,s) = s + (1-s)\gamma(t)$

between γ and the constant map $t \in [0,1] \mapsto 1$. Put $\tilde{H} = f \circ H$

$$\begin{tikzcd}
{[0,1] \times [0,1]} \arrow[r, "H"] \arrow[dr, "\tilde{H}"'] & \mathbb{C} \arrow[d, "f"] \\
& \mathbb{C}
\end{tikzcd}$$

Then $\tilde{H}$ is a homotopy between $\tilde{\gamma} = f \circ \gamma$ and the constant map $t \in [0,1] \mapsto f(1)$. Since $f(z)$ is never zero, $\tilde{H} : [0,1] \times [0,1] \to \mathbb{C} \setminus \{0\}$. Thus $\tilde{\gamma}$ and $f(1)$ are path-homotopic in $\mathbb{C} \setminus \{0\}$, which concludes the proof.

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Combining Prob. 1, 2. of Homework 6 and Prob. 2 of Homework 4 to ~~show~~ prove the Fundamental theorem of Algebra:

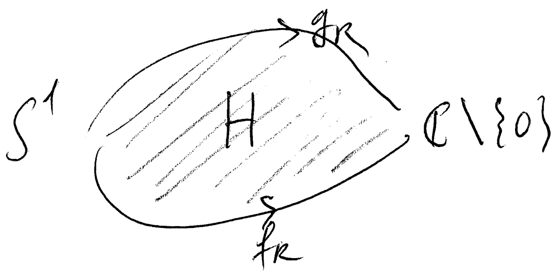
For $R > 0$, we put $g_R : S^1 \to \mathbb{C} \setminus \{0\}$

$$g_R(w) = (Rw)^n,$$

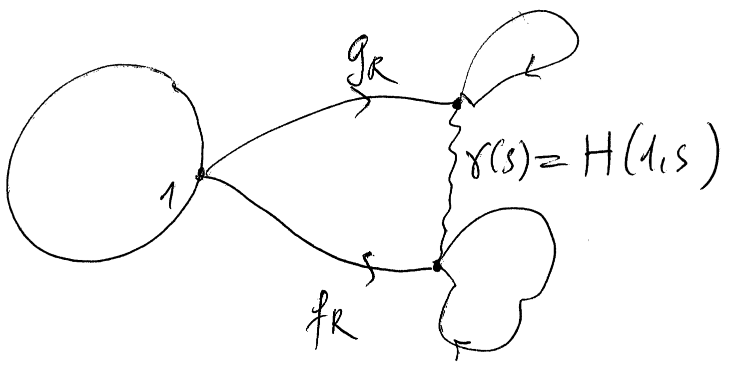
and $f_R : S^1 \to \mathbb{C} \setminus \{0\}$

$$f_R(w) = f(Rw),$$ where f is a polynomial of complex coefficients and of order $n \geq 1$. ~~which~~ Here we assumed that f had no zero in $\mathbb{C}$, and we will try to find a contradiction. For R sufficiently large, Problem 2 in Homework 4 said that there exists a

homotopy between them



where $H: S^1\times[0,1]\to\mathbb{C}\setminus\{0\}$, $H(x,0)=f_R(x)$ and $H(x,1)=g_R(x)$.



$\gamma(0)=H(1,0)=f_R(1)$

$\gamma(1)=H(1,1)=g_R(1)$

Then there exists a path γ in $\mathbb{C}\setminus\{0\}$ from $f_R(1)$ to $g_R(1)$. Then there exists ~~an~~ isomorphism ϕ such that $f_{R*}=g_{R*}\circ\phi$.

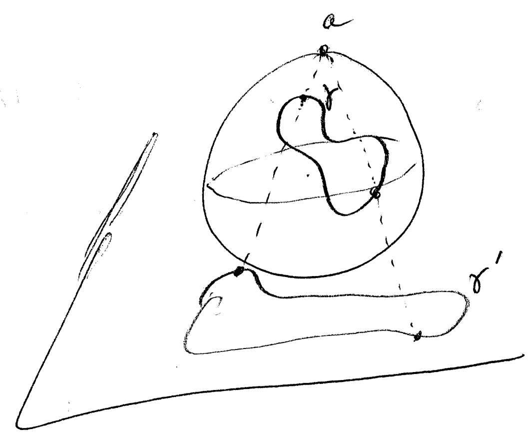
$$\begin{tikzcd}
 & \pi_1(\mathbb{C}\setminus\{0\}, g_R(1)) \arrow[d, "\phi(\alpha)=\gamma^{-1}\cdot\alpha\cdot\gamma"] \\
\pi_1(S^1,1) \arrow[ur, "g_{R*}"] \arrow[r, "f_{R*}"'] & \pi_1(\mathbb{C}\setminus\{0\}, f_R(1))
\end{tikzcd}$$

Problem 1 said that g_{R*} is injective (since $n\geq 1$). Thus, $f_{R*}=g_{R*}\circ\phi$ is injective. This contradicts problem 2, which says f_{R*} must be trivial on $\pi_1(S^1,1)$.

(3) Show that the fundamental group of the n-sphere S^n is trivial for $n>1$ by directly showing that any loop γ is homotopic to the trivial loop.

Proof: Here we will consider in detail the case $n=2$ because it will give us insight, probably sufficient, to deal with any case $n \geq 2$. Let $\gamma : [0,1] \to S^2$, $\gamma(0) = \gamma(1)$ be a loop on S^2. We will show that γ is homotopic to the trivial loop $t \in [0,1] \mapsto \gamma(0)$ ~~by~~ through the following steps:

1) If $\operatorname{Im} \gamma = \gamma([0,1]) \subsetneq S^2$ then there exists $a \in S^2 \setminus \gamma([0,1])$, or equivalently $\gamma([0,1]) \subset S^2 \setminus \{a\}$. The idea then is to ~~show~~ use the stereographic projection P at pole a the project γ onto a plane to a loop $\tilde{\gamma} = P \circ \gamma$ in $\mathbb{R}^2$.



Since $\mathbb{R}^2$ is strongly contractible, $\tilde{\gamma}$ is homotopic to the trivial loop $t \in [0,1] \mapsto \tilde{\gamma}(0)$. Then γ, as the image of $\tilde{\gamma}$ under P^{-1}, is also homotopic to the trivial loop $t \in [0,1] \mapsto \gamma(0)$ on S^2.

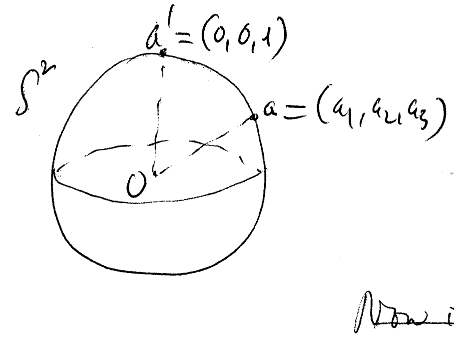
Note that S^2 in our circumstance is a modelling object, or mathematical object. It is described as a set of all points of distance 1 from the origin of a certain Cartesian coordinate system (x_1, x_2, x_3).

$$S^2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$$

Thus $S^2 \setminus \{a\}$ and $S^2 \setminus \{(0,0,1)\}$ are really different sets. The stereographic projection is meant for the latter. The real object (object in the real life) that S^2 models is the sphere in our intuition, in which the spheres punctured at any (one) point are topologically the same. To fit our intuition, it is necessary to show that $S^2 \setminus \{a\}$ and $S^2 \setminus \{(0,0,1)\}$ are homeomorphic.

2) Find a homeomorphism from $S^2 \setminus \{a\}$ to $S^2 \setminus \{(0,0,1)\}$:

To do so, we only need to find a rotation, or a chain of rotations on S^2 that maps a to $(0,0,1)$.



We will rotate in the (x_1, x_2)-plane first, then rotate in the (x_2, x_3)-plane. First, if $a = (0,0,1)$, the rotation is just the identity map.

~~Now if $a \neq (0,0,1)$ then $a(a_1, a_2, a_3)$ has $a_1^2 + a_2^2 \neq 0$.~~

If $a = (0,0,-1)$ then we simply have the symmetry

$$\begin{aligned} S^2 \setminus \{a\} &\longrightarrow S^2 \setminus \{(0,0,1)\} \\ w &\longmapsto -w \end{aligned}$$

, which is a homeomorphism.

If $a \neq (0,0,\pm 1)$, then $a=(a_1,a_2,a_3)$ with $a_1^2+a_2^2 \neq 0$. We will find two rotations as follow

$$a \xrightarrow{A} a'' \xrightarrow{A'} a' \xrightarrow{A''} (0,0,1)$$

$$(a_1,a_2,a_3) \mapsto (0,a_2'',a_3) \mapsto (0,0,a_3') \mapsto (0,0,1)$$

Here $A = \begin{pmatrix} \cos\alpha & -\sin\alpha & \\ \sin\alpha & \cos\alpha & \\ & & 1 \end{pmatrix}$

We need $Aa = a''$, which is equivalent to $a_1\cos\alpha - a_2\sin\alpha = 0$. We can choose such an α that $\cos\alpha = \dfrac{a_2}{\sqrt{a_1^2+a_2^2}}$ and $\sin\alpha = \dfrac{a_1}{\sqrt{a_1^2+a_2^2}}$.

If $a'' = (0,a_2'',a_3)$ is already $a'=(0,0,1)$, A' is just the identity map. Otherwise, we produce another rotation

$$A' = \begin{pmatrix} 1 & & \\ & \cos\beta & -\sin\beta \\ & \sin\beta & \cos\beta \end{pmatrix}$$

We need $A'a'' = a'$, which is equivalent to $a_2''\cos\beta - a_3\sin\beta = 0$. We can choose β such that $\cos\beta = \dfrac{a_3}{\sqrt{a_2''^2+a_3^2}}$ and $\sin\beta = \dfrac{a_2''}{\sqrt{a_2''^2+a_3^2}}$.

If $a' = (0,0,a_3')$ is already $(0,0,1)$ then A'' is just the identity map.

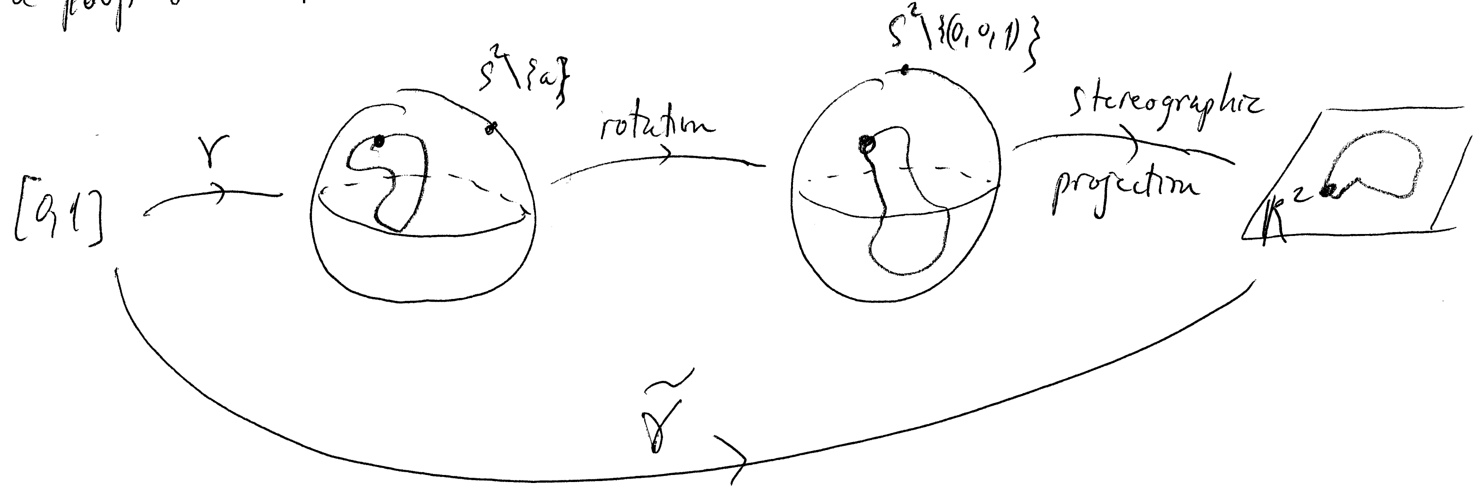
Otherwise, we take another 180°-rotation, or symmetry

$$A'' = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

Therefore, $A''A'A$ is a rotation in S^2 centered at 0 that maps a to $(0,0,1)$.

3) If $\gamma([0,1]) \subsetneq S^2$ then there exists $a \in S^2$ such that $\gamma([0,1]) \subset S^2 \setminus \{a\}$.

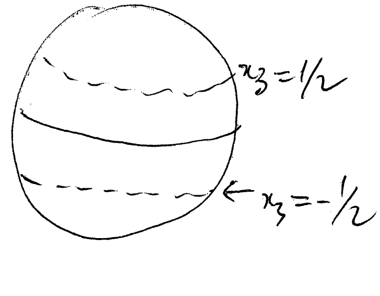
Then there exists a homeomorphism $\phi : S^2 \setminus \{a\} \to \mathbb{R}^2$ that maps γ into a loop $\tilde{\gamma}$ in $\mathbb{R}^2$.



Since $\tilde{\gamma} \sim \tilde{\gamma}(0)$, we get $\gamma \sim \gamma(0)$.
path-homotopic

4) If $\gamma([0,1]) = S^2$, i.e. γ is a sphere-filling curve, then all we need is another γ' in S^2 such that $\gamma' \sim \gamma$ and $\gamma'([0,1]) \subsetneq S^2$. The strategy is to divide $[0,1]$ into smaller segments and deal with γ on each segment.

5)



We define $U_1 = \{(x_1,x_2,x_3) \in S^2, \ x_3 > -1/2\}$

$U_2 = \{(x_1,x_2,x_3) \in S^2 : x_3 < 1/2\}$

Then $\{U_1, U_2\}$ is an open cover of S^2

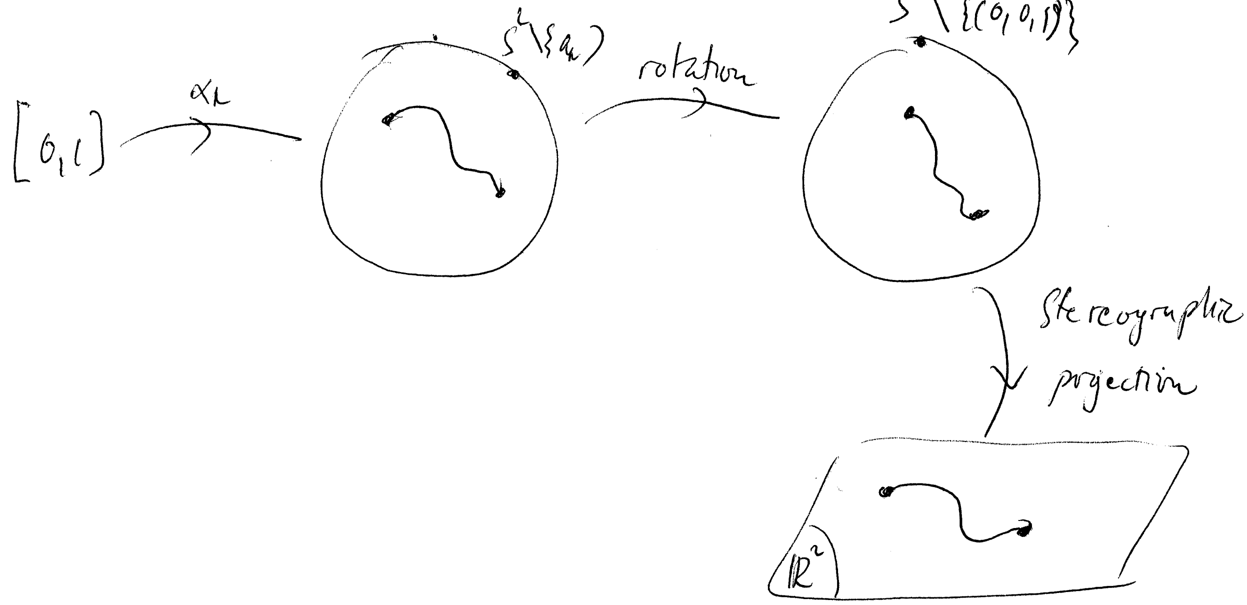
such that $U_1 \subsetneq S^2$ and $U_2 \subsetneq S^2$. Then $\{\gamma^{-1}(U_1), \gamma^{-1}(U_2)\}$ is an open cover of $[0,1]$. Since $[0,1]$ is compact, this cover has a Lebesgue number $\varepsilon = \frac{1}{N}$ for some $N \in \mathbb{N}$, i.e. for every subset A in $[0,1]$, if A has the diameter of at most $\frac{1}{N}$, then A is contained in either $\gamma^{-1}(U_1)$ or $\gamma^{-1}(U_2)$.

We subdivide $[0,1]$ into N intervals of equal length. For each $k = 0, \dots, N-1$, we define α_k as follow

$$\begin{tikzcd}[0,1] \arrow[r, "t \mapsto \frac{k}{N} + \frac{t}{N}"] \arrow[rr, bend right, "\alpha_k"'] & \left[\frac{k}{N}, \frac{k+1}{N}\right] \arrow[r, "\gamma"] & \gamma\left(\left[\frac{k}{N}, \frac{k+1}{N}\right]\right)\end{tikzcd}$$

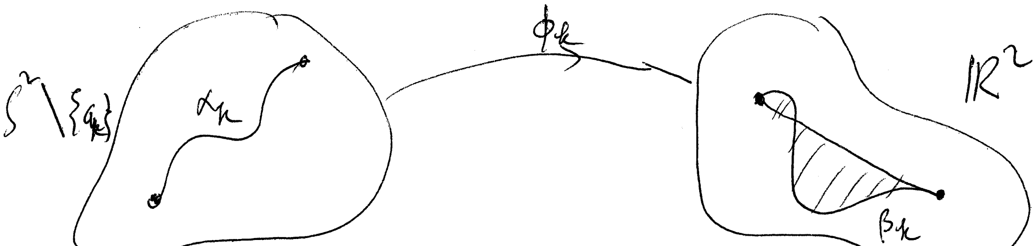
then $\alpha_k(0) = \gamma\left(\frac{k}{N}\right)$, $\alpha_k(1) = \gamma\left(\frac{k+1}{N}\right)$. As proved in Problem 3, Homework 4, $\gamma \sim \alpha_0 \cdot \alpha_1 \cdots \alpha_{N-1}$. Because γ is a sphere-filling curve, each α_k may be very "large", i.e. $\alpha_k([0,1])$ may contain a nonempty open subset of S^2.

6) For each $k = 0, \dots, N-1$, we try to replace α_k by a "smaller" path α_k'. Because $\alpha_k([0,1]) = \gamma\left(\left[\frac{k}{N}, \frac{k+1}{N}\right]\right)$, which is contained in either U_1 or U_2, there exists $a_k \in S^2$ such that $\alpha_k([0,1]) \subset S^2 \setminus \{a_k\}$. Then there exists a homeomorphism $\phi_k : S^2 \setminus \{a_k\} \to \mathbb{R}^2$ which maps α_k to a path β_k in $\mathbb{R}^2$.



β_k is homotopic to the ~~se~~ line segment connecting $\beta_k(0)$ to $\beta_k(1)$ by the following homotopy $H_k : [0,1]\times[0,1] \to \mathbb{R}^2$

$$H_k(t,s) = \big(t\beta_k(1) + (1-t)\beta_k(0)\big)s + (1-s)\beta_k(t)$$

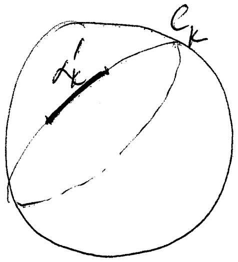


Then α_k is homotopic to $\alpha'_k = \phi_k^{-1}\beta_k$. Because the inverse image of a line under the stereographic projection is a circle passing through $(0,0,1)$ on S^2, and a circle passing through $(0,0,1)$ on S^2 become a circle passing through a on S^2 under backward rotation, $\alpha'_k([0,1])$ is contained in a circle C_k passing through point a in S^2.

7) We have $\alpha_k \sim \alpha'_k$ $\forall k = 0,1,\dots,N-1$. Thus

$\gamma = \alpha_1 \cdots \alpha_{N-1} \sim \alpha_1' \cdots \alpha_{N-1}' = \gamma'$.

What we need is to show that $\gamma'([0,1]) \neq S^2$.



We have

$$\gamma'([0,1]) = \bigcup_{k=0}^{N-1} \alpha_k'([0,1]) = \bigcup_{k=0}^{N-1} C_k$$

We'll show that ~~a fin~~ the union of finitely many circles passing through a is not equal to S^2. Since we already have a ~~tr~~ rotation on S^2 that maps a to $(0,0,1)$, it suffices to show that the union of finitely many circles C_k passing through $(0,0,1)$ is not equal to S^2. Put

$$C = \{ (x_1, x_2, x_3) \in S^2 : x_3 = 1/2 \}$$

Since C does not pass through $(0,0,1)$, ~~each~~ C is none of C_k's. Thus each C_k intersects C at at most two points. Thus $C \cap \left(\bigcup_{k=0}^{N-1} C_k \right)$ is finite.

Suppose by contradiction that $\bigcup_{k=0}^{N-1} C_k = S^2$, then

$C = C \cap S^2 = C \cap \left(\bigcup_{k=0}^{N-1} C_k \right)$, which is finite. For each $n \in \mathbb{N}$, $n \geq 2$, we have $\left(\frac{1}{n}, \sqrt{\frac{3}{4} - \frac{1}{n^2}}, \frac{1}{2} \right) \in C$, thus C must be infinite.

This contradiction completes the proof.

 #4? 0/4

(5) A graph is a simplicial complex with only vertices and edges, i.e. where no faces have dimension higher than one. A tree is a graph, with at least one vertex, such that for any vertices $p \neq q$, there exists a unique sequence $e_1, e_2, \ldots, e_n$ of edges such that

- $e_i \neq e_j$ for $i \neq j$
- e_i and e_{j+1} always share a common vertex
- p is a vertex of e_1, and
- q is a vertex of e_n.

Show that any tree gives rise to a space with trivial fundamental group.

Rough idea

We will examine the problem with simple case: finite set of vertices. Let $V = \{v_1, \ldots, v_n\}$ be the set of vertices and E be the set of edges of the simplicial complex. We consider the case V finite mainly because we do know that it has a geometric realization in $\mathbb{R}^n$, which is a very natural candidate for the topological space we are looking for. The following are steps to show that we can find a space that is contractible.

1) Let $u_i = (0, \ldots \underset{i}{1} \ldots 0)$ be the standard basis of $\mathbb{R}^n$. Consider the map $v_i \in V \mapsto u_i \in \mathbb{R}^n$.

this map gives rise to the geometric realization of (V, E) in $\mathbb{R}^n$ as follow
$$K = \bigcup_{\{v_i, v_j\} \in E} [u_i, u_j]$$
where $[u_i, u_j] = \{ t u_i + (1-t) u_j : 0 \le t \le 1 \}$.

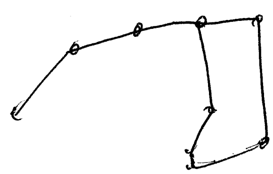
2) Then K is path-connected. Here is how to show it:

- Each point $x \in K$ is contained in a segment $[u_i, u_j]$. It is path connected to u_i by $t \mapsto u_i t + (1-t) x$.
- Two points u_i and u_j is connected by a path connecting v_i and v_j. Each each in this path in E share a comon vertex, thus this path gives rise to a path from u_i to u_j in K.

3) We show by induction with respect to n that there is a homotopy $H_n : K \times [0,1] \to K$ between id_K and the constant map $x \mapsto u_i$ (i is specified, but arbitrary).

- The base case is $n = 1$
- For $n > 1$; in V there exists one vertex that appears in only one edge. Why? Starting from an arbitrary vertex in V, we go to another vertex and keep going and avoid ~~meeting an already-visited vertex~~ ~~going on an old edge~~ meeting an already-visited vertex. Since there are only finitely many vertices, the process must stop. And it

stops because ~~of~~ ~~either one of two reasons: there is no new edge to~~ follow, ~~or a new edge the~~ there is no new vertex to come. The vertex v that is the terminal vertex either has no adjacent vertex or there is an edge connecting v to an odd edge. The second possibility couldn't happen because it will give a cycle in (V, E).



Thus v appears in only one edge. We can assume $v = v_n$ and the edge containing v is

$e = \{v_{n-1}, v_n\}$.

Put $V' = V \setminus \{v_n\}$ and $E' = E \setminus \{e\}$, $K' = K \setminus (u_{n-1}, u_n]$. Then (V', E') is a graph of $n-1$ vertices still ~~st~~ satisfying the problem's axioms and K' is a geometric realization of (V', E') in $\mathbb{R}^{n-1}$ (since we omitted v_n). Then by induction, there exists a homotopy $H_{n-1} : K' \times [0,1] \to K'$, between $id_{K'}$ and $x \in K' \mapsto u_{n-1}$. We define $H_n : K \times [0,1] \to K$ as follow

$$\begin{cases} H_n(x,t) = H_n(x,t) & \forall x \in K' \\ H_n(x,t) = (1-t)x + t u_{n-1} & \forall x \in [u_{n-1}, u_n] \end{cases}$$

Since the two formulas agree at a_{n-1}, H_n is continuous.

What about infinite trees? 3/4