Name: Tuan Pham | Math 8301: Topology and Manifolds

ID: 4652218 | Homework 8 $^{18}/_{20}$

(1) Show that S^2 is isomorphic to the universal covering space of $\mathbb{RP}^2$.

Proof A universal covering space of a path-connected space X was defined to be a space $\tilde{X}$ with a covering map $p: \tilde{X} \to X$ such that $\tilde{X}$ is path-connected and simply connected. Thus, in order to speak of the universal covering space of $\mathbb{RP}^2$, we need to show that it is path-connected. Then by the fact that if (Y_1, p_1) and (Y_2, p_2) are two covering spaces of X such that $\pi_1(Y_1) = \{e\}$, $\pi_1(Y_2) = \{e\}$ then $Y_1 \simeq Y_2$, we only need to find a covering $S^2 \xrightarrow{p} \mathbb{RP}^2$ and show that $\pi_1(S^2) = \{e\}$.

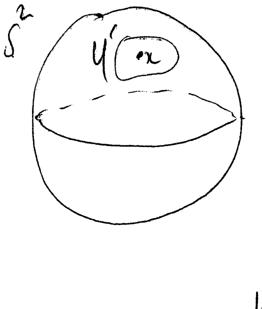
$$\begin{tikzcd} Y_1 \arrow[rr, "\sim"] \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Step 1: Show that $\mathbb{RP}^2$ is path-connected.

$\mathbb{RP}^2$ is by definition the quotient space $S^2/\sim$ where $x \sim y$ iff $x = x$ or $x = -y$. Let p be the projection $p: S^2 \to S^2/\sim$. Then p is continuous and surjective. Such a map maps a path-connected space onto a path-connected space, because if γ is a path from x to y in S^2 then $p \circ \gamma$ is a path from $p(x)$ to $p(y)$ in ~~$S^2/\sim$~~. Thus $S^2/\sim$ is path-connected if S^2 is path-connected. Any two points on S^2 can be connected by a geodesic paths.

<u>Step 2</u> We'll show that S^2 is simply connected and the projection map $p: S^2 \to S^2/\sim$ is a covering map. By Problem 3 of HW 6, each loop in S^2 is homotopic to the trivial loop. Thus S^2 is simply connected. Now we'll show that p is a covering map. There are two ways to prove it. The first way is to use the definition of covering maps i.e. for each $\tilde{x} \in S^2/\sim$, we find an open neighborhood U of $\tilde{x}$ such that $p^{-1}(U)$ is homeomorphic to a stack of pancakes (copies) of U. The second way is to find a group G such that G acts properly discontinuously on S^2 and $S^2/G = S^2/\sim$.

<u>The first way</u> Let $\tilde{x} = \{x, -x\} \in S^2/\sim$. Since $x \neq -x$ and S^2 is Hausdorff, x and $-x$ can be separated by two open sets. Since the family of balls on S^2 forms a basis of S^2, we can get a ball U' centered at x such that $U' \cap (-U') = \phi$. Then we put



~~$U = p(U') = U$~~ $U = \{\tilde{x} : x \in U' \text{ or } x \in -U'\}$. Then $p^{-1}(U) = U' \cup (-U')$. Since $U' \simeq (-U')$ and they are disjoint, we get the homeomorphism $\phi_1 : U' \cup (-U') \longrightarrow U' \times \{1, 2\}$

$$\phi(x) = \begin{cases} (x, 1) & \text{if } x \in U' \\ (x, 2) & \text{if } x \in -U' \end{cases}$$

Moreover, $U' \overset{p}{\simeq} U$ by the fact that ~~U is a~~ $U' \cup (-U')$ is a saturated set in S^2 and U' is open in S^2 (then $p : U' \to U$ is an open map). Thus we get the isomorphism $\phi_2 : U' \times \{1,2\} \to U \times \{1,2\}$

$$(x, i) \mapsto (\tilde{x}, i)$$

Then we get the commutative diagram

$$\begin{tikzcd}
U' \cup (-U') \arrow[r, "p"] \arrow[d, "\phi_1"] \arrow[dd, bend right=60, "\phi"'] & U \\
U' \times \{1,2\} \arrow[d, "\phi_2"] \arrow[ur, phantom, "\circlearrowleft"] & \\
U \times \{1,2\} \arrow[uur, "proj"'] &
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
x \arrow[r, "p"] \arrow[d, "\phi_1"] & \tilde{x} \\
(x, i) \arrow[d, "\phi_2"] & \\
(\tilde{x}, i) \arrow[uur, "proj"'] &
\end{tikzcd}$$

The second way Put $G = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. Let G acts on S^2 as follows

$$\bar{0} : (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3)$$

$$\bar{1} : (x_1, x_2, x_3) \mapsto (-x_1, -x_2, -x_3)$$

these maps are continuous from S^2 to S^2. Moreover, we can ~~we~~ write as follow

$$\bar{k} \cdot (x_1, x_2, x_3) = (-1)^k (x_1, x_2, x_3) \quad \text{for } k = 0 \text{ or } 1.$$

Then
$$\bar{k}_1 \cdot (\bar{k}_2 \cdot (x_1, x_2, x_3)) = \bar{k}_1 \cdot ((-1)^{k_2} (x_1, x_2, x_3))$$
$$= \bar{k}_1 \cdot (x_1 (-1)^{k_2}, x_2 (-1)^{k_2}, x_3 (-1)^{\cancel{k_3} k_2})$$
$$= (-1)^{k_1} (x_1 (-1)^{k_2}, x_2 (-1)^{k_2}, x_3 (-1)^{k_3})$$
$$= (-1)^{k_1 + k_2} (x_1, x_2, x_3)$$
$$= \cancel{(-1)} \, \overline{k_1 + k_2} \cdot (x_1, x_2, x_3)$$
$$= (\bar{k}_1 + \bar{k}_2) \cdot (x_1, x_2, x_3)$$

Moreover, $\bar{0}\cdot(x_1,x_2,x_3)=(x_1,x_2,x_3)$. Therefore G acts on the topological space S^2. Since S^2 is Hausdorff and G is finite, the action is properly discontinuous. Thus we have the covering map $p': S^2 \to S^2/G$ such that $p'(x)=[x]=Gx$. For each ~~$x\in S$~~ $x=(x_1,x_2,x_3)\in S^2$, we have

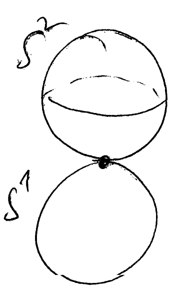
$$[x]=Gx=\{\bar{0}\cdot x,\ \bar{1}\cdot x\}=\{(x_1,x_2,x_3),(-x_1,-x_2,-x_3)\}=\{x,-x\}=\tilde{x}$$

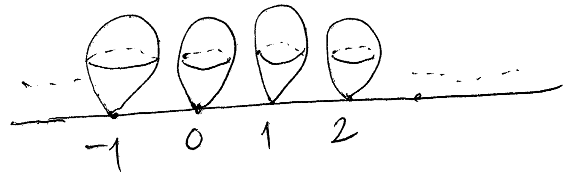
Thus $p=p'$, which is a covering map. 4/4

② Give a description of the universal cover of the space $S^2 \vee S^1$, obtained by gluing together S^2 and S^1 at a single point.

Proof

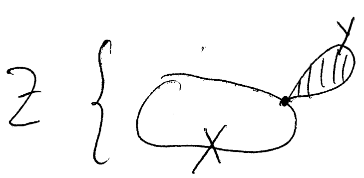
We know that $\mathbb{R}\xrightarrow{p} S^1$, where $p(t)=e^{2\pi i t}$, is the universal covering of S^1, and S^2 is simply connected. Then a universal covering space of $S^2 \vee S^1$ looks as follows

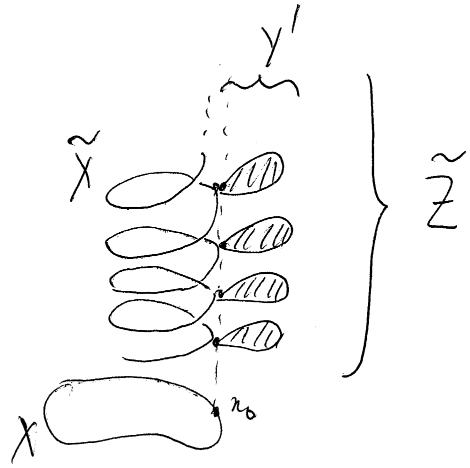




It's the Real line glued together with spheres at integer points. To give the exact covering map, we need many names. So we will deal with the general case which has the same description as follows: Let X be a path-connected topological space based at $x_0 \in X$, and Y be a ~~pathd~~ path-connected and simply connected space based at $y_0 \in Y$. Put $Z = X \coprod_{x_0 \sim y_0} Y$ to be the space obtained by gluing X and Y at x_0 and y_0. Let $\tilde{X} \xrightarrow{p} X$ be the universal covering of X.

We put $Y' = Y \times p^{-1}(x_0)$ where the topology on $p^{-1}(x_0)$ is discrete. Define the maps $f: p^{-1}(x_0) \to Y'$, $f(u) = (y_0, u)$ and $p_1: Y' \to Y$, $p_1(y, u) = y$. We define $\tilde{Z} = \tilde{X} \coprod_f Y'$ to be the adjunct space obtained by identifying u to $f(u)$ for every $u \in p^{-1}(x_0)$.





Define the map $p': \tilde{Z} \to Z$ such that $p'(u) = \begin{cases} p(u) & \text{if } u \in \tilde{X} \\ p_1(u) & \text{if } u \in Y' \end{cases}$

We will show that $\tilde{Z} \xrightarrow{p'} Z$ is the universal covering of Z.

First, we show that p' is well-defined:

We only need to check if the two formulas of $p'(u)$ agree when u coincides one of the gluing point. Let $v \in p^{-1}(x_0)$. Then $f(v) \in Y'$. We need to check $p(v) = p_1(f(v))$. We have $p(v) = x_0$ and $p_1(f(v)) = p_1(y_0, v) = y_0$. Since

x_0 and y_0 are viewed the same on Z, $p_f(v)$ and $p_1(f(v))$ are also consider the same.

Next, we show that p' is continuous. Since p' is continuous on $\tilde{X}$ and on Y' separately, and the two formulas agree on the overlap, p' is continuous on $Z = \tilde{X} \coprod_f Y'$ by Gluing lemma.

Next, we show that for every $v \in Z$, there exists an open neighborhood V of v such that there exists a discrete space F and a homeomorphism $\phi : (p')^{-1}(V) \to V \times F$ such that the following diagram commutes

$$\begin{tikzcd}
(p')^{-1}(V) \arrow[r, "p'"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & V \\
V \times F \arrow[ur, "\text{proj}"'] &
\end{tikzcd}$$

There are 3 cases for v: $v \in X \setminus \{x_0\}$, $v \in Y \setminus \{y_0\}$ or $v = \{x_0, y_0\}$.

$*$ Case 1: $v \in Y \setminus \{y_0\}$

Assuming that Y is a T_1-space, then $\{y_0\}$ is closed on Y. We choose $V = Y \setminus \{y_0\}$. Then V is open in Z. We have

$$\begin{aligned}
(p')^{-1}(V) &= \{u \in \tilde{Z} : p'(u) \in V\} \\
&= \{u \in Y' : p_1(u) \in V\} \\
&= \{u = (y, \alpha) : \alpha \in p^{-1}(x_0), y \in V\} \\
&= V \times p^{-1}(x_0)
\end{aligned}$$

Thus we can choose $F = p^{-1}(x_0)$ and $\phi : (p')^{-1}(V) \to V \times F$ to be the identity map.

* Case 2: $v \in X \setminus \{x_0\}$

Since $\tilde{Y} \xrightarrow{p} X$ is a covering space, there exists an open neighborhood U of v in X such that there exists a discrete space G together with a homeomorphism $\psi: p^{-1}(U) \to U \times G$ such that the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "p"] \arrow[d, "\psi"'] & U \\ U \times G \arrow[ur, "proj"'] & \end{tikzcd}$$

If $U \subset X \setminus \{x_0\}$ then we can choose $V = U$, $F = G$ and $\phi = \psi|_{p^{-1}(V)}$.

If $x_0 \in U$, then by assuming that X is a T_1-space, we have $\{x_0\}$ closed in X and $V = U \setminus \{x_0\}$ open in Z. We have the following diagram.

$$\begin{tikzcd} p^{-1}(x_0) \arrow[r, "p"] \arrow[d, "\psi"'] & x_0 \\ x_0 \times G \arrow[ur, "proj"'] & \end{tikzcd}$$

This diagram implies

$$\begin{tikzcd} p^{-1}(V) \arrow[r, "p"] \arrow[d, "\psi"'] & V \\ V \times G \arrow[ur, "proj"'] & \end{tikzcd}$$

Thus we can choose $F = G$ and $\phi = \psi|_{p^{-1}(V)}$.

* Case 3: $v = \{x_0, y_0\}$

Since $\tilde{Y} \xrightarrow{p} X$ is a covering space, there exists an open neighborhood U of x_0 in X, ~~such that~~ a discrete space G together with a homeomorphism $\phi_1 : p^{-1}(U) \to U \times G$ such that the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "p"] \arrow[d, "\phi_1"'] & U \\ U \times G \arrow[ur, "proj"'] & \end{tikzcd}$$

~~Because~~ As a consequence, we have the following diagram

$$\begin{tikzcd} p^{-1}(x_0) \arrow[r] \arrow[d, "\phi_1"'] & x_0 \\ \{x_0\}\times G \arrow[ur] \arrow[u, phantom, "\circlearrowleft", near start, shift left=3] & \end{tikzcd}$$

Thus $p^{-1}(x_0) \simeq G$ and we can choose $G = p^{-1}(x_0)$.

Then $Y' = Y \times G$. We have two following ~~isomorphisms~~ homeomorphisms:

$$\phi_1 : p^{-1}(U) \to U \times G \quad \text{and} \quad \phi_2 = id : Y' \to Y \times G$$

Gluing $p^{-1}(U)$ and Y' together by identifying $\alpha \sim f(\alpha)$ for every $\alpha \in G$ corresponds to gluing $U \times G$ and $Y \times G$ together by identifying $(x_0, \alpha) \sim (y_0, \alpha)$ for every $\alpha \in G$. This way of gluing spacing induces a homeomorphism between two gluing spaces:

$$\phi : p^{-1}(U) \underset{f}{\amalg} Y' \to (U \times G) \underset{\substack{(x_0,\alpha)\sim(y_0,\alpha) \\ \forall \alpha \in G}}{\amalg} (Y \times G)$$

We have to check if ϕ_1 and ϕ_2 agree on the identified points. To do so, let $\alpha \in p^{-1}(x_0)$, we'll check if $\phi_1(\alpha) = \phi_2(f(\alpha))$. By the diagram on the top of the page and the choice $G = p^{-1}(x_0)$, we get $\phi_1(\alpha) = (x_0, \alpha)$. Moreover,

$$\phi_2(f(\alpha)) = f(\alpha) = (y_0, \alpha).$$

Because (x_0, α) and (y_0, α) are identified, we can say that ϕ_1 and ϕ_2 agree ~~on the~~ at the identified points. We put $V = U \underset{x_0 \sim y_0}{\amalg} Y$. Then V is an open neighborhood of $x=(x_0,y_0)$ in Z. We see that

$$(U\times G)\coprod_{\substack{(x_0,\alpha)\sim(y_0,\alpha)\\ \forall\alpha\in G}}(Y\times G)=\Big(U\coprod_{x_0\sim y_0}Y\Big)\times G=V\times G.$$

On the other hand,

$$(p')^{-1}(V)=\{u\in\tilde{X}\coprod_f Y' : p'(u)\in V\}$$

$$=\{u\in\tilde{X}\coprod_f Y' : p'(u)\in U\coprod_{x_0\sim y_0}Y\}$$

$$=\{u\in\tilde{X} : p_X(u)\in U\coprod_{x_0\sim y_0}Y\}\coprod_f\{u\in Y' : p_1(u)\in \cancel{U}\coprod_{x_0\sim y_0}Y\}$$

$$=\{u\in\tilde{X} : p(u)\in U\}\coprod_f\{u\in Y' : p_1(u)\in Y\}$$

$$=p^{-1}(U)\coprod_f Y'$$

Therefore ϕ is ~~the~~ a map (homeomorphism) from $(p')^{-1}(V)$ to $V\times G$. Now we only need to check if the following diagram commutes:

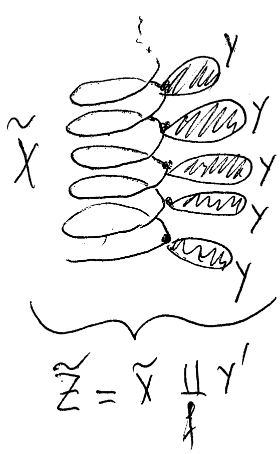
$$\begin{tikzcd}
(p')^{-1}(V) \arrow[r, "p'"] \arrow[d, "\phi"'] & V \\
V\times G \arrow[ur, "\text{proj}"'] &
\end{tikzcd}$$

We have $$\phi(u)=\begin{cases}\phi_1(u), & u\in p^{-1}(U)\\ \phi_2(u)=u, & u\in Y'\end{cases}$$

thus $$\text{proj}\circ\phi(u)=\begin{cases}\text{proj}\circ\phi_1(u), & u\in p^{-1}(U)\\ \text{proj}(u), & u\in Y\end{cases}=\begin{cases}p(u), & u\in p^{-1}(U)\\ p_1(u), & u\in Y\end{cases}=p'(u).$$

~~This completes the proof.~~

The last step is to show that $\tilde{Z}$ is simply connected. Using the fact that the fundamental group of a wedge ~~sum~~ equal the free ~~proproductu~~ product of two fundamental product, i.e. $\pi_1(U \vee V) \cong \pi_1(U) * \pi_1(V)$, we see that a wedge sum of two simply connected spaces is another simply connected space. Thus, by induction, we see that a wedge sum of finitely many simply connected spaces is also simply connected. For any closed path γ in $\tilde{Z}$, $\gamma([0,1])$ is a compact set and thus it must be contained in a ~~pro~~ wedge sum of $\tilde{X}$ and finitely many copies of Y's. This space is simply connected. Thus, γ is homotopic to a trivial loop.



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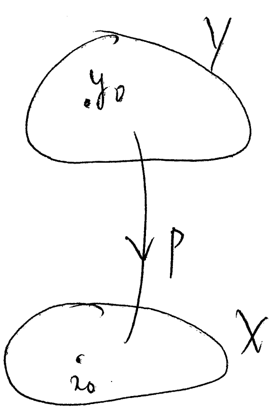
③ Let X and Y be connected spaces, and $p: Y \to X$ be a covering map and $y_0 \in Y$. The point was named y, instead of y_0, in the Problem Set by Prof. Lawson, but the name y is customarily reserved for a variable, not a fixed constant point; thus we rename it as y_0. We denote $x_0 = p(y_0)$ and put $G = \pi_1(X, x_0)$. Since p is a covering map, ~~$H = p_*(\pi_1(X, p($~~ $H = p_*(\pi_1(Y, y_0))$ is a subgroup of G. Let N_H be the normalizer of H in G. We recall the definition

of N_H:
$$N_H = \{x \in G : x^{-1} H x = H\}$$

By this definition, N_H is the largest subgroup of G, in sense of inclusion, in which H is normal. Then we can speak of the factor group N_H/H, which is the family of all cosets $N_H/H = \{xH : x \in N_H\}$.



Put $\mathcal{G} = \{\text{function } f : Y \to Y \text{ homeomorphism such that } pf = p\}$

First we'll show that $\mathcal{G}$ is a group under mapping composition.

- Check closure under composition: let f and g in $\mathcal{G}$. Then fg is also a homeomorphism and $p(fg) = (pf)g = pg = p$. Thus $fg \in \mathcal{G}$.
- Check associativity: since elements of $\mathcal{G}$ are maps, the associativity of composition is automatically satisfied.
- Check unitality: the identity map $id_Y \in \mathcal{G}$ because it's a homeomorphism and $p \circ id = p$.
- Check invertibility: for each $f \in \mathcal{G}$, f is a homeomorphism. Thus f^{-1} is also a homeomorphism. Since $pf = p$, we can multiply both sides by f^{-1} to the right and get $pff^{-1} = pf^{-1}$, or $p = pf^{-1}$. Thus $f^{-1} \in \mathcal{G}$. Therefore

$\mathcal{G}$ is a group.

For each $f \in \mathcal{G}$, we have $pf(y_0) = p(y_0) = x_0$. Thus, $f(y_0) \in p^{-1}(x_0)$. Since Y is connected, there exists a path γ in Y connecting y_0 to $f(y_0)$. The we define the following map

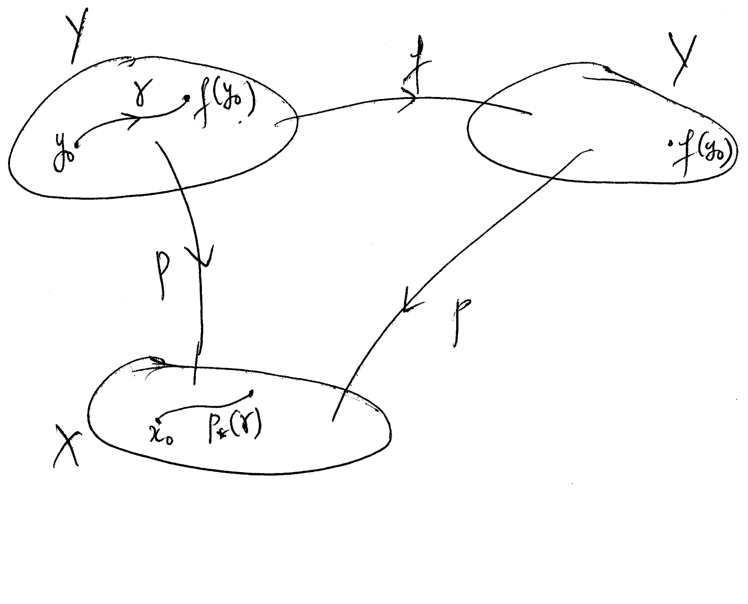
$$\phi : \mathcal{G} \to N_H/H,$$
$$f \mapsto [p\circ\gamma]H$$

hoping that ϕ is a group isomorphism. But first, there are several issues about the definition of ϕ. We need to verify two things:

- The path $p\circ\gamma$, or more precisely the ~~homotopic~~ class of paths homotopic to $p\circ\gamma$, is in N_H.
- the definition doesn't depend on the choice of γ.

<u>Show that $[p\circ\gamma] \in N_H$:</u>

This is equivalent to showing that $[p\circ\gamma]^{-1} H [p\circ\gamma] = H$. Note that $p\circ\gamma$ can be written as $p_*(\gamma)$.



Since f is a homeomorphism from (Y, y_0) to $(Y, f(y_0))$, we have the equality in terms of fundamental groups

$$f_*(\pi_1(Y, y_0)) = \pi_1(Y, f(y_0)).$$

We take p_* both sides and get $p_* f_*(\pi_1(Y, y_0)) = p_*(\pi_1(Y, f(y_0)))$ (1)

$LHS(1) = (p \circ f)_*(\pi_1(Y, y_0)) = p_*(\pi_1(Y, y_0)) = H$. Since γ is a path from y_0 to $f(y_0)$ in Y, each element in $\pi_1(Y, f(y_0))$ can be written as $[\gamma^{-1}.\alpha.\gamma]$ where $[\alpha] \in \pi_1(Y, f(y_0))$. Thus

$$RHS(1) = \{ p_*([\gamma^{-1}][\alpha].[\gamma]) : [\alpha] \in \pi_1(Y, y_0) \}$$

$$= \{ p_*([\gamma])^{-1} \cdot p_*([\alpha]) \cdot p_*([\gamma]) : [\alpha] \in \pi_1(Y, y_0) \}$$

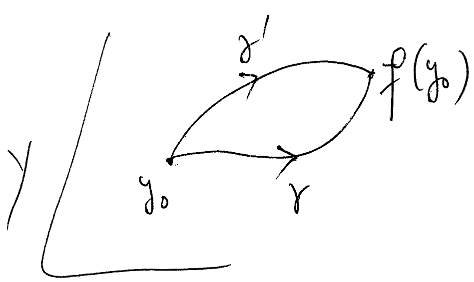
$$= p_*([\gamma])^{-1} \, p_*(\pi_1(Y, y_0)) \, p_*([\gamma])$$

$$= (p \circ \gamma)^{-1} H (p \circ \gamma)$$

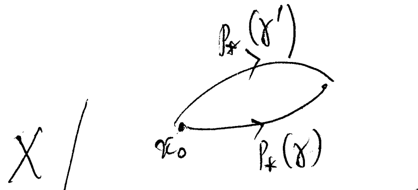
Thus (1) results in $H = (p \circ \gamma)^{-1} H (p \circ \gamma)$.

<u>Show that the definition of ϕ doesn't depend on the choice of γ</u>

Suppose that γ' is another path in Y connecting y_0 to $f(y_0)$. We need to show that $p_*(\gamma')H = p_*(\gamma)H$. We have $p_*(\gamma') = p_*(\gamma.\gamma^{-1}\gamma') = p_*(\gamma)\, p_*(\gamma^{-1}\gamma')$



Since $\gamma^{-1}.\gamma'$ is a loop at y_0 in Y, $\gamma^{-1}.\gamma' \in \pi_1(Y, y_0)$. Thus $p_*(\gamma^{-1}.\gamma') \in p_*(\pi(Y, y_0))$, which is equal to H. Thus $p_*(\gamma^{-1}.\gamma') \in H$ and $p_*(\gamma')H = p_*(\gamma)H$.

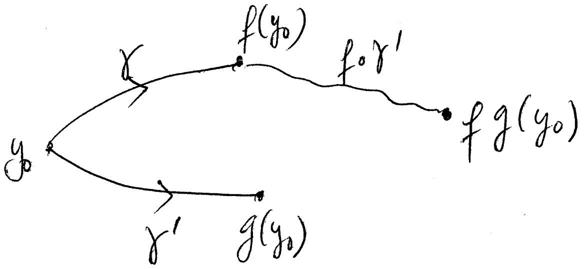


Therefore, ϕ is well-defined.

Next, we'll show in order that ϕ is a group homomorphism, injective and surjective.

<u>Show that ϕ is a group homomorphism</u>

Let $f, g \in \mathcal{G}$, we'll show that $\phi(fg) = \phi(f)\phi(g)$. We have $\cancel{\phi(f) = [p_*(\gamma)]H}$ $\cancel{\text{and } \phi(g) = [p_*(\gamma')]H \text{ where}}$ $\phi(f) = p_*([\gamma])H$ and $\phi(g) = p_*([\gamma'])H$, where γ is a path in Y from y_0 to $f(y_0)$, and γ' a path in Y from y_0 to $g(y_0)$.



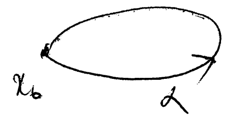
Then $f\circ\gamma'$ is a path from $f(y_0)$ to $fg(y_0)$. Thus $\gamma.(f\circ\gamma')$ is a path from y_0 to $fg(y_0)$. Thus,

$$\begin{aligned}
\phi(fg) = p_*([\gamma.(f\circ\gamma')])H &= p_*([\gamma])\,p_*([f\circ\gamma'])\,H \\
&= p_*([\gamma])\,p_*f_*([\gamma'])\,H \\
&= p_*([\gamma])\,(p\circ f)_*([\gamma'])\,H \\
&= p_*([\gamma])\,p_*([\gamma'])\,H \\
&= \left(p_*([\gamma])H\right)\left(p_*([\gamma'])H\right) \\
&= \phi(f)\,\phi(g)
\end{aligned}$$

Therefore, ϕ is a group homomorphism.

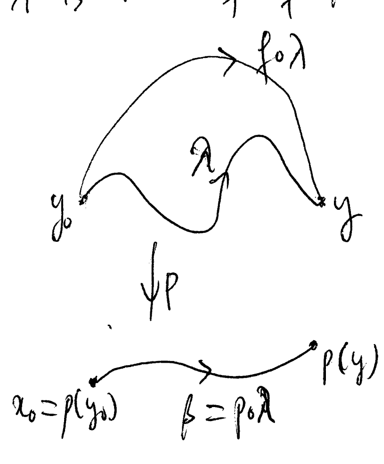
Show that ϕ is injective

Since ϕ is a group homomorphism, it suffices to show that $\ker\phi = \{id\}$.
We'll use the theorem of unique path-lifting several times. First, let $f \in \ker\phi$, we'll show that $f(y_0) = y_0$. Since $\phi(f) = p_*([\gamma])H = H$, we have $p_*([\gamma]) \in H = p_*(\pi_1(Y, y_0))$ where γ is a path in Y from y_0 to $f(y_0)$.



Then there exists $[\alpha] \in \pi_1(X, x_0)$ such that $p_*([\gamma]) = [\alpha]$, and there exists $[\gamma'] \in \pi_1(Y, y_0)$ such that $p_*([\gamma']) = [\alpha]$. Then γ and γ' are two lifts of α in Y at y_0. By the unique path-lifting theorem, $[\gamma] = [\gamma']$. In particular, $\gamma(1) = \gamma'(1) = y_0$. Thus $f(y_0) = y_0$.

Next we'll show that $f(y) = y$ for any $y \in Y$. Take $y \in Y$ and a path λ from y_0 to y. Put $\beta = p\circ\lambda$, be a path in X from $p(y_0) = x_0$ to $p(y)$. Then λ is the lift of β in Y at y_0. We see that $p\circ(f\circ\lambda) = (p\circ f)\circ\lambda = p\circ\lambda = \beta$



and $\cancel{p}(f(\lambda(0))) = \cancel{p}(f(y_0)) = \cancel{p}(y_0)$. Thus $f\circ\lambda$ is another lift of β at y_0. Therefore $f\circ\lambda = \lambda$.
In particular, $y = \lambda(1) = f(\lambda(1)) = f(y)$.

Therefore, $f = id_Y$.

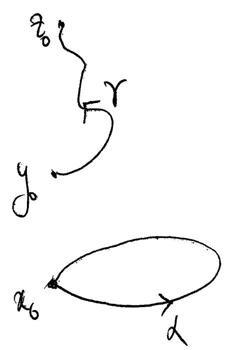
Show that ϕ is surjective

Take $[\alpha] \in N_H$. Then $[\alpha] \in \pi_1(X, x_0)$, i.e. α is a loop based at x_0 in X. Since $[\alpha] \in N_H$, we have $H = [\alpha]^{-1} H [\alpha]$. We'll look for a homeomorphism $Y \xrightarrow{f} Y$ such that the following diagram commutes

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

and $[\alpha]H = p_*([\gamma])H$, where γ is a path in Y from y_0 to $f(y_0)$.

Let γ be the (unique) lift of α in Y at y_0. Put $z_0 = \gamma(1)$. By the definition of γ, we have $\alpha = p \circ \gamma$ and thus $[\alpha] = p_*([\gamma])$.



Therefore, we only need to find a homeomorphism $f: Y \to Y$ such that $f(y_0) = z_0$ and the above diagram commutes. Consider the three base-point spaces (X, x_0), (Y, y_0) and (Y, z_0). We have the two covering mapping $p: (Y, y_0) \to (X, x_0)$ and $p: (Y, z_0) \to (X, x_0)$. Note that the choice of based points doesn't affect where p is a covering map or not. Nevertheless, to avoid confusion of notations, we shall denote the latter covering map by $p': (Y, z_0) \to (X, x_0)$ with the meaning $p'(y) = p(y)$ $\forall y \in Y$.

Since γ is a path from y_0 to z_0, every element of $\pi_1(Y, z_0)$ can be written in the form $[\gamma^{-1}.\beta.\gamma]$ where $[\beta]\in\pi_1(Y,y_0)$. Thus,

$$
\begin{aligned}
p'_*(\pi_1(Y,z_0)) &= \{ p_*([\gamma^{-1}.\beta.\gamma]) : [\beta]\in\pi_1(Y,y_0)\} \\
&= \{ p_*([\gamma])^{-1} p_*([\beta])\, p_*([\gamma]) : [\beta]\in\pi_1(Y,y_0)\} \\
&= \{ [\alpha]^{-1} p_*([\beta])\, \cancel{p_*}[\alpha] : [\beta]\in\pi_1(Y,y_0)\} \\
&= [\alpha]^{-1} p_*(\pi_1(Y,y_0))\, [\alpha] \\
&= [\alpha]^{-1} H [\alpha] \\
&= H
\end{aligned}
$$

Thus, $p'_*(\pi_1(Y,z_0)) = H = p_*(\pi_1(Y,y_0))$. In particular, $p'_*(\pi_1(Y,z_0)) \subset p_*(\pi_1(Y,y_0))$. Thus there exists a continuous map $f: Y\to Y$ that makes the following diagram commutative

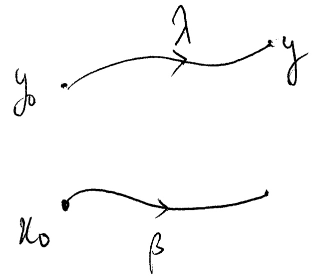
$$
\begin{tikzcd}
(Y,y_0) \arrow[rr, "f"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & (Y,z_0) \arrow[dl, "p'=p"] \\
& (X,x_0) &
\end{tikzcd}
$$

Since ~~$f(y_0)$~~ we have $f(y_0)=z_0$, all we need to do now is to show that f is a homeomorphism. Since $p_*(\pi_1(Y,y_0)) \subset p'_*(\pi_1(Y,z_0))$, there exists a continuous map $g:(Y,z_0)\to(Y,y_0)$ such that the following diagram commutes

$$
\begin{tikzcd}
(Y,z_0) \arrow[rr, "g"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & (Y,y_0) \arrow[dl, "p'=p"] \\
& (X,x_0) &
\end{tikzcd}
$$

Here we have $g(x_0) = y_0$. Thus $gf(y_0) = y_0$. We need to show that $gf = id_Y$. For each $y \in Y$, let λ be a path from y_0 to y and $\beta = p \circ \lambda$. ~~do the~~ Then λ is the lift of β by p at y_0. We see that $gf\lambda(0) = gf(y_0) = y_0$

and $p \circ (gf\lambda) = (pg) f\lambda = pf\lambda = p\lambda = \beta$.



Thus $gf\lambda$ is another lift of β by p at y_0. By the uniqueness of path-lifting, we have $gf\lambda = \lambda$. In particular, $gf\lambda(1) = \lambda(1)$. Thus $gf(y) = y$. Thus,

$gf = id_Y$, and therefore f is a homeomorphism.

<u>Remark</u>

The phenomenon above can be explained by an important property of covering homomorphisms, namely if $q_1 : E_1 \to X$ and $q_2 : E_2 \to X$ are covering maps then any two covering homomorphisms agreeing at one point of E_1 are equal.

$$\begin{tikzcd} E_1 \arrow[rr, bend left, "\phi"] \arrow[rr, "\psi"] \arrow[dr, "q_1"'] & & E_2 \arrow[dl, "q_2"] \\ & X & \end{tikzcd}$$

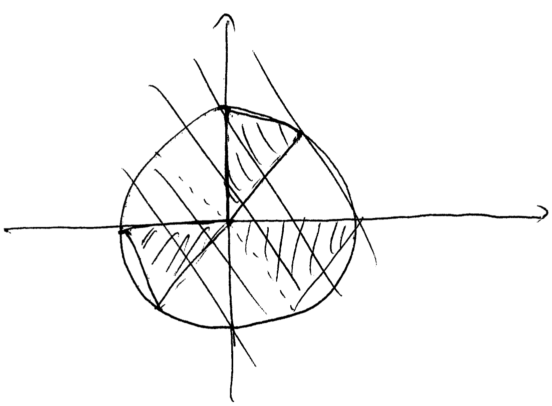
In symbols; if two continuous function ϕ and ψ making the beside diagram commutative then, if there exists $c \in E_1$ such that $\phi(c) = \psi(c)$ then $\phi(y) = \psi(y)$, $\forall y \in E_1$.

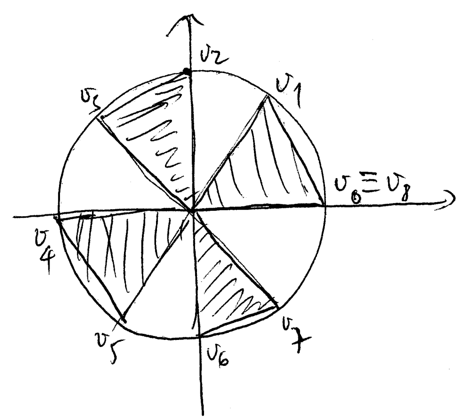
Return ~~to~~ the the problem above, we see that there are two commutative diagrams

$$\begin{tikzcd}(Y,y_0) \arrow[rr,"gf"] \arrow[dr,"p"'] & \arrow[d, phantom, "\circlearrowleft"] & (Y,y_0) \arrow[dl,"p"] \\ & (X,x_0) & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}(Y,y_0) \arrow[rr,"id_Y"] \arrow[dr,"p"'] & \arrow[d, phantom, "\circlearrowleft"] & (Y,y_0) \arrow[dl,"p"] \\ & (X,x_0) & \end{tikzcd}$$

Moreover, $gf(y_0) = y_0 = id_Y(y_0)$. Thus $gf = id_Y$. 4/4

(5) Assume that we have the result in problem 4: the fundamental group of a graph of v vertices, and e edges, is a free group with $1-v+e$ generators. Let F be a free group with n generators. To find a graph whose fundamental group is isomorphic to F, we only need to find a graph such that $1-v+e=n$.





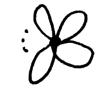
Dividing the unit circle ~~by~~ into $2n$ equal arcs by points $e^{k\frac{\pi}{n}i}$, $0 \le k \le 2n-1$.

We put $v_k = e^{k\frac{\pi}{n}i}$ and the set of vertices is $V = \{0, v_0, v_1, \dots, v_{2n-1}\}$. The edges are the line segments connecting from v_{2k} to v_{2k+1}, from 0 to v_k.

$$\mathcal{E} = \{\{0, v_{2k}\}, \{0, v_{2k+1}\}, \{v_{2k}, v_{2k+1}\} : 0 \le k \le n-1\}$$

This gives us a graph of $(2n+1)$ vertices and $3n$ edges. Thus $1-(2n+1)+3n = n$, Easier to just take $v=1$, $e=n$:

which is the number we need.



Denote the realization of this graph by a topological space X. Then $F=\pi_1(X,x_0)$ is a free group of n generator. Let $H<F$ such that $[F:H]=m$. Then there exists a covering map $p:Y\to X$ such that $p_*(\pi_1(Y,y_0))=H$. Since p_* is an injective, we have $\pi_1(Y,y_0)\cong H$. Thus we need to show that $\pi_1(Y,y_0)$ is a free group with $1+m(n-1)$ generators. With this purpose, we can regard $\pi_1(Y,y_0)=H$. By problem 4, that will be achieved if we can show that Y is homeomorphic to ~~gr~~ connected graph with m vertices and mn edges.

The monodromy action of $F=\pi_1(X,x_0)$ on $p^{-1}(x_0)$ is transitive because Y is ~~con~~ path-connected. Note that $y_0\in p^{-1}(x_0)$. Since H is the image of p_*, we have $H=\mathrm{stab}(y_0)$ - the stablizer of y_0. Moreover, since the action is transitive, $|p^{-1}(x_0)|=|\mathrm{orb}(y_0)|=[F:\mathrm{stab}(y_0)]=[F:H]=m$. Thus $p^{-1}(x_0)$ has m elements. We'll construct a graph whose vertices are these m elements.

This graph won't cover X: each of the v vertices of X has m elements in the fiber, so any covering space of X has mv vertices.

We need to define edges. Since $F=\pi_1(X,x_0)$ has n generators, we call them $[\gamma_1],\dots,[\gamma_n]$. For each $y_j\in p^{-1}(x_0)$, there exists, for each $1\le k\le n$, a unique

lifting path of $[\gamma_k]$ at y_j. The monodromy action gives us $(y_j \cdot [\gamma_k]) \in p^{-1}(x_0)$.
That path, called $[\gamma_{jk}]$, defines an edge from y_j to $y_j \cdot [\gamma_k]$. Thus, we obtain a graph with the vertex-set and edges-set as follow.

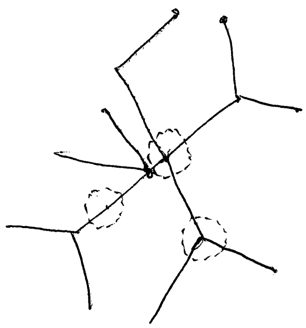
$$\mathcal{V} = \{ y_1, y_2, \dots, y_m \} \;(= p^{-1}(x_0))$$

$$\mathcal{E} = \{ \{y_j, y_j \cdot [\gamma_k]\} : 1 \le j \le m,\ 1 \le k \le n \}$$

this is a graph of m vertices and mn edges. Therefore, what we need to show is that Y is isomorphic to this graph. In otherwords, we'll show that Y is homeomorphic to the realization of this graph. (Not finished) 3/4

4 We'll show that the fundamental group of a connected graph with v vertices and e edges is a free group of $1-v+e$ generators. Let X be the topological space associating with the given graph. We want to show that $\pi_1(X, x_0)$ is a free group with ~~n generators~~ $1-v+e$ generators. Since the given graph is connected, every vertex belongs to some edge. Thus we can travel from one vertex to any other vertices along the edges. Thus X is a connected space and $\pi_1(X, x_0)$ is the same, up to a group homomorphism, if we choose different basepoints. Since the graph has finitely many vertices and edges, every point on the edges (including the vertices) has a simply connected neighborhood. Thus X is locally

Simply connected. In particular, X is semilocally simply connected. Also,



each point on X has a ~~conn~~ path-connected neighborhood. Thus X is locally path-connected. Thus X has a covering space $\tilde{X} \xrightarrow{p} X$. We apply the result of Problem 3 for $Y = \tilde{X}$. Here $H = p_*(\pi_1(\tilde{X}, y_0)) = \{e\}$ because $\tilde{X}$ is simply connected. Then $N_H = N_{\{e\}} = \pi_1(X, x_0)$. Thus $N_H / H = \pi_1(X, x_0)/\{e\} \simeq \pi_1(X, x_0)$. Thus the group isomorphism ϕ is now between the group of all deck transformations $\mathcal{G}$ to $\pi_1(X, x_0)$.

Hence, to show that $\pi_1(X, x_0)$ is free and has $1 - v + e$ generators is equivalent to show that $\mathcal{G}$ is free and has $1 - v + e$ generators.

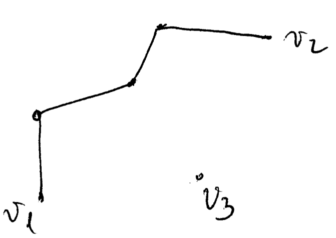
In oder to show that a group is free, we have only one tool so far. That is to show that every element in this group is repepented uniquely in a reduced form of elements in the set of generators. That means, first we should know what the generators are. This means we must use the fact that X is a graph of v vertices and e edges. We'll find them by means of maximal tree. A tree is a connected graph without cycles, as defined in Problem 5, HW6.

We begin with a vertex v_1. If there is another vertex, called v_2, then v_2 must be connected to v_1 by a path of edges. This path ~~may~~ can be viewed as a sequence of vertices it passes by. ~~Th~~In this sequence of vertices, one ~~vertex~~ vertex may occur more than once. However, we can reduce our path such that each vertex occurs at most once. We do so by collapsing

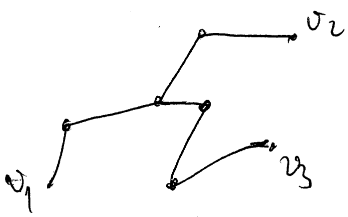
$$a\underbrace{bcdefb}_{\text{collapse}}g \longrightarrow abg$$

subpaths withe the same starting point ~~and~~ and ending point. Thus, we can connect v_2 to v_1 with a path such that each vertex occurs at most once. Consequently, each edge also occur at most once (because if it occurs twice in the path then it two terminal points will occur twice).

Now we have a tree with vertices are those on the paths and edges are those on the path. If there are still some vertex, called v_3, outside of the tree, then there exists a path from v_3 to v_1. We ~~travel from v_3 along the path~~ reduce this path like above so that every vertex occurs at most once on the path. Then we travel from v_3 to v_1 along the reduced path until we first meet ~~the~~ one ~~element~~ vertex in

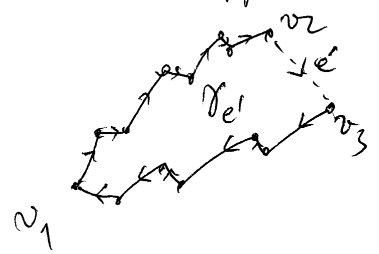


the existing tree. The existing tree together with the new paths form another tree. We continue this procedure. This must stop because we have only finitely many vertices and edges.

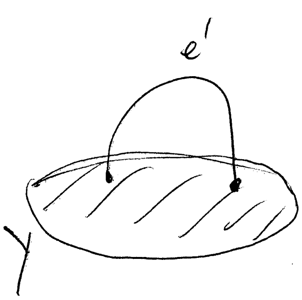


Thus, at the end there is no vertices lying outside the existing tree. At that time, the tree encompasses ~~passes~~ ~~through~~ all vertices, and is called a maximal tree. ~~Sin~~ This tree has v vertices. Since the tree has no cycles, the number of edges is $v-1$. We can show this by induction as follows: we start from any vertex v_1 and choose any edge to go to ~~the~~ a next vertex and keep doing so ~~with~~ respecting the rule that no vertices are met twice. The tour must end at v_2, for example, because the numbers of vertices and edges are finite. Then v_2 only belongs to one edge. The tree with v_2 and this edge excluded is also a tree with the number of vertices decreasing by 1. With the base case "the tree with one vertex has no edge", we conclude that a tree with v vertices has $v-1$ edges. Therefore, there are $e-(v-1) = 1-v+e$ edges in the graph X not contained in the maximal tree. For each of these edges, added into the tree, we will have a cycle.

Indeed, suppose that the added edge is the edge from v_2 to v_3.



Then we have a cycle; traveling from v_1 to v_2 by ~~pat~~ edges in the tree, then going from v_2 to v_3 by e', and then going from v_3 back to v_1 by edges in the tree. The tree with this edges e' added is topologically the same as the topological space obtained by gluing a simply



connected space Y (which is the tree in our case) and an interval (line segment) at two terminal points. By the result of problem 5, HW7, this space has the fundamental group $\pi_1(Y) * \mathbb{Z}$, which is isomorphic to $\mathbb{Z}$ (since $\pi_1(Y) = \{e\}$). Thus it is a cyclic group with one generator. The path $\gamma^*_{e'}$ above must be a generator of this group because any other cycle that is not a generator will pass through some vertex more than once. Therefore, ~~wid~~ we have $1-v+e$ candidates for generators of $\pi_1(X, x_0)$. Equivalently, we have $1-v+e$ candidates for generators of $\mathcal{G}$ - the group of deck transformations.

(Not finished)

This is enough: by induction $\pi_1(X,x_0) \cong \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{1-v+e}$ and you know that a free product of free groups is free.

3/4