Name: Tuan Pham
ID: 4652218

Math 8301: Topology & Manifolds

Homework 7 18/20

① $F = \langle x, y\rangle$ is a free group on two generators x and y. For each $n \in \mathbb{Z}$, we put $z_n = y^{-n} x y^{n}$ and set $S = \{z_n : n \in \mathbb{Z}\}$ as a subset of F. We put $H = \langle S\rangle$ - the subgroup of F generated by S. We'll show that H is a free group generated by S, i.e. $H = F(S)$.

$$\begin{tikzcd} S \arrow[r, "\lambda"] \arrow[d, "i"'] & F(S) \arrow[dl, "\exists! h"] \\ H & \end{tikzcd}$$

There are a canonical injection $\lambda : S \to F(S)$ and an set inclusion $i : S \to H$. S generates $F(S)$ freely, while it generates H by the rule in group F. We need to show that these two ways of generating result in the same (up to group-isomorphism) group. By the universal property of free group $(F(S), \lambda)$, there exists a unique group-homomorphism $h : F(S) \to H$ such that $i = h\lambda$. We'll show that h is an isomorphism.

We couldn't simply use the universal property of $F(S)$ and F to show that, because S is "special". In other words, if ~~we~~ randomly choose a subset S in F, we may not have the result $\langle S\rangle \cong F(S)$, for example $S = \{x, y, xy\}$. Thus we need to actually touch the elements in S, $F(S)$ and H, i.e. we need

Consider the construction of free groups. To show that h is a group-homomorphism, we'll show that h is injective and surjective.

- <u>Show that h is surjective</u>

$$\begin{tikzcd} S \arrow[r, "\lambda"] \arrow[d, "i"'] & F(S) \arrow[dl, "h"] \\ H & \end{tikzcd}$$

For each $x \in H$, there exists $x_1, \dots, x_n \in S$ and $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$ such that $x = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The multiplication is in F. We have

$$x_1 = i(x_1) = h\lambda(x_1), \quad \dots, \quad x_n = i(x_n) = h\lambda(x_n)$$

Thus
$$\begin{aligned} x &= h(\lambda(x_1))^{\alpha_1} \cdots h(\lambda(x_n))^{\alpha_n} \\ &= h(\lambda(x_1)^{\alpha_1}) \cdots h(\lambda(x_n)^{\alpha_n}) \\ &= h(\lambda(x_1)^{\alpha_1} \cdots \lambda(x_n)^{\alpha_n}) \end{aligned}$$

therefore h is surjective.

- <u>Show that h is in~~sur~~jective</u>

It suffices to show that $\ker h = \{e\}$. Let $y \in \ker h$. Since $F(S)$ is a free group generated by S, $\lambda(S)$ generates $F(S)$. We want to show $\overset{z}{\cancel{y}} = e$. Suppose by contradiction that $g \neq \{e\}$. Then there exists $k > 0$ and $z_{n_1}, \dots, z_{n_k} \in S$ and $\alpha_1, \dots, \alpha_k \in \mathbb{Z}$ such that $z\cancel{y} = \lambda(z_{n_1})^{\alpha_1} \cdots \lambda(z_{n_k})^{\alpha_k}$. Here we assume that the word on the right hand side is already in the reduced form, i.e. $n_i \neq n_{i+1}$ and $\alpha_i \neq 0$ $\forall i = 1, \dots, k$.

- If $k=1$ then $z = \lambda(z_{n_1})^{\alpha_1}$ =

$$e = h(z) = h\left(\lambda(z_{n_1})^{\alpha_1}\right) = \left(h\lambda(z_{n_1})\right)^{\alpha_1} = z_{n_1}^{\alpha_1} = (y^{-n_1}xy^{n_1})^{\alpha_1} = y^{-n_1}x^{\alpha_1}y^{n_1}$$

This is impossible because e is a word of length 0 in F while ~~$z_{n_1}^{\alpha_1}$ is of length 1~~. $y^{-n_1}x^{\alpha_1}y^{n_1}$ is of length 3 if $n_1 \neq 0$, or of length 1 if $n_1 = 0$.

- If $k \geq 2$ then

$$\begin{aligned}
e = h(z) = h\left(\lambda(z_1)^{\alpha_1}\cdots\lambda(z_{n_k})^{\alpha_k}\right) &= h(\lambda(z_{n_1}))^{\alpha_1}\cdots h(\lambda(z_{n_k}))^{\alpha_k} \\
&= z_{n_1}^{\alpha_1}\cdots z_{n_k}^{\alpha_k} \\
&= (y^{-n_1}xy^{n_1})^{\alpha_1}\cdots(y^{-n_k}xy^{n_k})^{\alpha_k} \\
&= (y^{-n_1}x^{\alpha_1}y^{n_1})(y^{-n_2}x^{\alpha_2}y^{n_2})\cdots(y^{-n_k}x^{\alpha_k}y^{n_k}) \\
&= y^{-n_1}x^{\alpha_1}y^{n_1-n_2}x^{\alpha_2}y^{n_2-n_3}\cdots x^{\alpha_{k-1}}y^{n_{k-1}-n_k}x^{\alpha_k}y^{n_k} \\
&= w
\end{aligned}$$

the word w is almost in the reduced form because $\alpha_i \neq 0$ and $n_i - n_{i+1} \neq 0$.

From here, we get

$$y^{n_1-n_k} = x^{\alpha_1}y^{n_1-n_2}x^{\alpha_2}\cdots y^{n_{k-1}-n_k}x^{\alpha_k}$$

the right-hand-side word is now in the reduced form of length ≥ 3 because $k \geq 2$. The word on the left-hand-side has the reduce form of at most length 1. Thus we cannot have an identity. Therefore h must be injective. 4/4

Since h is a group-isomorphism, (H, i) is a free group generated by S.

(2) We'll show that a free product of two free groups is also a free group. Let $\mathcal{A}$ be the ~~CAT~~ Category of sets and $\mathcal{B}$ the Category of groups. Let $F: \mathcal{A} \to \mathcal{B}$ be a functor such that

- For each $S \in \mathcal{A}$, $F(S)$ is a free group generated by S together with an injection $\lambda_S : S \to F(S)$.
- For each map $f: S \to S'$ in $\mathcal{A}$, $\lambda_{S'} f$ is a map from S to group $F(S')$.

$$\begin{tikzcd} S \arrow[r, "\lambda_S"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & F(S) \arrow[d, dashed, "\bar{f} = F(f)"] \\ S' \arrow[r, "\lambda_{S'}"'] & F(S') \end{tikzcd}$$

By the universal property of $F(S)$, there exists a unique $\bar{f} \in Mor_{\mathcal{B}}(F(S), F(S'))$ that makes the diagram commutative.

We define $F(f) := \bar{f}$.

We'll double check if F is a functor.

- $$\begin{tikzcd} S \arrow[r, "\lambda_S"] \arrow[d, "id"'] \arrow[dr, phantom, "\circlearrowleft"] & F(S) \arrow[d, "id_{F(S)}"] \\ S^* \arrow[r, "\lambda_S"'] & F(S^*) \end{tikzcd}$$

  Since $id_{F(S)}$ makes the above diagram commutative, $F(id_S) = id_{F(S)}$.

- $$\begin{tikzcd} S \arrow[r, "\lambda_S"] \arrow[d, "f"'] & F(S) \arrow[d, "F(f)"] \\ S' \arrow[r, "\lambda_{S'}"] \arrow[d, "f'"'] & F(S') \arrow[d, "F(f')"] \\ S'' \arrow[r, "\lambda_{S''}"'] & F(S'') \end{tikzcd}$$

  Since the concatenation of two diagrams is also a commutative diagram, $F(f') \circ F(f) = F(f' \circ f)$.

Return to the problem. Let $F(S)$ and $F(S')$ be two free groups. We want to show that the free product $F(S) * F(S')$ is also a free group. The free product, by its formulation, is nothing but the coproduct of two groups in the category of groups: $F(S) * F(S') = F(S) \amalg F(S')$. Thus it's temptatious to show that $F(S \amalg S') = F(S) \amalg F(S')$, whence the problem is solved. In other words, we'll show that functor F is commutative with coproduct.

Let $S, S' \in \mathfrak{a}$ and $(T, i, i') = S \amalg S'$. We'll show that $(F(T), F(i), F(i')) = F(S) \amalg F(S')$

$$\begin{tikzcd}
S \arrow[rr, "\lambda^*_S"] \arrow[dr, "i"'] & & F(S) \arrow[dr, dashed, "j"] & \\
& T \arrow[rr, "\lambda_T"] & & F(T) \\
S' \arrow[ur, "i'"] \arrow[rr, "\lambda_{S'}"] & & F(S') \arrow[ur, "j'"'] &
\end{tikzcd}$$

Put $j = F(i)$ and $j' = F(i')$. We obtain two commutative diagrams shown above. We'll show that $(F(T), j, j')$ satisfies the universal property characterizing the coproduct of $F(S)$ and $F(S')$. Let (H, f, f') be such that H is a group, f and f' are group-morphisms from $F(S)$ to H, and $F(S')$ to H respectively. We'll show that there exists a unique group-morphism $h: F(T) \to H$ that makes the two triangles commutative, i.e. $hj = f$ and $hj' = f'$.

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$$\begin{tikzcd}
S \arrow[rr, "\lambda_S"] \arrow[dr, "i"'] & & F(S) \arrow[d, "j"] \arrow[dr, "f"] & \\
& T \arrow[r, "\lambda_T"] & F(T) \arrow[r, dashed, "h"] & H \\
S' \arrow[ur, "i'"] \arrow[rr, "\lambda_{S'}"'] & & F(S') \arrow[u, "j'"'] \arrow[ur, "f'"'] &
\end{tikzcd}$$

Fig 1

First we'll show that if such an h exists, it is unique. To show that, it suffices to show that $j(F(S)) \cup j'(F(S'))$ can generate $F(T)$. We have

$$j(F(S)) \supset j(\lambda_S(S)) = (j\lambda_S)(S) = (\lambda_T i)(S) = \lambda_T(i(S))$$

Similarly, $j'(F(S')) \supset \lambda_T(i'(S'))$. Thus

$$\cancel{\lambda}\ j(F(S)) \cup j'(F(S')) \supset \left(\lambda_T(i(S)) \cup \lambda_T(i'(S'))\right) \supset \lambda_T\left(i(S) \cup i'(S')\right).$$

Since T is just a disjoint union of S and S', we have $T = i(S) \cup i'(S')$. Since $F(T)$ is a free group generated by T, $\lambda_T(T)$ generates $F(T)$. Therefore, h if exists is unique.

We see that $(H, f\lambda_S, f'\lambda_{S'})$ is such that H is a group, $f\lambda_S$ and $f'\lambda_{S'}$ are respectively maps from S to H and S' to H. By the universal property of (T, i, i'), there exists a ? T is a set ~~group~~-morphism $k: T \to H$ that makes the following diagram commutative

$$\begin{tikzcd}
S \arrow[r, "\lambda_S"] \arrow[dr, "i"'] & F(S) \arrow[dr, "f"] & \\
& T \arrow[r, dashed, "k"] & H \\
S' \arrow[ur, "i'"] \arrow[r, "\lambda_{S'}"'] & F(S') \arrow[ur, "f'"'] &
\end{tikzcd}$$

Fig. 2

Also, we see that (H, k) is such that H is a group and k is a set map from T to H. By the universal property of $(F(T), \lambda_T)$, there exists $h: F(T) \to H$ such that

$$\begin{tikzcd} T \arrow[r, "\lambda_T"] \arrow[rr, bend right, "k"'] & F(T) \arrow[r, "h"] & H \end{tikzcd}$$

Commutes. Fig. 3

Combining Fig. 2 and Fig. 3, we have found h such that the two parallellograms and trapezoids in Fig. 1 are commute. Now we'll show that the upper little triangle commutes. The second will commutes by similar approach.

We'll show that $hj = f$. Since $\lambda_S(S)$ generates $F(S)$, it suffices to show that they are equal on $\lambda_S(S)$. Since λ_S is injective, it suffices to show that $hj\lambda_S = f\lambda_S$. We have

$$hj\lambda_S = h(j\lambda_S) = h(\lambda_T i) = (h\lambda_T)i = ki = f\lambda_S.$$

this completes the proof.

4/4

③ Let $f: H \to G$ be a group-morphism, and put $R = \operatorname{Im} f$. We'll show that the amalgamated product of $(f, H \to \{e\})$ is isomorphic to a quotient group of G.

$$\begin{tikzcd} & H \arrow[dl, "f"'] \arrow[dr] & \\ G \arrow[dr, "g"'] \arrow[rr, phantom, "\circlearrowleft"] & & \{e\} \arrow[dl] \\ & M & \end{tikzcd}$$

The action of giving two push-outs from G and $\{e\}$ to M that make the diagram commutative is the same as the action of giving a morphism $g: G \to M$ such that $g(x) = e$ $\forall x \in R$. The quadrilateral diagram is just another way to say the latter action. We know that the action of giving a morphism $G \xrightarrow{g} M$ such that $g(x) = e$ $\forall x \in R$ has a universal initial manner $G \xrightarrow{p} G/N$ where N is the normal closure of R in G and p is the canonical projection. In other words, for each morphism $G \xrightarrow{g} M$ satisfying $g(x) = e$ for every $x \in R$, there exists a unique morphism $G/N \xrightarrow{h} M$ such that the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "p"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & G/N \arrow[dl, dashed, "h"] \\ M & \end{tikzcd}$$

Tautologically, $(p, \{e\} \to G/N)$ is an amalgamated product of $(f, H \to \{e\})$. For each $G \xrightarrow{g} M$ and $\{e\} \to M$ that make the big quadrilateral diagram commutes, there exists a unique $h: G/N \to M$ such that two triangles commute. Thus $(p, \{e\} \to G/N)$ is an amalgamated product of $(f, H \to \{e\})$.

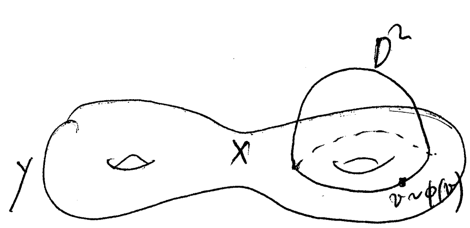
$$\begin{tikzcd} & H \arrow[dl, "f"'] \arrow[dr] & \\ G \arrow[dr, "p"] \arrow[ddr, bend right, "g"'] & & \{e\} \arrow[dl] \arrow[ddl, bend left] \\ & G/N \arrow[d, dashed, "h"] & \\ & M & \end{tikzcd}$$

In other words, G/N is always isomorphism to an amalgamated product of f and $H \to \{e\}$.

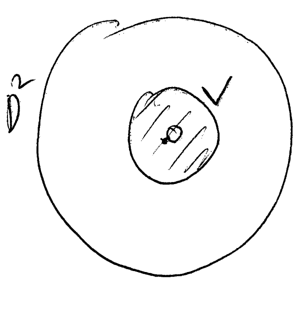
4/4

④ Let X be a topological space and $\phi: S^{n-1} \to X$ is a continuous function, where $n \geq 1$. We put $Y = X \cup_{\phi} e^n$, which means $Y = (X \amalg D^n)/v \sim \phi(v)$, the adjoint space with the gluing function $\phi : S^{n-1} \to X$. We'll use the Seifert-van Kampen theorem to describe the fundamental group of Y in terms of the fundamental group of X and the map ϕ in the cases $n > 1$.

First, we'll consider <u>the case $n = 2$</u>. Then $\phi : S^1 \to X$.



Put $V = B(0, \frac{1}{2})$, the open disk centered at 0 with radius $\frac{1}{2}$ in D^2. Put $U = Y \setminus \{0\}$, where $\{0\}$ is the center of D^2, and assumed to be not in X without loss of generality. Then U and V are open in Y and $U \cup V = Y$. We have $U \cap V = B(0, \frac{1}{2}) \setminus \{0\}$ - path-connected. V is also path-connected and simply connected : $\pi_1(V) = \{e\}$. We have



$$U = Y \setminus \{0\} = \big(X \amalg (D^2 \setminus \{0\}) \big) / v \sim \phi(v)$$

path-conn   path-conn

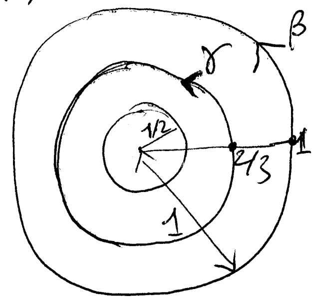
having nonempty intersection by taking quotient

thus U is also path-connected. Then the Seifert-van Kampen theorem says that $\pi_1(Y)$ is an amalgamated product of $\pi_1(U \cap V) \xrightarrow{i} \pi_1(U)$ and $\pi_1(U \cap V) \xrightarrow{j} \pi_1(V)$.

$$\begin{tikzcd}
 & \pi_1(U \cap V) \arrow[dl, "i"'] \arrow[dr, "j"] & \\
\pi_1(U) \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"] & & \pi_1(V) \arrow[dl] \\
 & \pi_1(Y) &
\end{tikzcd}$$

Fig. 4

To specify all group-morphisms, we need to work with a generator of $\pi_1(U\cap V)$.



Put $\gamma : [0,1] \to U\cap V$

$$\gamma(t) = \frac{2}{3} e^{i2\pi t}$$

and

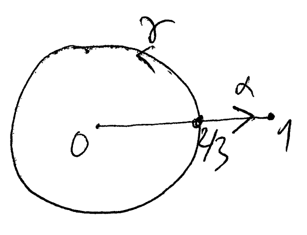
$$\beta(t) = e^{i2\pi t} \quad \forall t \in [0,1].$$

Since $U\cap V$ can be retracted onto γ, we have

$$\pi_1(U\cap V, 2/3) = \pi_1(\gamma, 2/3) = \{\gamma^k : k \in \mathbb{Z}\}.$$

Thus $\pi_1(U\cap V, 2/3)$ is an infinite cyclic group with generator γ. Now we understand $\pi_1(U\cap V)$ in Fig. 4. $\pi_1(V)$ is also known: $\pi_1(V)=\{e\}$. The only task is to try to understand $\pi_1(U, 2/3)$ ~~by connecting it w~~ in the correlation with $\pi_1(X, \phi(1))$. ~~and~~

Since $2/3$ and 1 are connected in U by a linear path α, we have the isomorphism



$$\phi : \pi_1(U, 2/3) \longrightarrow \pi_1(U, 1)$$

$$[\sigma] \longmapsto [\alpha^{-1}\cdot\sigma\cdot\alpha]$$

Next we'll show that $\pi_1(U,1) = \pi_1(X, \phi(1))$, where we have used interchangably u and $\phi(u)$ when $u \in S^1$. We'll show ~~that~~ it by showing that the is a retraction from U onto X. First, by using u and $\phi(u)$

interchangably in U, we can consider X as a subspace of U. We define the map $f : U \to X$ as follow

$$f(\tilde{u}) = \begin{cases} \phi(r(u)) & \text{if } u \in D^2 \setminus \{0\} \\ u & \text{if } u \in X \end{cases}$$

where $\tilde{u}$ denotes the equivalence class of u:

$$\tilde{u} = \begin{cases} \{u\} & \text{if } u \in (D^2 \setminus \{0\}) \setminus S^1 \text{ or } u \in X \setminus \mathrm{Im}\,\phi \\ \{u, \phi(u)\} & \text{if } u \in S^1 \end{cases}$$

and r is the retraction of $D^2 \setminus \{0\}$ onto S^1:

$$r : D^2 \setminus \{0\} \longrightarrow S^1$$
$$x \longmapsto \frac{x}{|x|}$$

For $u \in S^1$, the first formula of f yields $f(\tilde{u}) = \phi(r(u)) = \phi(u)$. The second yields $f(\tilde{u}) = f(\widetilde{\phi(u)}) = \phi(u)$. Thees two fomulas agree and thus f is continuous. By definition, f is a retraction of U onto X.

We have the following homotopy ~~mapping~~ from $id_{D^2 \setminus \{0\}}$ to r:

$$H(x, s) = x(1-s) + s\,r(x) \quad \forall x \in D^2 \setminus \{0\}$$

We define a homotopy from id_U to f as follow

$$H' : U \times [0,1] \to U$$

$$H'(\tilde{u}, s) = \begin{cases} \widetilde{H(u,s)} & \text{if } u \in D^2 \setminus \{0\} \\ \tilde{u} & \text{if } u \in X \end{cases}$$

If $\tilde{u} \in S'$, the first formula yields
$$H'(\tilde{u},s) = \widetilde{H(u,s)}$$
$$= \cancel{(1-u)s+1}$$
$$= \widetilde{(1-s)u + s\,r(u)}$$
$$= \widetilde{(1-s)u + su}$$
$$= \tilde{u},$$
which is consistent with the second formula. Thus H' is continuous. We have
$$H'(\tilde{u},0) = \begin{cases} \widetilde{H(u,0)} & \text{if } u \in D^n \setminus \{0\} \\ \tilde{u} & \text{if } u \in X \end{cases}$$
$$= \begin{cases} \tilde{u} & \text{if } u \in D^n \setminus \{0\} \\ \tilde{u} & \text{if } u \in X \end{cases}$$
$$= \tilde{u} = \mathrm{id}_U(\tilde{u})$$
Moreover,
$$H'(\tilde{u},1) = \begin{cases} \widetilde{H(u,1)} & \text{if } u \in D^n \setminus \{0\} \\ \tilde{u} & \text{if } u \in X \end{cases}$$
$$= \begin{cases} \widetilde{r(u)} & \text{if } u \in D^n \setminus \{0\} \\ \tilde{u} & \text{if } u \in X \end{cases}$$
$$= \begin{cases} \phi(r(u)) & \text{if } u \in D^n \setminus \{0\} \\ u & \text{if } u \in X \end{cases}$$
$$= f(\tilde{u})$$
Thus H' is a homotopy from id_U to f. Moreover, $H'(\tilde{1},s) = \tilde{1} = \phi(1)$ $\forall s \in [0,1]$.

Thus H' is basepoint-preserving, and hence $\pi_1(U,1) = \pi_1(X,\phi(1))$.

therefore, now we have the diagram

$$\begin{tikzcd}
 & & \pi_1(U\cap V, 2/3) \arrow[dl, "i"'] \arrow[dr, "j"] & \\
 & \pi_1(U,2/3) \arrow[dl, "\psi"'] \arrow[dr] \arrow[rr, phantom, "\circlearrowright"] & & \{e\} \arrow[dl] \\
\pi_1(X,\phi(1)) & & \pi_1(Y,2/3) &
\end{tikzcd}$$

Put $A = \pi_1(U\cap V, 2/3) = \{\gamma^k : k\in\mathbb{Z}\}$

$B = \pi_1(U, 2/3)$

$C = \pi_1(X, \phi(1))$

$D = \pi_1(Y, 2/3)$

we have

$$\begin{tikzcd}
 & & A \arrow[dl, "i"'] \arrow[dr] & \\
 & B \arrow[dl, "\psi"'] \arrow[dr] \arrow[rr, phantom, "\circlearrowright"] & & \{e\} \arrow[dl] \\
C & & D &
\end{tikzcd}$$

The commutative square, as in problem 3 suggested, is just a tautological way to say $D = B/\overline{i(A)}$, where $\overline{i(A)}$ is the normal closure of $i(A)$ in B. Since ψ is an isomorphism, we have

$$B/\overline{i(A)} \simeq C/\psi(\overline{i(A)})$$

this isomorphism is $\overline{\psi}$ in the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \overline{i(A)} \arrow[r] \arrow[d, "\psi"] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r] \arrow[d, "\psi"] \arrow[dr, phantom, "\circlearrowleft"] & B/\overline{i(A)} \arrow[r] \arrow[d, "\overline{\Psi}"] & 0 \\
0 \arrow[r] & \psi(\overline{i(A)}) \arrow[r] & C \arrow[r] & C/\psi(\overline{i(A)}) \arrow[r] & 0
\end{tikzcd}
$$

Thus, $D = (C/\psi(\overline{i(A)}), \overline{\Psi})$. Since ψ is an isomorphism, $\psi(\overline{i(A)}) = \overline{\psi(i(A))}$.

we have
$$A \longrightarrow i(A) \subset B \xrightarrow{\psi} C$$
$$[\gamma^k]_{U\cap V} \longmapsto [\gamma^k]_U \longmapsto [\alpha_*^{-1}\gamma^k . \alpha]_U = [\beta^k]_U = [\phi\circ\beta^k]_X$$

Thus $\psi(i(A)) = \{[\phi\circ\beta^k]_X : k\in\mathbb{Z}\}$. We can choose a representative of $[\phi\circ\beta^k]$ as follow
$$(\phi\circ\beta^k)(t) := \phi(e^{2i\pi t k}) \quad \forall\, t\in[0,1],\ \forall k\in\mathbb{Z}.$$

With this definition, we have
$$
\begin{aligned}
D &= (C/\overline{\psi(i(A))}, \overline{\Psi}) \\
&\simeq C/\overline{\psi(i(A))} \\
&\simeq \pi_1(X, \phi(1)) / \overline{\{\phi\circ\beta^k : k\in\mathbb{Z}\}}
\end{aligned}
$$

Therefore,
$$\pi_1(Y, 2/3) \simeq \pi_1(X, \phi(1)) / \overline{\{\phi\circ\beta^k : k\in\mathbb{Z}\}},$$

and the isomorphism is $\overline{\Psi}$ as stated above.

<u>For the case $n\geq 3$:</u>

We'll follow the same initial steps as in case $n=2$. We still put $V = B(0, 1/2)$ - the open ball centered at 0 with radius $1/2$. We still put

$U = \cancel{X}\, Y \setminus \{0\}$ and get:

- U and V are open in Y
- $U \cup V = Y$
- $U \cap V = B(0, 1/2) \setminus \{0\}$ path-connected
- Both U and V are path-connected.
- $\pi_1(V) = \{e\}$

Then for some $x_0 \in U \cap V$, the Seifert-van Kampen theorem says that

$$\pi_1(Y, x_0) = \pi_1(U, x_0) \ast_{\pi_1(U \cap V, x_0)} \{e\}.$$

$$\begin{tikzcd} & \pi_1(U \cap V, x_0) \arrow[dl] \arrow[dr] & \\ \pi_1(U, x_0) \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"] & & \cancel{\pi_1(V, x_0)}\ \{e\} \arrow[dl] \\ & \pi_1(Y, x_0) & \end{tikzcd}$$

Then we follow the same approaches as the previous case to reach the diagram:

$$\begin{tikzcd} & & \pi_1(U \cap V, x_0) \arrow[dl, "i"'] \arrow[dr, "j"] & \\ & \pi_1(U, x_0) \arrow[dl, "\psi"'] \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"] & & \{e\} \arrow[dl] \\ \pi_1(X, 1) & & \pi_1(Y, x_0) & \end{tikzcd}$$

where ψ is an isomorphism

$$\psi : \pi_1(U, x_0) \longrightarrow \pi_1(X, 1)$$
$$[\sigma] \longmapsto [\alpha^{-1} \cdot \sigma \cdot \alpha]$$

where α is a (fixed) path connecting x_0 to 1 in D^n.

Then we still get $\pi_1(Y, x_0) \simeq \pi_1(X, \phi(1)) / \overline{\varphi(i(\pi_1(U\cap V, x_0)))}$

and the isomorphism is $\overline{\varphi} : \pi_1(\cancel{X}, \cancel{\phi(1)}\, x_0) / \overline{i(\pi_1(U\cap V, x_0))}$

$$\to \pi_1(X, \phi(1)) / \overline{\varphi(i(\pi_1(U\cap V, x_0)))}$$

as in the previous case. We realize that $\cancel{\pi_1}$ $U\cap V = B(0,1) \setminus \{0\}$ having a retraction on $\partial B(0, 1/2)$, which is isomorphic to S^{n-1}. Since $n \geq 3$, S^{n-1} is simply-connected. This is actually a homework problem last week.

Thus $U\cap V$ is simply connected, and thus $\pi_1(U\cap V, x_0) = \{e\}$. Thus $\overline{\varphi} : \pi_1(U, x_0)/\{e\} \to \pi_1(X, \phi(1))/\{e\}$, thus $\overline{\varphi} = \varphi$. Moreover,

$$\pi_1(Y, x_0) \simeq \pi_1(X, \phi(1))/\{e\} = \pi_1(X, \phi(1))$$

Therefore, $\pi_1(Y, x_0) \simeq \pi(X, \phi(1))$ and this isomorphism is given by φ.

<u>Note</u>:

In both cases, we can eliminate the isomorphism $\overline{\varphi}$ or φ by converting them into identity. Equivalently, this says

$$\pi_1(Y, \phi(1)) = \pi_1(X, \phi(1)) / \overline{\{\phi\circ\beta^k ; k \in \mathbb{Z}\}} \quad \text{for } n = 2,$$

and

$$\pi_1(Y, \phi(1)) = \pi_1(X, \phi(1)) \quad \text{for } n \geq 3.$$

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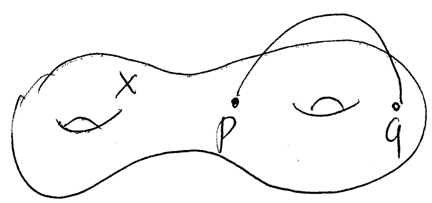
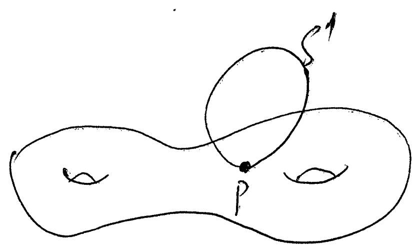
(5) Let X be a path-connected space and $p, q \in X$. A map $\phi: S^0 \to X$ is determined by $\phi(-1) = p$ and $\phi(1) = q$. Put $A = X \cup_\phi e^1$.

We'll show that $\pi_1(A, p) \simeq \pi_1(X, p) * \mathbb{Z}$.

First we'll consider a simpler case where $p = q$.

Then $A \cong (X \amalg S^1)/p \sim 1$.

A 



Let $\mathcal{A}$ be the category of pointed-topological spaces, and $\mathcal{B}$ the category of groups. Then

$$(A, p) = (X, p) \amalg (S^1, 1)$$

Here the coproduct is taken in $\mathcal{A}$. We have the following functor: $F: \mathcal{A} \to \mathcal{B}$ such that

- For each pointed space (X, p), $F(X, p) = \pi_1(X, p)$.
- For each morphism $(X, p) \xrightarrow{f} (Y, q)$, $F(f) = \pi_1(X, p) \xrightarrow{f_*} \pi_1(Y, q)$

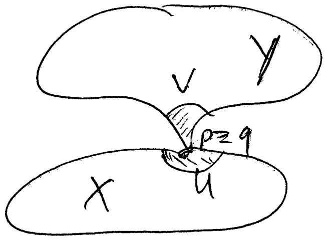
Then we'll show that F commutes with coproduct, i.e.

$$F((X, p) \amalg (Y, q)) = F(X, p) \amalg F(Y, q)$$

If we obtain that, we have

$$\pi_1(A, p) = F(A, p) = F((X, p) \amalg (S^1, 1)) = \pi_1(X, p) \amalg \pi_1(S^1, 1)$$
$$= \pi_1(X, p) * \mathbb{Z}$$

$$
\begin{tikzcd}
(X,p) \arrow[rr, "F"] \arrow[dr, "i"] & & \pi_1(X,p) \arrow[d, dashed, "i_*"] \\
 & (Z,r) \arrow[r, "F"] & \pi_1(Z,r) \\
(Y,q) \arrow[ur, "j"] \arrow[rr, "F"] & & \pi_1(Y,q) \arrow[u, "j_*"]
\end{tikzcd}
$$



In case X and Y are path-connected, we'll take a neighborhood $U \ni p$ in X and $V \ni q$ in Y such that $X \cup V$ has a retraction onto X, with all points in V mapped to p; and $Y \cup U$ has a retraction onto Y, with all points in U mapped to q. Yes we couldn't do so in general but we can find such U and V in case X and Y are manifolds. Then, due to the retractions,

(because in that case we can choose U and V such that $U, V \simeq \mathbb{R}^n$)

$$\pi_1(X \cup V, p) = \pi_1(X, p)$$

$$\pi_1(Y \cup U, q) = \pi_1(Y, q)$$

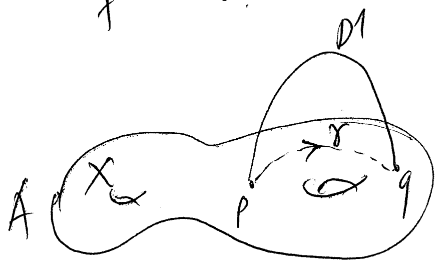
We have $(X \cup V) \cup (Y \cup U) = Z$ and $(X \cup V) \cap (Y \cup U) = (U \cap V)_{p \sim q}$ is path-connected. By Seifert-van Kampen theorem, we have

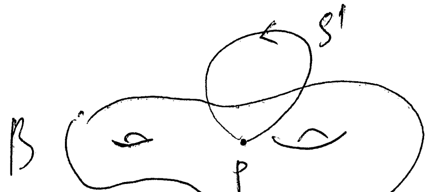
$$
\begin{tikzcd}
 & \pi_1(U \cap V, p) \arrow[dl, "i"'] \arrow[dr, "j"] & \\
\pi_1(X,p) \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"] & & \pi_1(Y,q) \arrow[dl] \\
 & \pi_1(Z,p) &
\end{tikzcd}
$$

Since $U \cap V$ can be retracted to a single point p, $U \cap V$ is simply connected.

Therefore, $\pi_1(U\cap V, p) = \{e\}$. Then the amalgamated product is just a free product and we get $\pi_1(Z,p) \cong \pi_1(X,p) * \pi_1(Y,q)$. In other words we can show that $\pi_1(A,p) \simeq \pi_1(X,p) * \mathbb{Z}$ in case X is a manifold and $p=q$ by using Seifert-van Kampen theorem.

In case that $p \neq q$, we couldn't directly use this theorem because whatever way we divide A into a union of U and V will result in the non-path connectedness of $U \cap V$:





Put $B = (X \amalg S^1)_{p\sim 1}$. We need to show that
$$\pi_1(A,p) \simeq \pi_1(B,p).$$
To do so, we'll introduce two maps $f: A \to B$ and $g: B \to A$ such that
$$\begin{cases} f\circ g \text{ is homotopic to } id_B, \\ g\circ f \text{ is homotopic to } id_A. \end{cases}$$

Since X is ~~con~~ path-connected, there exists a path γ from p to q. Then S^1 is homeomorphic to $D^1 \cup \gamma([0,1])$ by the following homeomorphism
$$\varphi: S^1 \longrightarrow D^1 \cup \gamma([0,1]) / -1\sim p,\ 1\sim q$$
$$\varphi(e^{2\pi i t}) = \begin{cases} \gamma(2t) \text{ if } 0 \le t \le 1/2 \\ \gamma(\cancel{2t-1/2}) \ \cancel{\text{if } \tfrac12 \le t \le 1} \ -4t+3,\ \tfrac12 \le t \le 1 \end{cases}$$

Thus A is homeomorphic to a quotient space of B, i.e.

$$A \simeq B / e^{2\pi i t} \sim \gamma(2t) \qquad \forall 0 \leq t \leq \frac{1}{2}$$

thus the map $g: B \to A$ is naturally chosen as the projection $u \mapsto [u]$.

A path in A is also a path in B (this claim is just intuitive, not rigorous), we can choose $f: A \to B$ to be the identity map. We then define the following homotopy $H: B \times [0,1] \to B$

$$H(u,s) = \begin{cases} \{u, \gamma(t)\} & \text{if } u = e^{2\pi i t}, 0 \leq t \leq s/2 \\ u & \text{elsewhere} \end{cases}$$

Then $H(u,0) = u$ and $H(u,1) = f \circ g$. Thus $f \circ g \sim id_B$.

By the definition of f and g above, we already have $g \circ f = id_A$.

Since $f(p) = p$, we have the isomorphism $f_*: \pi_1(A,p) \to \pi_1(B,p)$, and

$$\pi_1(A,p) \simeq \pi_1(X,p) * \mathbb{Z}.$$

The problem left is how to deal with the case where X is not a manifold.

Easier to just follow the
hint and use Seifert-van Kampen.

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