Name: Tuan Pham
ID: 4652218

Math 8301: Manifolds and Topology
Homework 9 14/20

(1) We'll find the homology groups of the complete graph on 4 vertices.
The procedure of finding the homology groups of a topological space is as follow.

1) Find a triangulation of its, which gives rise to a simplicial complex representation of the topological space.

2) Compute the groups of n-dimensional chains C_n for each $n = 0, 1, 2, \ldots$

3) Compute the boundary operators $\partial : C_n \to C_{n-1}$ for each $n = 1, 2, \ldots$

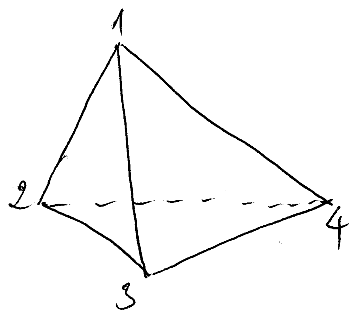
4) Find n-cycles Z_n and n-boundaries B_n for each $n = 0, 1, 2, \ldots$

5) Find the homology group $H_n = Z_n / B_n$ for each $n = 0, 1, 2, \ldots$

Return to our problem. We'll follow exactly the above procedure.

<u>Step 1</u> Find a triangulation.

The complete graph on 4 vertices has the visualization as a tetrahedron whose vertices are label 1, 2, 3, 4 as in the picture. Thus, the vertex set is $V = \{\{1\}, \{2\}, \{3\}, \{4\}\}$. The set of triangles is $F = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$



Here we realize that we misunderstood the problem. The given topological space is a graph, and thus it is a 1-simplicial complex. The set of vertices and edges are $V = \{\{1\},\{2\},\{3\},\{4\}\}$ and $F = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$.
The set of all faces can be obtained from F by adding all vertices to it.

Step 2 Compute the groups of n-chains

$$C_n = \left\{ \sum_{\substack{\sigma\in F\\ |\sigma|=n+1}} n_\sigma \sigma \;\middle|\; \begin{array}{l} n_\sigma \in \mathbb{Z}, \text{ only finitely} \\ \text{many } n_\sigma \text{ are nonzero} \end{array} \right\} = \bigoplus_{\substack{\sigma\in F\\ |\sigma|=n+1}} \mathbb{Z}.\sigma$$

Then $C_0 = \mathbb{Z}\{1\} \oplus \mathbb{Z}\{2\} \oplus \mathbb{Z}\{3\} \oplus \mathbb{Z}\{4\}$

$C_1 = \mathbb{Z}\{1,2\} \oplus \mathbb{Z}\{1,3\} \oplus \mathbb{Z}\{1,4\} \oplus \mathbb{Z}\{2,3\} \oplus \mathbb{Z}\{2,4\} \oplus \mathbb{Z}\{3,4\}$

$C_n = \{0\}$ for all $n \geq 2$ because F has no n-dimensional faces.

Step 3 Compute the boundary operators

For $n = 1$, we have $\partial : C_1 \to C_0$ given by

$$\partial\big(a_1\{1,2\} + a_2\{1,3\} + a_3\{1,4\} + a_4\{2,3\} + a_5\{2,4\} + a_6\{3,4\}\big)$$
$$= (-a_1 - a_2 - a_3)\{1\} + (a_1 - a_4 - a_5)\{2\} + (a_2 + a_4 - a_6)\{3\}$$
$$+ (a_3 + a_5 + a_6)\{4\}$$

because $\partial(\{i,j\}) = \{j\} - \{i\}$ for all pairs $i<j$.

For $n \geq 2$, we have $\partial : C_n \to C_{n-1}$ is a trivial group morphism because $C_n = \{0\}$.

$$\begin{array}{rc}
 & \vdots \\
C_3 = & 0 \\
 & \downarrow \\
C_2 = & 0 \\
 & \downarrow \\
C_1 = & \mathbb{Z}\{1,2\} \oplus \mathbb{Z}\{1,3\} \oplus \mathbb{Z}\{1,4\} \oplus \mathbb{Z}\{2,3\} \oplus \mathbb{Z}\{2,4\} \oplus \mathbb{Z}\{3,4\} \\
 & \downarrow \partial \\
C_0 = & \mathbb{Z}\{1\} \oplus \mathbb{Z}\{2\} \oplus \mathbb{Z}\{3\} \oplus \mathbb{Z}\{4\} \\
 & \downarrow \\
 & 0
\end{array}$$

<u>Step 4</u> Find n-cycles and n-boundaries.

For $n=0$, all chains are 0-cycles. Thus $Z_0 = C_0$. Take a generic 0-chain $n_1\{1\} + n_2\{2\} + n_3\{3\} + n_4\{4\}$. It is an 0-boundary if and only if there exists $a_1, \dots, a_6 \in \mathbb{Z}$ such that

$$\partial(a_1\{1,2\} + \dots + a_6\{3,4\}) = n_1\{1\} + \dots + n_4\{4\}$$

With the formula we found in Step 3, we get a system of equations

$$\begin{cases} -a_1 - a_2 - a_3 = n_1 \\ a_1 - a_4 - a_5 = n_2 \\ a_2 + a_4 - a_6 = n_3 \\ a_3 + a_5 + a_6 = n_4 \end{cases}$$

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In matrix form, we have

$$\left[\begin{array}{cccccc|c} -1 & -1 & -1 & 0 & 0 & 0 & n_1 \\ 1 & 0 & 0 & -1 & -1 & 0 & n_2 \\ 0 & 1 & 0 & 1 & 0 & -1 & n_3 \\ 0 & 0 & 1 & 0 & 1 & 1 & n_4 \end{array}\right] \xrightarrow[r_4 := r_4 + r_3]{\substack{r_2 := r_2 + r_1 \\ r_3 := r_3 + r_2}} \left[\begin{array}{cccccc|c} -1 & -1 & -1 & 0 & 0 & 0 & n_1 \\ 0 & -1 & -1 & -1 & -1 & 0 & n_2 + n_1 \\ 0 & 0 & -1 & 0 & -1 & -1 & n_3 + n_2 + n_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_4 + n_3 + n_2 + n_1 \end{array}\right]$$

Thus the system has integer solutions $a_1, \dots, a_6$ iff $n_1 + n_2 + n_3 + n_4 = 0$.

Therefore, $B_0 = \{ n_1\{1\} + n_2\{2\} + n_3\{3\} + n_4\{4\} \, ; \, n_1 + n_2 + n_3 + n_4 = 0 \}$.

$\simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

For $n = 1$, since $\partial : C_2 \to C_1$ is a trivial group morphism, $B_1 = \{0\}$.

We'll find Z_1. $a_1\{1,2\} + \dots + a_6\{3,4\}$ is a 1-cycle iff $\partial(a_1\{1,2\} + \dots + a_6\{3,4\}) = 0$,

which is equivalent to

$$\begin{cases} -a_1 - a_2 - a_3 = 0 \\ a_1 - a_4 - a_5 = 0 \\ a_2 + a_4 - a_6 = 0 \\ a_3 + a_5 + a_6 = 0 \end{cases}$$

The above matrix form is now equivalent to

$$\begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Thus $\cancel{a_3 = a,\ a_5 = b,\ a_6 = c}$

$$\cancel{a_2 =}\quad \begin{cases} a_5 = a,\ a_6 = b,\ a_3 = -a-b \\ a_4 = c,\ a_2 = -a_3 - a_4 - a_5 = b - c \\ a_1 = -a_2 - a_3 = c + a \end{cases}$$

Thus $a_1\{1,2\} + a_2\{1,3\} + a_3\{1,4\} + a_4\{2,3\} + a_5\{2,4\} + a_6\{3,4\}$

$$= (c+a)\{1,2\} + (b-c)\{1,3\} + (-a-b)\{1,4\} + c\{2,3\} + a\{2,4\} + b\{3,4\}$$

$$= a\big(-\{1,4\} + \{2,4\} + \{1,2\}\big) + b\big(\{1,3\} - \{1,4\} + \{3,4\}\big) + c\big(\{1,2\} - \{1,3\} + \{2,3\}\big)$$

Therefore Z_1 is generated by thee cycles

$$Z_1 = \mathbb{Z}\big(-\{1,4\} + \{2,4\} + \{1,2\}\big) \oplus \mathbb{Z}\big(\{1,3\} - \{1,4\} + \{3,4\}\big) \oplus \mathbb{Z}\big(\{1,2\} - \{1,3\} + \{2,3\}\big)$$

$$\simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

For $n \geq 2$, there is no n-cycles nor n-boundaries. Thus $Z_n = B_n = \{0\}$.

<u>Step 5</u> The homology groups are

$$H_0 = Z_0 / B_0 = \big(\mathbb{Z}\{1\} \oplus \mathbb{Z}\{2\} \oplus \mathbb{Z}\{3\} \oplus \mathbb{Z}\{4\}\big) / \big\{ n_1\{1\} + n_2\{2\} + n_3\{3\} + n_4\{4\} : n_1 + \dots + n_4 = 0 \big\}$$
$$\simeq \mathbb{Z},$$

$$H_1 = Z_1 / B_1 \simeq Z_1 \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},$$

$$H_n = \{0\} \quad \text{for all } n \geq 2.$$

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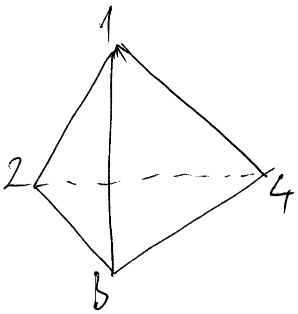
(2) We'll find the homology groups of the 2-sphere S^2.

<u>Step 1</u> Find a triangulation



The sphere is triangulated as the boundary of a tetrahedron. Thus, the vertex set is

$$V = \{1, 2, 3, 4\}$$

the set of triangles ~~are~~ is $K = \{123, 124, 134, 234\}$.

The set of faces F is obtained from K by following the rule: if $\sigma \in F$ then $\tau \in F$ for all $\tau \subset \sigma$.

$$F = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{13\}, \{14\}, \{23\}, \{24\}, \{34\}, \{123\}, \{124\}, \{134\}, \{234\}\}$$

<u>Step 2</u> Compute the groups of n-chains for $n = 0, 1, 2 \dots$

$$C_n = \left\{ \sum_{\substack{\sigma \in F \\ |\sigma| = n+1}} n_\sigma \sigma \;\middle|\; \begin{array}{l} n_\sigma \in \mathbb{Z}, \text{ only finitely many} \\ n_\sigma \text{ are nonzero.} \end{array} \right\} = \bigoplus_{\substack{\sigma \in F \\ |\sigma| = n+1}} \mathbb{Z}\sigma.$$

Then

$$C_0 = \mathbb{Z}\{1\} \oplus \mathbb{Z}\{2\} \oplus \mathbb{Z}\{3\} \oplus \mathbb{Z}\{4\}$$

$$C_1 = \mathbb{Z}\{12\} \oplus \mathbb{Z}\{13\} \oplus \mathbb{Z}\{14\} \oplus \mathbb{Z}\{23\} \oplus \mathbb{Z}\{24\} \oplus \mathbb{Z}\{34\}$$

$$C_2 = \mathbb{Z}\{123\} \oplus \mathbb{Z}\{124\} \oplus \mathbb{Z}\{134\} \oplus \mathbb{Z}\{234\}$$

$C_n = \{0\}$ for all $n \geq 3$ because F has no faces of n-dimension.

Step 3 Compute the boundary operators

- For $n=1$, we have $\partial : C_1 \to C_0$ given by

$$\partial\left(a_1\{12\} + a_2\{13\} + a_3\{14\} + a_4\{23\} + a_5\{24\} + a_6\{34\}\right)$$

$$= (-a_1 - a_2 - a_3)\{1\} + (a_1 - a_4 - a_5)\{2\} + (a_2 + a_4 - a_6)\{3\} + (a_3 + a_5 + a_6)\{4\}$$

because $\partial\{ij\} = \{j\} - \{i\}$ for all pairs $i<j$.

- For $n=2$, we have

$$\partial\{123\} = \{23\} - \{13\} + \{12\}$$

$$\partial\{124\} = \{24\} - \{14\} + \{12\}$$

$$\partial\{134\} = \{34\} - \{14\} + \{13\}$$

$$\partial\{234\} = \{34\} - \{24\} + \{23\}$$

Thus $\partial(b_1\{123\} + b_2\{124\} + b_3\{134\} + b_4\{234\})$

$$= (b_1 + b_2)\{12\} + (-b_1 + b_3)\{13\} + (-b_2 - b_3)\{14\} + (b_1 + b_4)\{23\} + (b_2 - b_4)\{24\} + (b_3 + b_4)\{34\}.$$

- For $n \geq 3$, since $C_n = \{0\}$, the boundary $\partial : C_n \to C_{n-1}$ is trivial.

Step 4 Find n-cycles Z_n and n-boundaries B_n for each $n = 0, 1, 2, \dots$

- For $n=0$: all 0-chains are cycles. Thus $Z_0 = C_0$. Take a generic 0-chain $n_1\{1\} + n_2\{2\} + n_3\{3\} + n_4\{4\}$. It is a 0-boundary if and only if there exists $a_1, \dots, a_6 \in \mathbb{Z}$ such that $\partial(a_1\{12\} + \dots + a_6\{13\}) = n_1\{1\} + \dots + n_4\{4\}$.

this is the same as in the case $n=0$ in the previous problem where we obtain $B_0 = \{n_1\{1\}+n_2\{2\}+n_3\{3\}+n_4\{4\} : n_1+n_2+n_3+n_4=0\}$

$$= \{n_1(\{1\}-\{4\}) + n_2(\{2\}-\{4\}) + n_3(\{3\}-\{4\}) : n_1, n_2, n_3 \in \mathbb{Z}\}$$

$$= \mathbb{Z}(\{1\}-\{4\}) \oplus \mathbb{Z}(\{2\}-\{4\}) \oplus \mathbb{Z}(\{3\}-\{4\})$$

$$\simeq \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}.$$

- For $n=1$: a 1-cycle is of the form $a_1\{12\}+\dots+a_6\{34\}$ such that $\partial(a_1\{12\}+\dots+a_6\{34\})=0$, which is equivalent to

$$\begin{cases} -a_1-a_2-a_3=0 \\ a_1-a_4-a_5=0 \\ a_2+a_4-a_6=0 \\ a_3+a_5+a_6=0 \end{cases}$$

this is the same case as in the previous problem, where we obtained

$$Z_1 = \mathbb{Z}(-\{14\}+\{24\}+\{12\}) \oplus \mathbb{Z}(\{13\}-\{14\}+\{34\}) \oplus \mathbb{Z}(\{12\}-\{13\}+\{23\})$$

$$\simeq \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}.$$

A 1-boundary is of the form $a_1\{12\}+\dots+a_6\{34\}$ such that there are integers b_1, b_2, b_3, b_4 such that $\partial(b_1\{123\}+b_2\{124\}+b_3\{134\}+b_4\{134\}) = a_1\{12\}+\dots+a_6\{34\}$.

This equation results in 6 linear equations as follow.

$$\begin{cases} b_1 + b_2 = a_1 \\ -b_1 + b_3 = a_2 \\ -b_2 - b_3 = a_3 \\ b_1 + b_4 = a_4 \\ b_2 - b_4 = a_5 \\ b_3 + b_4 = a_6 \end{cases}$$

or in matrix form:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a_1 \\ -1 & 0 & 1 & 0 & a_2 \\ 0 & -1 & -1 & 0 & a_3 \\ 1 & 0 & 0 & 1 & a_4 \\ 0 & 1 & 0 & -1 & a_5 \\ 0 & 0 & 1 & 1 & a_6 \end{array}\right]$$

$$\xrightarrow[r_4 = r_4 - r_1]{r_2 = r_2 + r_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a_1 \\ 0 & 1 & 1 & 0 & a_2 + a_1 \\ 0 & -1 & -1 & 0 & a_3 \\ 0 & -1 & 0 & 1 & a_4 - a_1 \\ 0 & 1 & 0 & -1 & a_5 \\ 0 & 0 & 1 & 1 & a_6 \end{array}\right] \xrightarrow[\substack{r_4 = r_4 + r_2 \\ r_5 = r_5 - r_2}]{r_3 = r_3 + r_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a_1 \\ 0 & 1 & 1 & 0 & a_2 + a_1 \\ 0 & 0 & 0 & 0 & a_3 + a_2 + a_1 \\ 0 & 0 & 1 & 1 & a_4 + a_2 \\ 0 & 0 & -1 & -1 & a_5 - a_2 - a_1 \\ 0 & 0 & 1 & 1 & a_6 \end{array}\right]$$

$$\xrightarrow[r_6 = r_6 - r_4]{r_5 = r_5 + r_4} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a_1 \\ 0 & 1 & 1 & 0 & a_2 + a_1 \\ 0 & 0 & 0 & 0 & a_3 + a_2 + a_1 \\ 0 & 0 & 1 & 1 & a_4 + a_2 \\ 0 & 0 & 0 & 0 & a_5 - a_1 + a_4 \\ 0 & 0 & 0 & 0 & a_6 - a_4 - a_2 \end{array}\right] \xrightarrow{r_3 \leftrightarrow r_4} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a_1 \\ 0 & 1 & 1 & 0 & a_2 + a_1 \\ 0 & 0 & 1 & 1 & a_4 + a_2 \\ 0 & 0 & 0 & 0 & a_3 + a_2 + a_1 \\ 0 & 0 & 0 & 0 & a_5 - a_1 + a_4 \\ 0 & 0 & 0 & 0 & a_6 - a_4 - a_2 \end{array}\right]$$

The system has integer solutions b_1, b_2, b_3, b_4 if and only if

$$\begin{cases} a_1 + a_2 + a_3 = 0 \\ -a_1 + a_4 + a_5 = 0 \\ -a_2 \mp a_4 + a_6 = 0 \end{cases} \xrightarrow[r_3 = r_3 + r_4]{r_2 = r_2 + r_1} \begin{cases} a_1 + a_2 + a_3 = 0 \\ a_2 + a_3 + a_4 + a_5 = 0 \\ a_3 + a_5 + a_6 = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} a_6 = a \\ a_5 = b \\ a_4 = c \\ a_3 = -a-b \\ a_2 = -c-b-(-a-b) = a-c \\ a_1 = -(a-c)-(-a-b) = c-a+a+b = b+c \end{cases}$$

Thus, $B_1 = \{ (b+c)\{12\} + (a-c)\{13\} + (-a-b)\{14\} + c\{23\} + b\{24\} + a\{34\} \,/\, a,b,c \in \mathbb{Z} \}$

$= \{ a(\{34\} - \{14\} + \{13\}) + b(\{1,2\} - \{14\} + \{24\}) + c(-\{13\} + \{23\} + \{12\}) \,/\, a,b,c \in \mathbb{Z} \}$

$= \mathbb{Z}(\{34\} - \{14\} + \{13\}) \oplus \mathbb{Z}(\{12\} - \{14\} + \{24\}) \oplus \mathbb{Z}(-\{13\} + \{23\} + \{12\})$

$= Z_1$.

- For $n = 2$:

Because there is no 3-dimensional face in F, there is no 2-boundary. Thus $B_2 = \{0\}$.

We'll find Z_2. A chain $b_1\{123\} + b_2\{124\} + b_3\{134\} + b_4\{234\}$ is a cycle iff

$$\partial(b_1\{123\} + b_2\{124\} + b_3\{134\} + b_4\{234\}) = 0$$

This results in a system of ~~four~~ 6 equations as in the previous case where $a_1, \dots, a_6$

are replaced by 0's. Then the matrix form at the very end is

$$\begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the solutions are

$$\begin{cases} b_4 = a \\ b_3 = -a \\ b_2 = a \\ b_1 = -a \end{cases}$$

Thus $Z_2 = \{-a\{123\} + a\{124\} - a\{134\} + a\{234\} \,/\, a \in \mathbb{Z}\}$

$= \mathbb{Z}(-\{123\} + \{124\} - \{134\} + \{234\})$

Step 5 The groups of homology are

$H_0 = Z_0/B_0 = \left(\mathbb{Z}\{1\} \oplus \mathbb{Z}\{2\} \oplus \mathbb{Z}\{3\} \oplus \mathbb{Z}\{4\}\right) \Big/ \left(\mathbb{Z}(\{1\}-\{4\}) \oplus \mathbb{Z}(\{2\}-\{4\}) \oplus \mathbb{Z}(\{3\}-\{4\})\right)$
$\cong \mathbb{Z}$

$H_1 = Z_1/B_1 = \{0\}$ because $Z_1 = B_1$,

$H_2 = Z_2/B_2 = Z_2 = \mathbb{Z}(-\{123\} + \{124\} - \{134\} + \{234\})$

$H_n = Z_n/B_n = \{0\}$ for all $n \geq 3$ because $Z_n = \{0\}$.

3/4

(4) We'll find the homology groups of the projective plane $\mathbb{R}P^2$

Step 1: Find a triangulation.

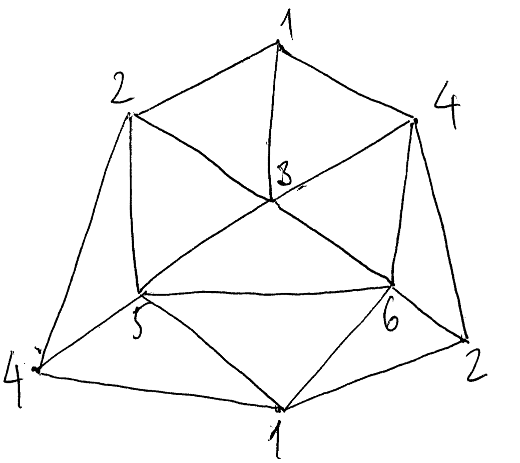
By Problem 4, HW2, $\mathbb{R}P^2$ can be ~~trang~~ triangulated by 6 vertices and 10 faces.

The set of vertices ~~are~~ is

$V = \{1,2,3,4,5,6\}$

The set of triangles is $K = \{123, 126, 134, 145, 156, 235, 245, 246, 346, 356\}$

The set of all faces is obtained by adding subfaces to K following the rule: if $\sigma \in F$ then $\tau \in F$ for all $\tau \subset \sigma$.



<u>Step 2</u> Compute the groups of n-chains for $n = 0, 1, 2, \dots$

$$C_n = \left\{ \sum_{\substack{\sigma\in F \\ |\sigma|=n+1}} n_\sigma \sigma \;\middle|\; \begin{array}{c} n_\sigma \in \mathbb{Z}, \text{ only finitely many} \\ n_\sigma \text{ are nonzero} \end{array} \right\} = \bigoplus_{\substack{\sigma\in F \\ |\sigma|=n+1}} \mathbb{Z}\sigma$$

then $C_0 = \mathbb{Z}\{1\} \oplus \dots \oplus \mathbb{Z}\{6\}$

$$C_1 = \mathbb{Z}\{12\} \oplus \mathbb{Z}\{13\} \oplus \mathbb{Z}\{14\} \oplus \mathbb{Z}\{15\} \oplus \mathbb{Z}\{16\} \oplus \mathbb{Z}\{23\} \oplus \mathbb{Z}\{24\} \oplus \mathbb{Z}\{25\} \oplus \mathbb{Z}\{26\} \oplus \mathbb{Z}\{34\} \oplus \mathbb{Z}\{35\} \oplus \mathbb{Z}\{36\} \oplus \mathbb{Z}\{45\} \oplus \mathbb{Z}\{46\} \oplus \mathbb{Z}\{56\}$$

$$C_2 = \mathbb{Z}\{123\} \oplus \mathbb{Z}\{126\} \oplus \mathbb{Z}\{134\} \oplus \mathbb{Z}\{145\} \oplus \mathbb{Z}\{156\} \oplus \mathbb{Z}\{235\} \oplus \mathbb{Z}\{245\} \oplus \mathbb{Z}\{246\} \oplus \mathbb{Z}\{346\} \oplus \mathbb{Z}\{356\}$$

$C_n = \{0\}$ for all $n \geq 3$.

Step 3 Compute the boundary operators.

- For $n = 0$, since $\partial\{ij\} = \{j\} - \{i\}$ for all pairs $i<j$, we get

$$
\begin{aligned}
\partial(a_1\{12\} + \cdots + a_{15}\{56\}) = {} & (-a_1 - a_2 - a_3 - a_4 - a_5)\{1\} \\
& + (a_1 - a_6 - a_7 - a_8 - a_9)\{2\} \\
& + (a_2 + a_6 - a_{10} - a_{11} - a_{12})\{3\} \\
& + (a_3 + a_7 + a_{10} - a_{13} - a_{14})\{4\} \\
& + (a_4 + a_8 + a_{11} + a_{13} - a_{15})\{5\} \\
& + (a_5 + a_9 + a_{12} + a_{14} + a_{15})\{6\}
\end{aligned}
$$

- For $n = 1$, since $\partial\{ijk\} = \partial\{jk\} - \partial\{ik\} + \partial\{ij\}$ for all $i<j<k$, we get

$$
\begin{aligned}
\partial(b_1\{123\} + \cdots + b_{10}\{356\}) = {} & (b_1 + b_2)\{12\} + (-b_1 + b_3)\{13\} + (-b_3 + b_4)\{14\} \\
& + (-b_4 + b_5)\{15\} + (-b_2 - b_5)\{16\} + (b_1 + b_6)\{23\} \\
& + (b_7 + b_8)\{24\} + (-b_6 - b_7)\{25\} + (b_2 - b_8)\{26\} \\
& + (b_3 + b_9)\{34\} + (b_6 + b_{10})\{35\} + (-b_9 - b_{10})\{36\} \\
& + (b_4 + b_7)\{45\} + (b_8 + b_9)\{46\} + (b_5 + b_{10})\{56\}
\end{aligned}
$$

- For $n \geq 2$, $\partial = 0$ because there's no face of n-dimension.

Step 4 Find Z_n and B_n for each $n = 0, 1, 2, \ldots$

$$Z_n = \ker(C_n \to C_{n-1}) \ , \ B_n = \ker(\partial : C_{n+1} \to C_n)$$

Thus,
$$Z_0 = C_0$$
$$B_0 = ?$$
$$Z_1 = ?$$
$$B_1 = ?$$
$$Z_2 = ?$$
$$B_2 = 0$$
$$Z_n = B_n = \{0\} \quad \forall n \geq 3$$

• Compute B_0

$$B_0 = \{\partial(a_1\{12\} + \cdots + a_{15}\{56\}) \,/\, a_i \in \mathbb{Z}\}$$
$$= \{a_1(\{2\} - \{1\}) + \cdots + a_{15}(\{6\} - \{5\}) \,/\, a_i \in \mathbb{Z}\}$$
$$= \langle \{2\}-\{1\}, \{3\}-\{1\}, \ldots, \{6\}-\{5\} \rangle$$ - the abelian group generated by 15 elements.

This is a subgroup of a free ~~abel~~ abelian group generated by 6 elements, so it is also a free abelian group. To find a basis for B_0, we only need to find the linear independent rows of the following matrix by means of Gaussian elimination.

$$\begin{array}{c} \begin{array}{cccccc} \{1\} & \{2\} & \{3\} & \{4\} & \{5\} & \{6\} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} -1 & +1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \xrightarrow[\text{transforms}]{\text{row}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ & & 0 & & & \end{bmatrix}$$

Therefore, B_0 is the free abelian group generated by $\{\{1\}-\{6\}, \{2\}-\{6\}, \{3\}-\{6\}, \{4\}-\{6\}, \{5\}-\{6\}\}$

$$\simeq \bigoplus_{i=1}^{5} \mathbb{Z}.$$

- Compute Z_1

$$Z_1 = \{ a_1\{12\} + \dots + a_{15}\{56\} \;/\; \partial(a_1\{12\} + \dots + a_{15}\{56\}) = 0 \}$$

Thus we need to solve a system of 6 equations in 15 unknowns. The system in matrix form is

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$$
\begin{array}{ccccccccccccccc}
a_1 & a_2 & & & & \cdots & & & & & & & & & a_{15} \\
\downarrow & \downarrow & & & & & & & & & & & & & \downarrow
\end{array}
$$

$$
\begin{bmatrix}
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
$$

$$
\Longleftrightarrow
\begin{cases}
a_1 = r_7 + r_8 + r_9 + r_{10} \\
a_2 = r_4 + r_5 + r_6 - r_{10} \\
a_3 = r_2 + r_3 - r_6 - r_9 \\
a_4 = r_1 - r_3 - r_5 - r_8 \\
a_5 = -r_1 - r_2 - r_4 - r_7 \\
a_6 = r_{10} \\
a_7 = r_9 \\
a_8 = r_8 \\
a_9 = r_7 \\
a_{10} = r_6 \\
a_{11} = r_5 \\
a_{12} = r_4 \\
a_{13} = r_3 \\
a_{14} = r_2 \\
a_{15} = r_1
\end{cases}
$$

where $r_i \in \mathbb{Z}$.

Thus, $a_1\{12\} + \dots + a_{15}\{56\}$

$$
\begin{aligned}
&= r_1(\{15\} - \{16\} + \{56\}) + r_2(\{14\} - \{16\} + \{46\}) \\
&+ r_3(\{14\} - \{15\} + \{45\}) + r_4(\{13\} - \{16\} + \{36\}) \\
&+ r_5(\{13\} - \{15\} + \{35\}) + r_6(\{13\} - \{14\} + \{34\}) \\
&+ r_7(\{12\} - \{16\} + \{26\}) + r_8(\{12\} - \{15\} + \{25\}) \\
&+ r_9(\{12\} - \{14\} + \{24\}) + r_{10}(\{12\} - \{13\} + \{23\})
\end{aligned}
$$

Thus Z_1 is the free abelian group generated by the above ten cycles. $Z_1 \simeq \bigoplus_{i=1}^{10} \mathbb{Z}$.

- $\underset{\sim\sim\sim}{\text{Compute } B_1}$

$$B_1 = \{\partial(b_1\{123\} + \dots + b_{10}\{356\}) \,/\, b_i \in \mathbb{Z}\}$$

$$= \{ b_1(\{12\} - \{13\} + \{23\}) + \dots + b_{10}(\{35\} + \{56\} - \{36\}) \,/\, b_i \in \mathbb{Z}\}$$

$$= \langle \{12\} - \{13\} + \{23\}, \dots, \{35\} + \{56\} - \{3,6\} \rangle$$ – ~~the~~ an abelian subgroup of C_1 generated by 10 elements. Since C_1 is free, so is B_1. To find a basis for B_1, we only need to find the linearly independent rows of the following matrix by means of Gauss elimination.

$$\begin{array}{c}
\begin{bmatrix}
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \\
\begin{array}{ccccccccccccccc}
\uparrow & \uparrow & & & & & & & & & & & & & \uparrow \\
\{12\} & \{13\} & & \dots & & \dots & & \dots & & & & & & & \{56\}
\end{array}
\end{array}$$

Using the command rref(A) in Matlab to reduce this matrix by Gauss elimination, we get

$$
\begin{array}{c}
\begin{array}{ccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15
\end{array} \\
\left[\begin{array}{ccccccccccccccc}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1
\end{array}\right]
\end{array}
$$

Thus B_1 is the free abelian group generated by 10 elements

$\{\{12\}-\{16\}+\{26\}, \{13\}-\{16\}+\{36\}, \{14\}-\{16\}+\{46\},$

$\{15\}-\{16\}+\{56\}, \{23\}-\{26\}+\{36\}, \{24\}-\{26\}+\{46\},$

$\{25\}-\{26\}+\{56\}, \{34\}-\{36\}+\{46\}, \{35\}-\{36\}+\{56\},$

$\{45\}-\{46\}+\{56\}\}$

Note that B_1 ~~may~~ not be Z_1 although they are both abelian groups generated by 10 elements. That is because the generators of B_1 and the generators of Z_1 are not the same, e.g. $\{45\}-\{46\}+\{56\}$ is a generator of B_1 but not of Z_1. This is like the case of $\mathbb{Z}$ and $2\mathbb{Z}$. Both are free abelian groups. $\mathbb{Z}=\langle 1\rangle$ and $2\mathbb{Z}=\langle 2\rangle$. But we don't have $\mathbb{Z}=2\mathbb{Z}$.

- Compute Z_2:

$$Z_2 = \{ b_1\{123\} + \dots + b_{10}\{356\} \,/\, \partial(b_1\{123\} + \dots + b_{10}\{356\}) = 0 \}$$

$$= \{ b_1\{123\} + \dots + b_{10}\{356\} \,/\, (b_1 + b_2)\{12\} + \dots + (b_5 + b_{10})\{56\} = 0 \}$$

This equation is equivalent to a system of 15 equation of 10 unknowns. The matrix form is

$$\begin{array}{c}
\begin{matrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{matrix} \\
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}$$

Using the command rank(A) in Matlab, we get rank(A) = 10. Thus the system gives unique solution $(b_1, \dots, b_{10}) = (0, 0, \dots, 0)$. Therefore, $Z_2 = \{0\}$.

Step 5

$$H_0 = Z_0/B_0 = \overbrace{\big(\mathbb{Z}\{1\} \oplus \cdots \oplus \mathbb{Z}\{6\}\big) \big/ \big(\mathbb{Z}(\{1\}-\{6\}) \oplus \mathbb{Z}(\{2\}-\{6\}) \oplus \cdots \oplus \mathbb{Z}(\{5\}-\{6\})\big)}^{\cong \mathbb{Z}}.$$

$$H_1 = Z_1/B_1 = \Big(\mathbb{Z}(\{15\}-\{16\}+\{56\}) \oplus \mathbb{Z}(\{14\}-\{16\}+\{46\}) \oplus \mathbb{Z}(\{14\}-\{15\}+\{45\})$$
$$\oplus\, \mathbb{Z}(\{13\}-\{16\}+\{36\}) \oplus \mathbb{Z}(\{13\}-\{15\}+\{35\}) \oplus \mathbb{Z}(\{13\}-\{14\}+\{34\})$$
$$\oplus\, \mathbb{Z}(\{12\}-\{16\}+\{26\}) \oplus \mathbb{Z}(\{12\}-\{15\}+\{25\}) \oplus \mathbb{Z}(\{12\}-\{14\}+\{24\})$$
$$\oplus\, \mathbb{Z}(\{12\}-\{13\}+\{23\})\Big) \Big/ \Big(\mathbb{Z}(\{12\}-\{16\}+\{26\}) \oplus$$
$$\mathbb{Z}(\{13\}-\{16\}+\{36\}) \oplus \mathbb{Z}(\{14\}-\{16\}+\{46\}) \oplus \mathbb{Z}(\{15\}-\{16\}+\{56\})$$
$$\oplus\, \mathbb{Z}(\{23\}-\{26\}+\{36\}) \oplus \mathbb{Z}(\{24\}-\{26\}+\{46\}) \oplus \mathbb{Z}(\{25\}-\{26\}+\{56\})$$
$$\oplus\, \mathbb{Z}(\{34\}-\{36\}+\{46\}) \oplus \mathbb{Z}(\{35\}-\{36\}+\{56\}) \oplus \mathbb{Z}(\{45\}-\{46\}+\{56\})\Big)$$
$$\cong \mathbb{Z}/2$$

$$H_2 = Z_2/B_1 = \{0\}$$

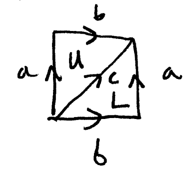
$$H_n = \{0\} \text{ for all } n \geq 3.$$

$2/4$

(3) We'll find the homology groups of the torus $S^1 \times S^1$

Step 1 We choose a triangulation with 7 vertrices

Prof. Lawson said that using the easier "triangulation"



is ok.

$$V = \{1, 2, 3, 4, 5, 6, 7\}$$

and 14 faces

$$K = \{\overset{1}{123}, \overset{2}{127}, \overset{3}{134}, \overset{4}{145}, \overset{5}{156}, \overset{6}{167}, \overset{7}{236}, \overset{8}{245}, \overset{9}{246}, \overset{10}{257}, \overset{11}{347},$$
$$\overset{12}{356}, \overset{13}{357}, \overset{14}{467}\}$$

The set of all faces $\mathcal{F}$ is obtained by adding vertices and edges to $\mathcal{K}$.

Step 2

$C_0 = \cancel{\mathbb{Z}\oplus}\, \mathbb{Z}\{1\} \oplus \cdots \oplus \mathbb{Z}\{7\}$

$$C_1 = \overset{1}{\mathbb{Z}\{12\}} \oplus \overset{2}{\mathbb{Z}\{13\}} \oplus \overset{3}{\mathbb{Z}\{14\}} \oplus \overset{4}{\mathbb{Z}\{15\}} \oplus \overset{5}{\mathbb{Z}\{16\}} \oplus \overset{6}{\mathbb{Z}\{17\}}$$
$$\oplus \overset{7}{\mathbb{Z}\{23\}} \oplus \overset{8}{\mathbb{Z}\{24\}} \oplus \overset{9}{\mathbb{Z}\{25\}} \oplus \overset{10}{\mathbb{Z}\{26\}} \oplus \overset{11}{\mathbb{Z}\{27\}} \oplus \overset{12}{\mathbb{Z}\{34\}} \oplus \overset{13}{\mathbb{Z}\{35\}} \oplus \overset{14}{\mathbb{Z}\{36\}}$$
$$\oplus \overset{15}{\mathbb{Z}\{37\}} \oplus \overset{16}{\mathbb{Z}\{45\}} \oplus \overset{17}{\mathbb{Z}\{46\}} \oplus \overset{18}{\mathbb{Z}\{47\}} \oplus \overset{19}{\mathbb{Z}\{56\}} \oplus \overset{20}{\mathbb{Z}\{57\}} \oplus \overset{21}{\mathbb{Z}\{67\}}$$

$$C_2 = \overset{1}{\mathbb{Z}\{123\}} \oplus \overset{2}{\mathbb{Z}\{127\}} \oplus \overset{3}{\mathbb{Z}\{134\}} \oplus \overset{4}{\mathbb{Z}\{145\}} \oplus \overset{5}{\mathbb{Z}\{156\}} \oplus \overset{6}{\mathbb{Z}\{167\}} \oplus \overset{7}{\mathbb{Z}\{236\}}$$
$$\oplus \overset{8}{\mathbb{Z}\{245\}} \oplus \overset{9}{\mathbb{Z}\{246\}} \oplus \overset{10}{\mathbb{Z}\{257\}} \oplus \overset{11}{\mathbb{Z}\{347\}} \oplus \overset{12}{\mathbb{Z}\{356\}} \oplus \overset{13}{\mathbb{Z}\{357\}} \oplus \overset{14}{\mathbb{Z}\{467\}}$$

$C_n = \{0\} \quad \forall n \geq 3$

There are 7 vertices, 21 edges and 14 triangles in total.

Step 3

- For $n = \cancel{0}\ 1$:

$$\begin{aligned}
\partial(a_1\{12\} + \cdots + a_{21}\{67\}) = &(-a_1 - a_2 - a_3 - a_4 - a_5 - a_6)\{1\} \\
&+ (a_1 - a_7 - a_8 - a_9 - a_{10} - a_{11})\{2\} \\
&\cancel{+ (a_2 - a_{12} - a_{13} - a_{14} - a_{15} - a_{16}} + (a_2 + a_7 - a_{12} - a_{13} - a_{14} - a_{15})\{3\} \\
&+ (a_3 + a_8 + a_{12} - a_{16} - a_{17} - a_{18})\{4\} \\
&+ (a_4 + a_9 + a_{13} + a_{16} - a_{19} - a_{20})\{5\} \\
&+ (a_5 + a_{10} + a_{14} + a_{17} + a_{19} - a_{21})\{6\} \\
&+ (a_6 + a_{11} + a_{15} + a_{18} + a_{20} + a_{21})\{7\}
\end{aligned}$$

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- For $n=2$:

$$\partial(b_1\{123\}+\dots+b_{14}\{467\}) = (b_1+b_2)\{12\} + (-b_1+b_3)\{13\} + (-b_3+b_4)\{14\} + (-b_4+b_5)\{15\}$$
$$+ (-b_5+b_6)\{16\} + (-b_2-b_6)\{17\} + (b_1+b_7)\{23\} + (b_8+b_9)\{24\}$$
$$+ (-b_8+b_{10})\{25\} + (-b_7-b_9)\{26\} + (b_2-b_{10})\{27\} + (b_3+b_{11})\{34\}$$
$$+ (b_{12}+b_{13})\{35\} + (b_7-b_{12})\{36\} + (-b_{11}-b_{13})\{37\} + (b_4+b_8)\{45\}$$
$$+ (b_9+b_{14})\{46\} + (b_{11}-b_{14})\{47\} + (b_5+b_{12})\{56\} + (b_{10}+b_{13})\{57\}$$
$$+ (b_6+b_{14})\{67\}$$

- For $n \geq 3$: $\partial = 0$ because $C_n = \{0\}$.

<u>Step 4</u> $Z_n = \ker(\partial : C_n \to C_{n-1})$, $B_n = \operatorname{Im}(\partial : C_{n+1} \to C_n)$

Thus,
$$Z_0 = C_0$$
$$B_0 = ?$$
$$Z_1 = ?$$
$$B_1 = ?$$
$$Z_2 = ?$$
$$B_2 = 0$$
$Z_n = B_n = \{0\}$ for all $n \geq 3$.

- <u>Compute B_0</u>

$$B_0 = \{\partial(a_1\{12\}+\dots+a_{21}\{67\}) \ / \ a_i \in \mathbb{Z}\}$$
$$= \{a_1(\{2\}-\{1\})+\dots+a_{21}(\{7\}-\{6\}) \ / \ a_i \in \mathbb{Z}\}$$
$$= \langle \{2\}-\{1\}, \{3\}-\{1\}, \dots, \{7\}-\{6\} \rangle \ \cancel{/ a_i \in \mathbb{Z}}$$

$= \langle \{j\} - \{i\} \,/\, 1 \leq i < j \leq 7 \rangle$

Since $\{j\}-\{i\} = (\{j\}-\{1\}) - (\{i\}-\{1\})$, B_0 ~~has a basis of at most~~

$$B_0 = \langle \{j\} - \{1\} \,/\, 1 < j \leq 7 \rangle$$

These six vectors are linearly independent because following matrix

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

has rank 6 (by using the command rank(A) in Matlab). Thus,

$$B_0 = \bigoplus_{j=2}^{7} \mathbb{Z}(\{j\}-\{1\})$$

• Compute Z_1:

$$Z_1 = \{ a_1\{12\} + \dots + a_{21}\{67\} \,/\, \partial(a_1\{12\} + \dots + a_{21}\{67\}) = 0 \}$$

$$= \{ a_1\{12\} + \dots + a_{21}\{67\} \,/\, (-a_1 - a_2 - a_3 - a_4 - a_5 - a_6)\{1\} + \dots + (a_6 + a_{11} + a_{15} + a_{18} + a_{20} + a_{21})\{7\} = 0 \}$$

Then we have the matrix form of the equation. It is a matrix with 7 rows, corresponding to 7 equations, and 21 columns, corresponding to 21 unknowns. ~~Matlab~~ Matlab doesn't provide a solver for such a system. Thus we use an online solver called "Linear Solver" in the website wims.unice.fr/ This is the first result by Google when we type "Linear solver". We input the matrix (which consists of only 1's, 0's and -1's) into it. Then we get the solution as follow:

$$
\begin{cases}
a_1 = r_{11} + r_{12} + r_{13} + r_{14} + r_{15} \\
a_2 = r_7 + r_8 + r_9 + r_{10} - r_{15} \\
a_3 = r_4 + r_5 + r_6 - r_{10} - r_{14} \\
a_4 = r_2 + r_3 \mp r_6 - r_9 - r_{13} \\
a_5 = r_1 - r_3 - r_5 - r_8 - r_{12} \\
a_6 = -r_1 - r_2 - r_4 - r_7 - r_{11} \\
a_7 = r_{15} \\
a_8 = r_{14} \\
a_9 = r_{13} \\
\vdots \qquad \vdots \\
a_{21} = r_1
\end{cases}
$$

where $r_i \in \mathbb{Z}$. By replacing these solutions in $a_1\{12\} + \cdots + a_{67}\{67\}$ and then group terms with common factor r_i together (like $r_1 \boxed{\ } + r_2 \boxed{\ } + \cdots + r_{15} \boxed{\ }$) we get: Z_1 is a free abelian group generated by 15 cycles as follow.

$\{16\} - \{17\} + \{67\}$, $\{14\} - \{15\} + \{45\}$, $\{12\} - \{17\} + \{27\}$,

$\{15\} - \{17\} + \{57\}$, $\{13\} - \{17\} + \{37\}$, $\{12\} - \{16\} + \{26\}$,

$-\{16\} + \{15\} + \{56\}$, $\{13\} - \{16\} + \{36\}$, $\{12\} - \{15\} + \{25\}$,

$\{14\} - \{17\} + \{47\}$, $\{13\} - \{15\} + \{35\}$, $\{12\} - \{14\} + \{24\}$,

$\{14\} - \{16\} + \{46\}$, $\{13\} - \{14\} + \{34\}$, $\{12\} - \{13\} + \{23\}$.

- Compute B_1

$$B_1 = \{ \partial (b_1\{123\} + \cdots + b_{14}\{467\}) / b_i \in \mathbb{Z} \}$$

$$= \{ b_1 (\{12\} - \{13\} + \{23\}) + \cdots + b_{14} (\{46\} - \{47\} + \{67\}) / b_i \in \mathbb{Z} \}$$

$$= \langle \{12\} - \{13\} + \{23\}, \ldots, \{46\} - \{47\} + \{67\} \rangle$$

Then B_1 is generated by 14 elements (cycles). To get a basis of B_1, we need to use Gaussian elimination for a matrix whose rows are these cycles. Thus we consider a matrix of 14 rows, which represent 14 cycles, and 21 columns which present $\{12\}, \{13\}, \ldots, \{67\}$. We input this matrix into Matlab and use the command rref(A) for Gaussian elimination and obtain:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & \cdots & 1 & \cdots & & & \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & \cdots & & \cdots & 1 & & \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & & & & & & \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & \cdots & & \cdots & & \cdots & \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & \cdots & & \cdots & & \cdots & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & -1 & & 1 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \cdots & -1 & & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \cdots & -1 & & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots 1 \cdots & -1 & \cdots & 0 & \cdots & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & \cdots & -1 & \cdots & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 & & -1 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & 1 & -1 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & 1 \cdots 1 & & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & 1 \; -1 \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & 1 \; -1 & 0\;0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & 0\;0 & 0
\end{bmatrix}$$

This is ~~ma~~ a matrix with rank equal 13. Thus B_2 is a free abelian group generated by the following 13 cycles.

$\{12\} - \{17\} + \{27\}$,

$\{13\} - \{17\} + \{37\} + \{45\} + \{56\} - \{47\} + \{67\}$,

$\{14\} - \{17\} + \{45\} + \{56\} + \{67\}$,

$\{15\} - \{17\} + \{56\} + \{67\}$,

$\{16\} - \{17\} + \{67\}$,

$\{23\} - \{27\} + \{37\} + \{45\} - \{47\} + \{56\} + \{67\}$,

$\{24\} - \{27\} + \{45\} + \{57\}$,

$\{25\} - \{27\} + \{57\}$,

$\{26\} - \{27\} + \{45\} - \{47\} + \{57\} + \{67\}$,

$\{34\} - \{37\} + \{47\}$,

$\{35\} - \{37\} + \{57\}$,

$\{36\} - \{37\} - \{56\} + \{57\}$,

$\{46\} - \{47\} + \{67\}$.

Compute Z_2:

$$Z_2 = \{ b_1\{123\} + \dots + b_{14}\{467\} \,/\, \partial(b_1\{123\} + \dots + b_{14}\{467\}) = 0 \}$$

$$= \{ b_1\{123\} + \dots + b_{14}\{467\} \,/\, (b_1+b_2)\{12\} + (-b_1+b_3)\{13\} + \dots + (b_6+b_{14})\{67\} = 0 \}$$

The above equation is equivalent to a system of 21 equations of 14 unknowns.

The corresponding matrix has 21 rows, corresponding to 21 equations, and 14 columns, corresponding to 14 unknowns. We input this matrix into the linear solver mentioned above and get the solution:

$$\begin{cases} b_1 = -r \\ b_2 = r \\ b_3 = -r \\ b_4 = -r \\ b_5 = -r \\ b_6 = -r \\ b_7 = r \\ b_8 = r \\ b_9 = -r \\ b_{10} = r \\ b_{11} = r \\ b_{12} = r \\ b_{13} = -r \\ b_{14} = r \end{cases}$$

where $r \in \mathbb{Z}$

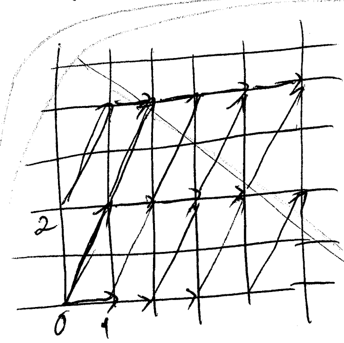
Thus, Z_2 contains only one cycle:

$$Z_2 = \mathbb{Z}\big(-\{123\} + \{127\} - \{134\} - \{145\} - \{156\} - \{167\} + \{236\} + \{245\} - \{246\} + \{257\} + \{347\} + \{356\} - \{357\} + \{467\}\big).$$

<u>Step 5</u> Determine the homology groups

$$H_0 = Z_0 / B_0 = \left(\bigoplus_{i=1}^{7} \mathbb{Z}\{i\}\right) \Big/ \left(\bigoplus_{j=2}^{7} \mathbb{Z}(\{j\} - \{1\})\right)$$

In general, we're not sure if $H_0 \simeq \mathbb{Z}$. A free ~~grou~~ subgroup generated by one element of a free group which is generated by two elements may not have a direct summand. For example,



$\mathbb{Z} \oplus \mathbb{Z}$ is a free abelian group generated by $(0,1)$ and $(1,0)$. $\mathbb{Z}(1,2)$ is a free subgroup generated by $(1,2)$. A direct sum $\mathbb{Z}(1,2) \oplus \mathbb{Z}v$ will always create a tilted mesh system, which is not equal to $\mathbb{Z} \oplus \mathbb{Z}$.

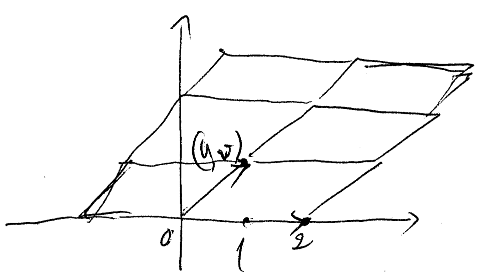
$H_1 = Z_1/B_1$, where Z_1 is a free abelian group generated by 15 elements,
B_1 " 13 " .

$H_2 = Z_2/B_2 \cong Z_2 \simeq \mathbb{Z}$ ✓

$H_n = \{0\}$ for all $n \geq 3$. ✓ This is true

The reason we hesitated to write $H_0 \simeq \mathbb{Z}$ and $H_1 \simeq \mathbb{Z}\oplus\mathbb{Z}$ is that we only know B_i is a subgroup of Z_i. We don't know if B_i has a direct summand in Z_i. Let's look at the following example.

$\mathbb{Z}(2,0)$ is the subgroup of $\mathbb{Z}\oplus\mathbb{Z}$ generated by the element $(2,0)$.



Suppose by contradiction that $\mathbb{Z}(2,0)$ has a direct summand $\mathbb{Z}(u,v)$. Then there exist $k,l\in\mathbb{Z}$ such that $k(2,0)+l(u,v)=(1,0)$. Then

$$\begin{cases} 2k+lu = 1 \\ lv = 0 \end{cases}$$

The first equation say that $l \neq 0$. Thus $v=0$. Then $(2u,0) \in \mathbb{Z}(2,0) \cap \mathbb{Z}(u,0)$. This contradicts the assumption ~~fact~~ that we have the direct sum.

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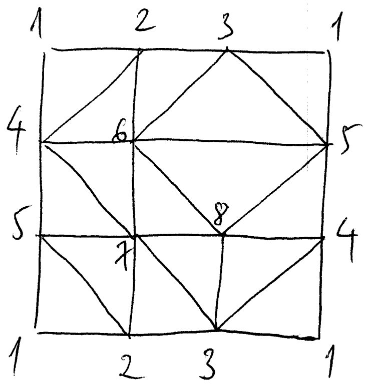
(5) We'll find the homology groups of the Klein bottle.

Step 1 The following triangulation is taken from the article "Vertex-Minimal Simplicial Immersions of the Klein bottle in Three Space" of Davide P. Cervone.



The set of vertices is

$V = \{1, 2, \dots, 8\}$

The set of triangles is

$K = \{124, 125, 134, 135, 236, 237, 246, 257, 348, 356, 378, 457, 458, 467, 568, 678\}$

The set of all faces F is obtained by adding all vertices and edges to K.

<u>Step 2</u> Find the groups of n-chains.

$$C_0 = \mathbb{Z}\{1\} \oplus \cdots \oplus \mathbb{Z}\{8\}$$

$$C_1 = \overset{1}{\mathbb{Z}\{12\}} \oplus \overset{2}{\mathbb{Z}\{13\}} \oplus \overset{3}{\mathbb{Z}\{14\}} \oplus \overset{4}{\mathbb{Z}\{15\}} \oplus \overset{5}{\mathbb{Z}\{16\}} \oplus \overset{6}{\mathbb{Z}\{17\}} \oplus \overset{7}{\mathbb{Z}\{18\}} \oplus \overset{8}{\mathbb{Z}\{2,3\}}$$
$$\oplus \overset{9}{\mathbb{Z}\{2,4\}} \oplus \overset{10}{\mathbb{Z}\{25\}} \oplus \overset{11}{\mathbb{Z}\{26\}} \oplus \overset{12}{\mathbb{Z}\{27\}} \oplus \overset{13}{\mathbb{Z}\{28\}} \oplus \overset{14}{\mathbb{Z}\{34\}} \oplus \overset{15}{\mathbb{Z}\{35\}} \oplus \overset{16}{\mathbb{Z}\{36\}} \oplus \overset{17}{\mathbb{Z}\{37\}}$$
$$\oplus \overset{18}{\mathbb{Z}\{38\}} \oplus \overset{19}{\mathbb{Z}\{45\}} \oplus \overset{20}{\mathbb{Z}\{46\}} \oplus \overset{21}{\mathbb{Z}\{47\}} \oplus \overset{22}{\mathbb{Z}\{48\}} \oplus \overset{23}{\mathbb{Z}\{56\}} \oplus \overset{24}{\mathbb{Z}\{57\}} \oplus \overset{25}{\mathbb{Z}\{58\}} \oplus \overset{26}{\mathbb{Z}\{67\}}$$
$$\oplus \overset{27}{\mathbb{Z}\{68\}} \oplus \overset{28}{\mathbb{Z}\{78\}}$$

$$C_2 = \overset{1}{\mathbb{Z}\{124\}} \oplus \overset{2}{\mathbb{Z}\{125\}} \oplus \overset{3}{\mathbb{Z}\{134\}} \oplus \overset{4}{\mathbb{Z}\{135\}} \oplus \overset{5}{\mathbb{Z}\{236\}} \oplus \overset{6}{\mathbb{Z}\{237\}} \oplus \overset{7}{\mathbb{Z}\{246\}} \oplus \overset{8}{\mathbb{Z}\{257\}} \oplus \overset{9}{\mathbb{Z}\{348\}}$$
$$\oplus \overset{10}{\mathbb{Z}\{356\}} \oplus \overset{11}{\mathbb{Z}\{378\}} \oplus \overset{12}{\mathbb{Z}\{457\}} \oplus \overset{13}{\mathbb{Z}\{458\}} \oplus \overset{14}{\mathbb{Z}\{467\}} \oplus \overset{15}{\mathbb{Z}\{568\}} \oplus \overset{16}{\mathbb{Z}\{678\}}$$

$C_n = \{0\}$ for all $n \geq 3$.

<u>Step 3</u> Compute the boundary operators

- For $n = 1$:

$$\partial(a_1\{12\}+\dots+a_{28}\{78\}) = (-a_1-a_2-a_3-a_4-a_5-a_6-a_7)\{1\}$$
$$+(a_1-a_8-a_9-a_{10}-a_{11}-a_{12}-a_{13})\{2\}$$
$$+(a_2+a_8-a_{14}-a_{15}-a_{16}-a_{17}-a_{18})\{3\}$$
$$+(a_3+a_9+a_{14}-a_{19}-a_{20}-a_{21}-a_{22})\{4\}$$
$$+(a_4+a_{10}+a_{15}+a_{19}-a_{23}-a_{24}-a_{25})\{5\}$$
$$+(a_5+a_{11}+a_{16}+a_{20}+a_{23}-a_{26}-a_{27})\{6\}$$
$$+(a_6+a_{12}+a_{17}+a_{21}+a_{24}+a_{26}-a_{28})\{7\}$$
$$+(a_7+a_{13}+a_{18}+a_{22}+a_{25}+a_{27}+a_{28})\{8\}.$$

For $n=2$:

$$\partial(b_1\{124\}+\dots+b_{16}\{678\}) = (b_1+b_2)\{12\}+(b_3+b_4)\{13\}+(-b_1-b_3)\{14$$
$$+(-b_2-b_4)\{15\}+(b_5+b_6)\{23\}+(b_1+b_7)\{24\}$$
$$+(b_2+b_8)\{25\}+(-b_5-b_7)\{26\}+(-b_6-b_8)\{27\}$$
$$+(b_3+b_9)\{34\}+(b_4+b_{10})\{35\}+(b_5-b_{10})\{36\}$$
$$+(b_6+b_{11})\{37\}+(-b_9-b_{11})\{38\}+(b_{12}+b_{13})\{45\}$$
$$+(b_7+b_{14})\{46\}+(-b_{12}-b_{14})\{47\}+(b_9-b_{13})\{48\}$$
$$+(b_{10}+b_{15})\{56\}+(b_8+b_{12})\{57\}+(b_{13}-b_{15})\{58\}$$
$$+(b_{14}+b_{16})\{67\}+(b_{15}-b_{16})\{68\}+(b_{11}+b_{16})\{7$$

- For $n \geq 3$: $\partial = 0$ because $C_n = \{0\}$.

Step 4 Find Z_n and B_n for $n = 0, 1, 2, \dots$

$Z_0 = C_0$

$B_0 = ?$

$Z_1 = ?$

$B_1 = ?$

$Z_2 = ?$

$B_2 = 0$

$Z_n = B_n = 0$ for all $n \geq 3$.

- Compute B_0

$$B_0 = \{ \partial(a_1\{12\} + \dots + a_{28}\{78\}) / a_i \in \mathbb{Z} \}$$

$$= \{ a_1(\{2\} - \{1\}) + \dots + a_{28}(\{8\} - \{7\}) / a_i \in \mathbb{Z} \}$$

$$= \langle \{2\} - \{1\}, \{3\} - \{1\}, \dots, \{8\} - \{7\} \rangle$$

Since $\{j\} - \{i\} = (\{j\} - \{1\}) - (\{i\} - \{1\})$, we get $B_0 = \langle \{j\} - \{1\} / 1 < j \leq 8 \rangle$

these 7 vectors are linearly independent because the following matrix has rank 7. Thus,

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B_0 = \bigoplus_{j=2}^{8} \mathbb{Z}(\{j\} - \{1\}).$$

Compute Z_1

$$Z_1 = \{ a_1\{12\} + \cdots + a_{28}\{78\} \ / \ \partial(a_1\{12\} + \cdots + a_{28}\{78\}) = 0 \}$$

$$= \{ a_1\{12\} + \cdots + a_{28}\{78\} \ / \ (-a_1 - a_2 - a_3 - a_4 - a_5 - a_6 - a_7)\{1\} + \cdots + (a_7 + a_{13} + a_{18} + a_{22} + a_{25} + a_{27} + a_{28})\{8\} = 0 \}$$

Then we have the matrix form of the equation. It's a matrix with 8 rows, corresponding to 8 equations, and 28 columns, corresponding to 28 unknowns.

We input this matrix (which consists of 1's, 0's and -1's) into the linear solver mentioned above and get:

$$\begin{cases} a_1 = r_{16} + r_{17} + r_{18} + r_{19} + r_{20} + r_{21} \\ a_2 = r_{11} + r_{12} + r_{13} + r_{14} + r_{15} - r_{21} \\ a_3 = r_7 + r_8 + r_9 + r_{10} - r_{15} - r_{20} \\ a_4 = r_4 + r_5 + r_6 - r_{10} - r_{14} - r_{19} \\ a_5 = r_2 + r_3 - r_6 - r_9 - r_{13} - r_{18} \\ a_6 = +r_1 - r_3 - r_5 - r_8 - r_{12} - r_{17} \\ a_7 = -r_1 - r_2 - r_4 - r_7 - r_{11} - r_{16} \\ a_8 = r_{21} \\ \vdots \\ a_{28} = r_1 \end{cases} \qquad \text{where } r_i \in \mathbb{Z}$$

By replacing these solutions in $a_1\{12\}+\dots+a_{28}\{78\}$ and then group terms with common factor r_i together (like $r_1\square+r_2\square+\dots+r_{21}\square$), we get: Z_1 is a free abelian group generated by 21 cycles as follow.

$\{17\}-\{18\}+\{78\}$,
$\{16\}-\{18\}+\{68\}$,
$\{16\}-\{17\}+\{67\}$,
$\{15\}-\{18\}+\{58\}$,
$\{15\}-\{17\}+\{57\}$,
$\{15\}-\{16\}+\{56\}$,
$\{14\}-\{18\}+\{48\}$,
$\{14\}-\{17\}+\{47\}$,
$\{14\}-\{16\}+\{46\}$,
$\{14\}-\{15\}+\{45\}$,
$\{13\}-\{18\}+\{38\}$,
$\{13\}-\{17\}+\{37\}$,
$\{13\}-\{16\}+\{36\}$,
$\{13\}-\{15\}+\{35\}$,
$\{13\}-\{14\}+\{34\}$,
$\{12\}-\{18\}+\{28\}$,
$\{12\}-\{17\}+\{27\}$,
$\{12\}-\{16\}+\{26\}$,
$\{12\}-\{15\}+\{25\}$,
$\{12\}-\{14\}+\{24\}$,
$\{12\}-\{13\}+\{23\}$.

- Compute B_1

$$B_1=\{\partial(b_1\{124\}+\dots+b_{16}\{678\})\ /\ b_i\in\mathbb{Z}\}$$
$$=\{b_1(\{12\}-\{14\}+\{24\})+\dots+b_{16}(\{67\}-\{68\}+\{78\})\ /\ b_i\in\mathbb{Z}\}$$
$$=\langle\{12\}-\{14\}+\{24\},\ \dots,\ \{67\}-\{68\}+\{78\}\rangle$$

Then B_1 is generated by 16 cycles. To get a basis of B_1, we need to use Gaussian elimination for a matrix whose rows are these cycles. Thus, we need to consider a matrix of 16 rows, which represents 16 cycles, and 28 columns, which represents $\{12\},\{13\},\dots,\{78\}$. We input this matrix (which consists of 1's, 0's and -1's) into Matlab. Then we use the command rref(A) to get the Gaussian reduction form. We then see that this reduced form has rank 16. Therefore all of the original ~~at~~ generators of B_1 are linearly independent. Thus B_1 is a free abelian group generated by 16 following cycles:

$\{12\}-\{14\}+\{24\}$, $\{34\}-\{38\}+\{48\}$,

$\{12\}-\{15\}+\{25\}$, $\{35\}-\{36\}+\{56\}$,

$\{13\}-\{14\}+\{34\}$, $\{37\}-\{38\}+\{78\}$,

$\{13\}-\{15\}+\{35\}$, $\{45\}-\{47\}+\{57\}$,

$\{23\}-\{26\}+\{36\}$, $\{45\}-\{48\}+\{58\}$,

$\{23\}-\{27\}+\{37\}$, $\{46\}-\{47\}+\{67\}$,

$\{24\}-\{26\}+\{46\}$, $\{56\}-\{58\}+\{68\}$,

$\{25\}-\{27\}+\{57\}$, $\{67\}-\{68\}+\{78\}$.

- Compute Z_2:

$$Z_2 = \{ b_1\{124\} + \dots + b_{16}\{678\} \,/\, \partial(b_1\{124\} + \dots + b_{16}\{678\}) = 0 \}$$

$$= \{ b_1\{124\} + \dots + b_{16}\{678\} ;\ (b_1+b_2)\{12\} + (b_3+b_4)\{13\} + \dots + (b_{11}+b_{16})\{78\} = 0 \}$$

The above equation is equivalent to a system of 28 equations of 16 unknowns. The corresponding matrix has 28 rows, corresponding to 28 equations, and 16 columns, corresponding to 16 unknowns. We input this matrix into the linear solver mentioned above and get the solution:

$$b_1 = b_2 = \dots = b_{16} = 0.$$

This means $Z_2 = \{0\}$.

Step 5

$$H_0 = Z_0/B_0 = \left(\bigoplus_{i=1}^{8} \mathbb{Z}\{i\}\right) \Big/ \left(\bigoplus_{j=2}^{8} \mathbb{Z}(\{j\}-\{1\})\right) \cong \mathbb{Z}$$

$H_1 = Z_1/B_1$ where Z_1 is a free abelian group generated by 21 elements,

B_1 is a free abelian group generated by 16 elements.

$\cong \mathbb{Z} \oplus \mathbb{Z}/2$

$H_2 = Z_2/B_2 = \{0\}$ ✓

$H_n = \{0\}$ for all $n \geq 3$ because $Z_n = 0$. ✓

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