1

Name: Tuan Pham
ID: 4652218

Math 8302: Manifolds and Topology
Homework 1

$\begin{array}{c|c} c & 3 \\ \hline 10 & 9 \end{array}$ 19

(1) Let M be a manifold. Suppose that $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ and $\mathcal{B} = \{(V_\beta, \psi_\beta)\}_{\beta\in J}$ are two atlases on M. Denote $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ respectively the maximal atlases containing $\mathcal{A}, \mathcal{B}$. We'll show that

$$\bar{\mathcal{A}} = \bar{\mathcal{B}} \Leftrightarrow \mathcal{A} \cup \mathcal{B} \text{ is a (smooth) atlas}.$$

($\Rightarrow$) Suppose that $\bar{\mathcal{A}} = \bar{\mathcal{B}}$. Then $(U_\alpha, \varphi_\alpha) \in \bar{\mathcal{B}}$ for all $\alpha \in I$. Thus every coordinate chart of $\mathcal{A}$ is compatible with all coordinate charts of $\mathcal{B}$. Similary, $(V_\beta, \psi_\beta) \in \bar{\mathcal{A}}$ for all $\beta \in J$. Thus every coordinate chart of $\mathcal{B}$ is compatible with all coordinate charts of $\mathcal{A}$. Therefore, all coordinate charts of $\mathcal{A}\cup\mathcal{B}$ are compatible with each other. Moreover,

$$\left(\bigcup_{\alpha\in I} U_\alpha\right) \cup \left(\bigcup_{\beta\in J} V_\beta\right) = M \cup M = M.$$

Thus $\mathcal{A}\cup\mathcal{B}$ is an atlas on M.

($\Leftarrow$) Suppose $\mathcal{A}\cup\mathcal{B}$ is an atlas on M. Then every coordinate chart of $\mathcal{A}$ is compatible with all coordinate charts of $\mathcal{B}$. Thus $\mathcal{A} \subset \bar{\mathcal{B}}$. Thus every coordinate chart of $\bar{\mathcal{B}}$ is compatible with all coordinate charts of $\mathcal{A}$. Thus $\bar{\mathcal{B}} \subset \bar{\mathcal{A}}$. By symmetry, $\bar{\mathcal{A}} \subset \bar{\mathcal{B}}$. Thus $\bar{\mathcal{A}} = \bar{\mathcal{B}}$.

(2) First, we'll construct homeomorphisms from the open unit disk $\mathbb{B}^n$ to itself that are smooth on $\mathbb{B}^n \setminus \{0\}$, but not smooth at 0.

We have $\mathbb{B}^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_1^2+\dots+x_n^2 < 1\}$.

For each $\alpha > -1$, we define the following function $\Psi_\alpha : \mathbb{B}^n \to \mathbb{B}^n$

$$\Psi_\alpha(x) = \begin{cases} |x|^\alpha x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \text{ where } |x| = \sqrt{x_1^2+\dots+x_n^2}$$

Then Ψ_α is well-defined because $|\Psi_\alpha(x)| = |x|^{\alpha+1} < 1$ for all $x \in \mathbb{B}^n$.

For all $x \neq 0$, Ψ_α is continuous at x. At $x=0$, we have

$$|\Psi_\alpha(x)| = |x|^{\alpha+1} \longrightarrow 0 \quad \text{as } x \to 0 \quad \text{because } \alpha+1 > 0.$$

Thus Ψ_α is also continuous at 0. Thus Ψ_α is a continuous function.

For each $\alpha > -1$, we put $\beta = -\dfrac{\alpha}{\alpha+1} > -1$. We have

$$\Psi_\beta \circ \Psi_\alpha(x) = \Psi_\beta(|x|^\alpha x) = \left(\left||x|^\alpha x\right|\right)^\beta (|x|^\alpha x) = |x|^{(\alpha+1)\beta+\alpha} x = x$$

for all $x \neq 0$.

Because $\Psi_\beta \circ \Psi_\alpha$ is continuous, $\Psi_\beta \circ \Psi_\alpha(x) = x$ for all $x \in \mathbb{B}^n$. Similarly,

$$\Psi_\alpha \circ \Psi_\beta(x) = \Psi_\alpha(|x|^\beta x) = \left||x|^\beta x\right|^\alpha |x|^\beta x = |x|^{(\beta+1)\alpha+\beta} x = x$$

for all $x \neq 0$.

Because $\Psi_\alpha \circ \Psi_\beta$ is continuous, $\Psi_\alpha \circ \Psi_\beta(x) = x$ for all $x \in \mathbb{B}^n$. Therefore, Ψ_β is the inverse function of Ψ_α. Thus Ψ_α is a homeomorphism.

We see that Ψ_α is smooth on $\mathbb{B}^n \setminus \{0\}$ and Ψ_β is also smooth on $\mathbb{B}^n \setminus \{0\}$. Thus $\mathbb{B}^n \setminus \{0\} \xrightarrow{\Psi_\alpha} \mathbb{B}^n \setminus \{0\}$ is a diffeomorphism for all $\alpha > -1$. We'll show that Ψ_α is <u>not</u> a diffeomorphism on $\mathbb{B}^n$. Write $x = (x_1,\dots,x_n)$ and put $e_i = (0 \cdots \underset{i}{1} \cdots 0)$, we have

$$\frac{\partial \Psi_\alpha}{\partial x_i} = \frac{\partial}{\partial x_i}(r^\alpha x) \quad \left(\text{where } r = \sqrt{x_1^2 + \cdots + x_n^2}\right)$$

$$= r^\alpha x_i e_i + \alpha \frac{x_i}{r} r^{\alpha-1} x \quad \left(\text{we've used } \frac{\partial r}{\partial x_i} = \frac{x_i}{r}\right)$$

$$= r^{\alpha-2} x_i (r^2 e_i + \alpha x)$$

Consider $x = t e_i$ with $|t| < 1$, $t \in \mathbb{R}$, we have $r = |t|$.

$$\frac{\partial \Psi_\alpha}{\partial x_i}(t e_i) = |t|^{\alpha-2} t (t^2 e_i + \alpha t e_i) = |t|^\alpha (t + \alpha) e_i$$

If $-1 < \alpha < 0$ then $t + \alpha \to \alpha < 0$ and $|t|^\alpha \to \infty$ as $t \to 0$. Thus $\frac{\partial \Psi_\alpha}{\partial x_i}(t e_i) \to \infty$ as $t \to 0$. Then Ψ_α is not smooth at 0.

If $\alpha > 0$ then $\Psi_\alpha^{-1} = \Psi_\beta$ with $\beta = \frac{-\alpha}{\alpha+1} < 0$. Then $\frac{\partial \Psi_\beta}{\partial x_i}(t e_i) \to \infty$ as $t \to 0$. Then Ψ_β is not smooth at 0.

Therefore, Ψ_α is not diffeomorphic around 0, except for $\alpha = 0$.
(proof continues on page 14)

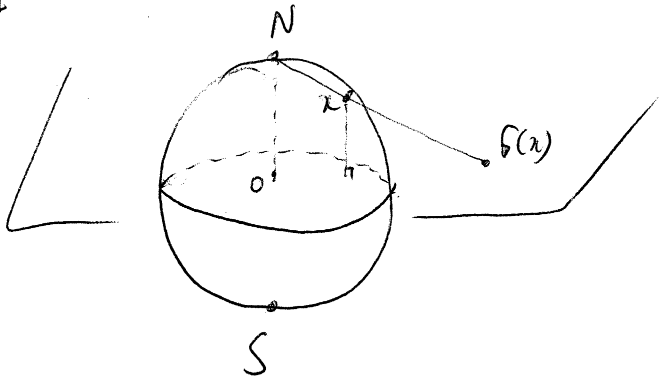
(3) $N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$

$x = (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1}$

$S = -N = (0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$

$\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$, $\quad \sigma(x^1, \ldots, x^{n+1}) = \dfrac{(x^1, \ldots, x^n)}{1 - x^{n+1}}$.

(a) We'll show that $\sigma(x)$ is the point where the line through N and x intersects the plane where $x^{n+1} = 0$.



The line passing through N and x has the following parametric equation

$$\begin{cases} y^1 = x^1 t \\ \vdots \\ y^n = x^n t \\ y^{n+1} = (x^{n+1}-1)t+1 \end{cases}$$

where $t \in \mathbb{R}$.

At the intercept with the plane $y^{n+1}=0$, we have $t = \dfrac{1}{1-x^{n+1}}$.

Thus the intercept is $(y^1,\dots,y^n,y^{n+1}) = \left(\dfrac{x^1}{1-x^{n+1}},\dots,\dfrac{x^n}{1-x^{n+1}},0\right) = \dfrac{(x^1,\dots,x^n,0)}{1-x^{n+1}}$.

If we ignore the last coordinate, the intercept is simply $\sigma(x)$.

(b) We'll show that $\sigma: S^n \setminus \{N\} \to \mathbb{R}^n$ is bijective.

Suppose that $\sigma(x^1,\dots,x^n,x^{n+1}) = \sigma(y^1,\dots,y^n,y^{n+1})$. Then

$$\frac{(x^1,\dots,x^n)}{1-x^{n+1}} = \frac{(y^1,\dots,y^n)}{1-y^{n+1}} \qquad (*)$$

Because $(x^1)^2+\dots+(x^n)^2+(x^{n+1})^2 = (y^1)^2+\dots+(y^n)^2+(y^{n+1})^2 = 1$, we have $x^{n+1}, y^{n+1} < 1$.

Taking the norm of both sides of $(*)$, we get $1-x^{n+1} = 1-y^{n+1}$. Thus $x^{n+1} = y^{n+1}$.

True, but quite a leap!

Then $(x^1,\dots,x^n) = (y^1,\dots,y^n)$. Thus $(x^1,\dots,x^{n+1}) = (y^1,\dots,y^{n+1})$. Therefore σ is injective.

Take $(u^1,\dots,u^n) \in \mathbb{R}^n$ and put $x^i = \dfrac{2u^i}{|u|^2+1}$ for $1 \le i \le n$, and

$$x^{n+1} = \frac{|u|^2-1}{|u|^2+1},$$ where $|u|^2 = u_1^2+\dots+u_n^2$.

we have $(x^1)^2 + \cdots + (x^n)^2 + (x^{n+1})^2 = \left(\frac{2u^1}{|u|^2+1}\right)^2 + \cdots + \left(\frac{2u^n}{|u|^2+1}\right)^2 + \left(\frac{|u|^2-1}{|u|^2+1}\right)^2$

$$= \frac{4|u|^2 + (|u|^2-1)^2}{(|u|^2+1)^2} = \frac{(|u|^2+1)^2}{((|u|^2+1))^2} = 1.$$

Thus $(x^1, \ldots, x^{n+1}) \in S^n$. Moreover, because $x^{n+1} \neq 1$, we have $(x^1, \ldots x^{n+1}) \in S^n \setminus \{N\}$

We have $\sigma(x^1, \ldots, x^n, x^{n+1}) = \frac{(x^1, \ldots, x^n)}{1-x^{n+1}} = \left(\frac{2u^1}{|u|^2+1}, \ldots, \frac{2u^n}{|u|^2+1}\right) \Big/ \left(1 - \frac{|u|^2-1}{|u|^2+1}\right)$

$$= \left(\frac{2u^1}{|u|^2+1}, \ldots, \frac{2u^n}{|u|^2+1}\right) \Big/ \frac{2}{|u|^2+1}$$

$$= (u^1, \ldots, u^n).$$

Thus σ is surjective. Consequently, σ is bijective and

$$\sigma^{-1}(u^1, \ldots, u^n) = (x^1, \ldots, x^n, x^{n+1}) = \frac{(2u^1, \ldots, 2u^n, |u|^2-1)}{|u|^2+1}$$

(c) Consider the transition map $\tilde{\sigma} \circ \sigma^{-1}$: $\underbrace{\sigma(S^n \setminus \{N\} \cap S^n \setminus \{S\})}_{\sigma(S^n \setminus \{N,S\})} \longrightarrow \underbrace{\tilde{\sigma}(S^n \setminus \{N\} \cap S^n \setminus \{S\})}_{\tilde{\sigma}(S^n \setminus \{N,S\})}$.

We have $\sigma(S^n \setminus \{N,S\}) = \mathbb{R}^n \setminus \{0\} = \tilde{\sigma}(S^n \setminus \{N,S\})$, where

$$\tilde{\sigma}(x) = -\sigma(-x).$$

Thus we have the transition map $\tilde{\sigma} \circ \sigma^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$.

For $u = (u^1, \ldots, u^n) \in \mathbb{R}^n \setminus \{0\}$, we have

$$\tilde{\sigma} \circ \sigma^{-1}(u) = -\sigma(-\sigma^{-1}(u)) = -\sigma\left(-\frac{(2u^1, \ldots, 2u^n, |u|^2-1)}{|u|^2+1}\right)$$

$$= -\sigma\left(\frac{(-2u^1, \ldots, -2u^n, 1-|u|^2)}{|u|^2+1}\right)$$

$$= -\frac{\left(\frac{-2u^1}{|u|^2+1}, \ldots, \frac{-2u^n}{|u|^2+1}\right)}{1 - \frac{1-|u|^2}{1+|u|^2}} = \frac{2u}{2|u|^2} = \frac{u}{|u|^2}$$

Thus $\tilde{\sigma}\circ\sigma^{-1}(u^1,\dots,u^n) = \left(\frac{u_1}{u_1^2+\dots+u_n^2},\dots,\frac{u_n}{u_1^2+\dots+u_n^2}\right)$.

This is a smooth function on $\mathbb{R}^n\setminus\{0\}$. Next, we'll show that the inverse of the map $\tilde{\sigma}\circ\sigma^{-1}$ is also smooth on $\mathbb{R}^n\setminus\{0\}$. We have

$$\sigma\circ\tilde{\sigma}^{-1}: \underbrace{\tilde{\sigma}(S^n\setminus\{N\}\cap S^n\setminus\{S\})}_{\mathbb{R}^n\setminus\{0\}} \to \underbrace{\sigma(S^n\setminus\{N\}\cap S^n\setminus\{S\})}_{\mathbb{R}^n\setminus\{0\}}$$

We have $\tilde{\sigma}\circ(-\sigma^{-1}(-u)) = -\sigma(\sigma^{-1}(-u)) = -(-u) = u$. Thus $\tilde{\sigma}^{-1}(u) = -\sigma^{-1}(-u)$.

For $u\in\mathbb{R}^n\setminus\{0\}$, we have

$$\sigma\circ\tilde{\sigma}^{-1}(u) = \sigma\left(-\sigma^{-1}(-u^1,\dots,-u^n)\right) = \sigma\left(-\frac{(-2u^1,\dots,-2u^n,|u|^2-1)}{|u|^2+1}\right)$$

$$= \sigma\left(\frac{(2u^1,\dots,2u^n,1-|u|^2)}{|u|^2+1}\right)$$

$$= \frac{\left(\frac{2u^1}{|u|^2+1},\dots,\frac{2u^n}{|u|^2+1}\right)}{1-\frac{1-|u|^2}{1+|u|^2}} = \frac{2u}{2|u|^2} = \frac{u}{|u|^2}$$

Thus $\sigma\circ\tilde{\sigma}(u_1,\dots,u^n) = \left(\frac{u_1}{u_1^2+\dots+u_n^2},\dots,\frac{u_n}{u_1^2+\dots+u_n^2}\right)$

This is a smooth function on $\mathbb{R}^n\setminus\{0\}$. Thus $\tilde{\sigma}\circ\sigma^{-1}: \mathbb{R}^n\setminus\{0\}\to\mathbb{R}^n\setminus\{0\}$ is a diffeomorphism. It's inverse is $\sigma\circ\tilde{\sigma}^{-1}: \mathbb{R}^n\setminus\{0\}\to\mathbb{R}^n\setminus\{0\}$, which is also a diffeomorphism. Moreover, $(S^n\setminus\{N\})\cup(S^n\setminus\{S\}) = S^n$. Thus,
$\{(S^n\setminus\{N\},\sigma),\ (S^n\setminus\{S\},\tilde{\sigma})\}$ is a smooth atlas on S^n.

(4) Suppose by contradiction that there is a continuous function $\theta: S^1 \to \mathbb{R}$ such that $e^{i\theta(p)} = p$ for all $p \in S^1$. For each $t \in \mathbb{R}$, we denote $\varphi(t) = e^{it}$.

Then $\theta \circ \varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover,

$$\varphi(t) = e^{i\theta(\varphi(t))} \iff e^{it} = e^{i\theta(\varphi(t))}$$

$$\iff t - \theta(\varphi(t)) \equiv 0 \pmod{2\pi}$$

$$\iff \frac{t-\theta(\varphi(t))}{2\pi} \in \mathbb{Z}$$

Because the function $t \mapsto \dfrac{t-\theta(\varphi(t))}{2\pi}$ is continuous on $\mathbb{R}$, its image is a connected space. This happens only if $\dfrac{t-\theta(\varphi(t))}{2\pi} = k \in \mathbb{Z}$ for all $t \in \mathbb{R}$.

Thus $\theta(\varphi(t)) = k2\pi + t$ for all $t \in \mathbb{R}$.

At $t=0$, we have $\theta(1) = \theta(\varphi(0)) = k2\pi + 0 = k2\pi$.

At $t = 2\pi$, we have $\theta(1) = \theta(\varphi(2\pi)) = k2\pi + 2\pi$. This is a contradiction.

Now suppose that U is an open subset of S^1 and $U \neq S^1$. We'll show that there is an angle function on U. Because $U \neq S^1$, there exists $q \in S^1 \setminus U$. There is $\alpha \in \mathbb{R}$ such that $q = e^{i\alpha}$. Then for every $p \in S^1 \setminus \{q\}$, there exists a unique number $t \in (\alpha, \alpha + 2\pi)$ such that $p = e^{it}$. We define $\theta(p) = t$. Then we have a function $\theta : S^1 \setminus \{q\} \to \mathbb{R}$.

To show θ is continuous, we take $p_0 \in S^1 \setminus \{q\}$ and a sequence (p_n) in $S^1 \setminus \{q\}$ such that $p_n \to p_0$. For each $n = 0, 1, 2, \ldots$ there exists a unique number $t_n \in (\alpha, \alpha+2\pi)$ such that $p_n = e^{it_n}$. Then $\theta(p_n) = t_n$. We'll show that $t_n \to t_0$. Suppose by contradiction that this is not true. Because the sequence (t_n) is bounded, there exists a subsequence (t_{n_k}) that converges to $\beta \in [\alpha, \alpha+2\pi]$ and $\beta \neq t_0$. Thus,

$$\underbrace{e^{it_{n_k}}}_{p_{n_k}} \longrightarrow e^{i\beta} \quad \text{as } k \to \infty$$

~~Thus $e^{i\beta} = \lim$~~. Because θ is continuous, $\theta(p_{n_k}) \longrightarrow \theta(e^{i\beta})$. Thus

$$\theta(e^{i\beta}) = \lim_{k\to\infty} \theta(p_{n_k}) = \theta(p_0) = t_0$$

. Thus,

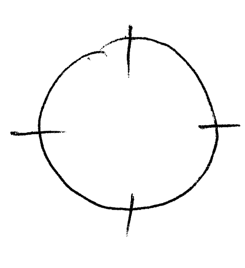
$$e^{i\beta} = e^{i\theta(e^{i\beta})} = e^{it_0}$$

Thus $\beta \equiv t_0 \pmod{2\pi}$. ~~Thus~~ Because $\beta \neq t_0$, $|\beta - t_0| \geq 2\pi$. This is a contradiction because $t_0 \in (\alpha, \alpha+2\pi)$, $\beta \in [\alpha, \alpha+2\pi]$.

Therefore, the function $\theta : S^1 \setminus \{q\} \to \mathbb{R}$ that we have defined is continuous. Because U is open in S^1 and $U \subset S^1 \setminus \{q\}$, it is also open in $S^1 \setminus \{q\}$. Thus $\theta|_U : U \to \mathbb{R}$ is a continuous, and hence an angle function on U.

Next, we'll show that the chart $(U, \theta : U \to \theta(U))$ is compatible with the standard smooth structure on S^1. The standard smooth structure on S^1 is an atlas with 4 charts.

$$\varphi_1 : \{(x,y) \in S^1, x > 0\} \to \mathbb{R}$$
$$\varphi_1(x,y) = y$$



$$\varphi_2 : \{(x,y) \in S^1,\ x < 0\} \to \mathbb{R}$$
$$\varphi_2(x,y) = y,$$
$$\varphi_3 : \{(x,y) \in S^1,\ y > 0\} \to \mathbb{R}$$
$$\varphi_3(x,y) = x,$$
$$\varphi_4 : \{(x,y) \in S^1,\ y < 0\} \to \mathbb{R}$$
$$\varphi_4(x,y) = x.$$

We'll show that $\theta : U \to \theta(U)$ is an homeomorphism and that (U, θ) is compatible with $\varphi_1, \varphi_2, \varphi_3, \varphi_4$. Put $\varphi : \mathbb{R} \to S^1$, $\varphi(t) = e^{it}$. Then $\varphi(\theta(t)) = e^{i\theta(t)} = t$. Thus $\varphi \circ \theta = id$. Thus θ is injective. Then $\theta : U \to \theta(U)$ is bijective. Then $\varphi : \theta(U) \to U$ is the inverse of θ.

~~The topology on S^1 is the same as the quotient topology $\mathbb{R}/\sim$ where $x \sim y \Leftrightarrow \frac{x-y}{2\pi} \in \mathbb{Z}$. The map $\varphi : \mathbb{R} \to S^1$, $t \mapsto e^{it}$ can be considered the quotient map $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$. Thus φ is a quotient map. We have $\varphi(\theta(U)) = U$, which is~~ We'll show that $\theta(U)$ is an open subset of $\mathbb{R}$. Decompose U into connected components $U = \bigcup_{\alpha \in I} U_\alpha$. We have

$$\theta(U) = \bigcup_\alpha \theta(U_\alpha)$$

Thus it suffices to show that each $\theta(U_\alpha)$ is open. Because S^1 is locally connected, U is also locally connected. Thus each connected component U_α is open in U. Since U is open in S^1, each U_α is open in S^1. Therefore, we can pose one more assumption in the original problem: U is connected.

With U an open connected subset of S^1, we'll show that $\theta(U)$ is open in $\mathbb{R}$. Since θ is continuous, $\theta(U)$ is a connected ~~component in~~ subspace of $\mathbb{R}$. Thus $\theta(U)$ is an interval.

We know that φ is injective on $\theta(U)$. Since φ is periodic of period 2π, the length of $\theta(U)$ must be less than or equal to 2π. Moreover, since the image of $\theta(U)$ under φ is $U \subset S^1 \setminus \{q\}$, there is $\alpha \in \mathbb{R}$ such that $\theta(U) \subset (\alpha, \alpha + 2\pi)$. We'll show that $\theta(U) = (\alpha, \alpha+2\pi) \cap \varphi^{-1}(U)$.

By what we have proved, $\theta(U) \subset (\alpha, \alpha+2\pi) \cap \varphi^{-1}(U)$. Now take $t \in (\alpha, \alpha+2\pi) \cap \varphi^{-1}(U)$. We have

$$e^{i\theta(\varphi(t))} = \varphi(t) = e^{it}$$

Thus $\theta(\varphi(t)) - t \in 2\pi\mathbb{Z}$. Thus there is $k \in \mathbb{Z}$ such that $\theta(\varphi(t)) = t + k2\pi$. Because $\theta(\varphi(t)) \in \theta(U) \subset (\alpha, \alpha+2\pi)$, $k = 0$. Thus $t = \theta(\varphi(t)) \in \theta(U)$.

Therefore $\theta(U) = (\alpha, \alpha+2\pi) \cap \varphi^{-1}(U)$. Since φ is continuous and U is open in S^1, $\varphi^{-1}(U)$ is open. Thus $\theta(U)$ is open in $\mathbb{R}$. Therefore, $(U, \theta : U \to \theta(U))$ is a chart on S^1.

Next, we'll show that (U, θ) is compatible with $\varphi_1, \varphi_2, \varphi_3, \varphi_4$. We denote short hand $\{x>0\}$ for $\{(x,y) \in S^1 : x > 0\}$, and similarly for $\{x<0\}$, $\{y>0\}$, $\{y<0\}$. We have

$$\varphi_1 \circ \theta^{-1} : \theta(U \cap \{x > 0\}) \longrightarrow \varphi_1(U \cap \{x > 0\})$$

$$\varphi_1 \circ \theta^{-1}(t) = \varphi_1(\varphi(t)) = \varphi_1(\cos t, \sin t) = \sin t,$$

$$\theta \circ \varphi_1^{-1} : \varphi_1(U \cap \{x > 0\}) \subset (-1,1) \longrightarrow \theta(U \cap \{x > 0\})$$

$$\theta \circ \varphi_1^{-1}(t) = \theta(\sqrt{1-t^2}, t) = \arcsin(t) + 2k_1\pi \quad (k_1 \in \mathbb{Z})$$

$$\varphi_2 \circ \theta^{-1} : \theta(U \cap \{x < 0\}) \longrightarrow \varphi_2(U \cap \{x < 0\})$$

$$\varphi_2 \circ \theta^{-1}(t) = \varphi_2(\varphi(t)) = \varphi_2(\cos t, \sin t) = \sin t,$$

$$\theta \circ \varphi_2^{-1} : \varphi_2(U \cap \{x < 0\}) \longrightarrow \theta(U \cap \{x < 0\})$$

$$\theta \circ \varphi_2^{-1}(t) = \theta(-\sqrt{1-t^2}, t) = \pi - \arcsin(t) + 2k_2\pi, \quad (k_2 \in \mathbb{Z})$$

$$\varphi_3 \circ \theta^{-1} : \theta(U \cap \{y > 0\}) \longrightarrow \varphi_3(U \cap \{y > 0\})$$

$$\varphi_3 \circ \theta^{-1}(t) = \varphi_3(\varphi(t)) = \varphi_3(\cos t, \sin t) = \cos t,$$

$$\theta \circ \varphi_3^{-1} : \varphi_3(U \cap \{y > 0\}) \longrightarrow \theta(U \cap \{y > 0\})$$

$$\theta \circ \varphi_3^{-1}(t) = \theta(t, \sqrt{1-t^2}) = \arccos(t) + 2k_3\pi, \quad (k_3 \in \mathbb{Z}),$$

$$\varphi_4 \circ \theta^{-1} : \theta(U \cap \{y < 0\}) \longrightarrow \varphi_4(U \cap \{y < 0\})$$

$$\varphi_4 \circ \theta^{-1}(t) = \varphi_4(\varphi(t)) = \varphi(\cos t, \sin t) = \cos t,$$

$$\theta \circ \varphi_4^{-1} : \varphi_4(U \cap \{y < 0\}) \subset (-1,1) \longrightarrow \theta(U \cap \{y < 0\})$$

$$\theta \circ \varphi_4^{-1}(t) = \theta(t, -\sqrt{1-t^2}) = -\arccos(t) + 2k_4\pi \quad (k_4 \in \mathbb{Z}).$$

All of these functions are smooth on their domains. Therefore, the chart $(U, \theta : U \to \theta(U))$ is compatible with the standard smooth structure on S^1.

⑤ $C^\infty(M)$ is the set of all smooth real-valued functions on M. On this set, we define the addition, multiplication by a real number and multiplication of two functions as follow

$$(f+g)(x) := f(x) + g(x) \quad \forall f, g \in C^\infty(M),\ x \in M,$$

$$(cf)(x) := c f(x) \quad \forall f \in C^\infty(M),\ x \in M,\ c \in \mathbb{R},$$

$$(fg)(x) := f(x)\, g(x) \quad \forall f, g \in C^\infty(M),\ x \in M.$$

We'll show that $f+g,\ cf,\ fg \in C^\infty(M)$.

- Check with $f+g$:

For each chart (U, φ) in M, for each $x \in \varphi(U)$, we have

$$\begin{tikzcd} M \arrow[r, "f"] & \mathbb{R} \\ \varphi(U) \arrow[u, "\varphi^{-1}"] \end{tikzcd} \qquad \begin{aligned} (f+g)\circ\varphi^{-1}(x) &= (f+g)(\varphi^{-1}(x)) \\ &= f\circ\varphi^{-1}(x) + g\circ\varphi^{-1}(x) \end{aligned}$$

As functions from $\varphi(U) \subset \mathbb{R}^n$ to $\mathbb{R}$, these functions give a relation $(f+g)\circ\varphi^{-1} = f\circ\varphi^{-1} + g\circ\varphi^{-1}$. Since f and g are smooths, $f\circ\varphi^{-1}, g\circ\varphi^{-1} \in C^\infty(\varphi(U))$. Thus there sum $(f+g)\circ\varphi^{-1}$ is also smooth. Thus $(f+g)\circ\varphi^{-1} \in C^\infty(\varphi(U))$, and $f+g \in C^\infty(M)$.

- Check with cf:

For each chart (U, φ) in M and $x \in \varphi(U)$, we have

$$(cf)\circ\varphi^{-1}(x) = (cf)(\varphi^{-1}(x)) = c\, f\circ\varphi^{-1}(x)$$

Because $f\circ\varphi^{-1}$ is smooth as a function from $\mathbb{R}^n$ to $\mathbb{R}$, $c(f\circ\varphi^{-1})$ is also smooth. Thus $(cf)\circ\varphi^{-1}$ is smooth. Thus $cf \in C^\infty(M)$.

- Check with fg:

For each chat (U, φ) on M and $x \in \varphi(U)$, we have

$$(fg)\circ\varphi^{-1}(x) = (fg)(\varphi^{-1}(x)) = f\circ\varphi^{-1}(x)\; g\circ\varphi^{-1}(x).$$

Because $f\circ\varphi^{-1}$ and $g\circ\varphi^{-1}$ are two smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$, their pointwise product is also a smooth function (by the product law of derivatives). Thus $(fg)\circ\varphi^{-1}$ is smooth. Thus $fg \in C^\infty(M)$.

Therefore, we have defined 3 operations on $C^\infty(M)$. Because $\mathbb{R}$ is a vector space, $C^\infty(M)$ becomes naturally a vector space over $\mathbb{R}$. We have

$$\begin{aligned}[(f+g)h](x) = (f+g)h(x) = (f+g)(x)\, h(x) &= (f(x)+g(x))h(x) \\ &= f(x)h(x) + g(x)h(x) \\ &= (f\cdot h)(x) + (g\cdot h)(x)\end{aligned}$$

Thus $(f+g)h = f\cdot h + g\cdot h$. Similarly, $f(g+h) = fg + fh$. Thus $C^\infty(M)$ becomes an algebra over $\mathbb{R}$. Moreover,

$$(fg)(x) = f(x)g(x) = g(x)f(x) = (gf)(x) \quad \forall x \in M$$

Thus $fg = gf$. Thus $C^\infty(M)$ is an commutative algebra. We have

$$((fg)h)(x) = (fg)(x)\,h(x) = (f(x)g(x))\,h(x) = f(x)(g(x)h(x)) = (f(gh))(x).\ \forall x \in M$$

thus $(fg)h = f(gh)$. Thus $C^\infty(M)$ is an associative algebra.

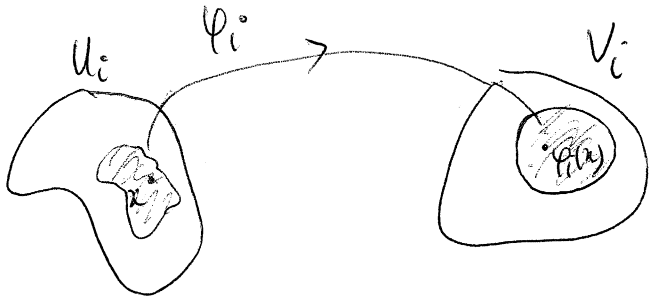
② (continue the solution of Problem 2)

For $\alpha, \beta > -1$, we can check easily that $\psi_\alpha \circ \psi_\beta = \psi_\beta \circ \psi_\alpha = \psi_{\alpha+\beta+\alpha\beta}$. Note that $\alpha+\beta+\alpha\beta > -1$. Therefore, we obtain a family of maps $\{\psi_\alpha\}_{\alpha>-1}$ from $\mathbb{B}^n$ to $\mathbb{B}^n$ that is smoothe everywhere when $\alpha = 0$, and smooth everywhere but at 0 when $\alpha \neq 0$. Moreover, the inverse function of each ψ_α is also in the family $\psi_\alpha^{-1} = \psi_{\alpha^*}$ where $\alpha^* = \dfrac{-\alpha}{\alpha+1}$.

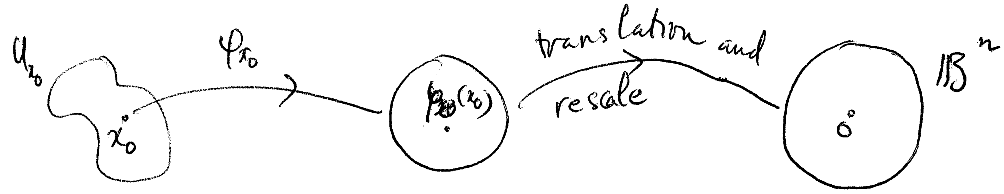
Moreover, $\psi_\alpha \circ \psi_\beta = \psi_{\alpha+\beta+\alpha\beta}$ which is smooth everywhere in $\mathbb{B}^n$ iff $\beta = \alpha^*$.

Now we return to the problem. Suppose that $(M, \mathcal{A})$ is a smooth structure with atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$. Each $\varphi_i : U_i \to V_i$ is a homeomorphism from U_i to an open subset V_i of $\mathbb{R}^n$. Ta~~ke $x_i \in V_i$ and a small open ball $B(x_i, \varepsilon_i)$ in $\mathbb{R}^n$ that is contained in~~ V_i. For each $x \in M$, x belongs to at least one U_i. Then $\varphi_i(x) \in V_i$. Take an open ball $B(\varphi_i(x), \varepsilon_i)$ in $\mathbb{R}^n$ that is small enough so that $B(\varphi_i(x), \varepsilon_i) \subset V_i$. Then we also a have a chart $\left(\varphi_i^{-1}(B(\varphi_i(x), \varepsilon)), \varphi_i\right)$. The colletion of all such charts as x

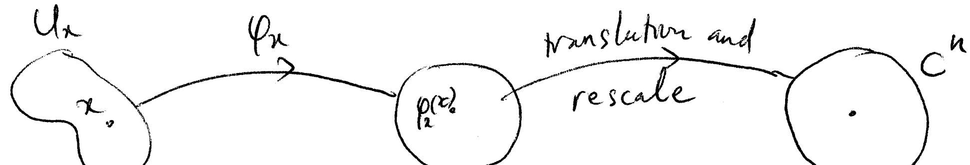
varies in M. These charts cover M and compatible to each other because they are simply restrictions of the given atlas φ_i's onto an open set. Thus these charts determine an atlas on M, which is equivalent



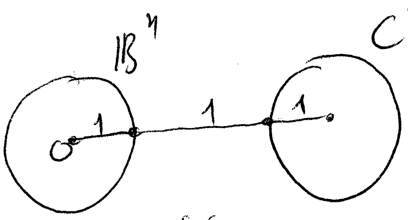
to $\mathcal{A}$. Thus, we can assume that the atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ from the beginning is indexed by $x \in M$, i.e. $\mathcal{A} = \{(U_x, \varphi_x)\}_{x\in M}$ such that $x \in U_x$, $\varphi_x : U_x \to V_x$ where V_x is an open ball in $\mathbb{R}^n$ centered at $\varphi_x(x)$. Fix $x_0 \in M$ (doable because $M \neq \emptyset$). By translation and



rescale, we can map U_{x_0} to the unit all $\mathbb{B}^n$. Then we get a chart (U_{x_0}, f_{x_0}) with $f_{x_0} : U_{x_0} \to \mathbb{B}^n$. For any other $x \in M \setminus \{x_0\}$, we use translation and rescaling to map U_x to C^n - the open ball centered at $(3,0,0\dots,0)$ with radius 1 in $\mathbb{R}^n$.



Then we get a chart (U_x, f_x) with $f_x : U_x \to C^n$.

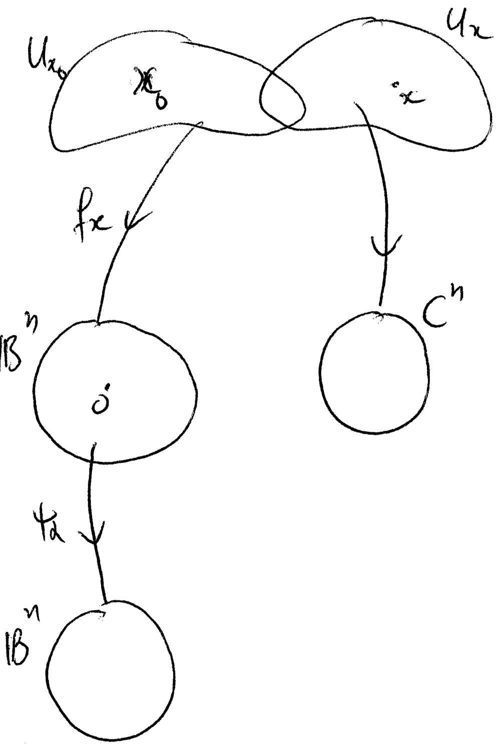


Then the family $\{(U_x, f_x)\}_{x \in M}$ is also compatible because we didn't destroy the smoothness of φ_x's. We just composed φ_x with a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$. Thus $\{(U_x, f_x)\}_{x \in M}$ is also an atlas, which is equivalent to A. For each $\alpha > -1$, we introduce the family

$$A_\alpha = \{(U_{x_0}, \Psi_\alpha \circ f_{x_0})\} \cup \{(U_x, f_x)\}_{x \in M \setminus \{x_0\}}.$$

where $\Psi_\alpha : \mathbb{B}^n \to \mathbb{B}^n$, $\Psi_\alpha(x) = |x|^\alpha x$.

We'll show that A_α is also an atlas on M. It's obvious that A_α covers M. For each $x \in M \setminus \{x_0\}$, f_x is compatible with $\Psi_\alpha \circ f_{x_0}$ because $\mathbb{B}^n$



and C^n are disjoint. Therefore, the coordinate charts in A_α are compatible. $\Psi_\alpha \circ f_{x_0}$ is a homeomorphism from U_{x_0} to $\mathbb{B}^n$. Thus A_α is an atlas on M. We'll show that for $\alpha, \beta > -1$, $\alpha \neq \beta$, A_α is not equivalent to A_β. To do so, we only need to check that two charts $(U_{x_0}, \Psi_\alpha \circ f_{x_0})$ and $(U_{x_0}, \Psi_\beta \circ f_{x_0})$ are incompatible.

We have $(\varphi_\alpha \circ f_{x_0}) \circ (\varphi_\beta \circ f_{x_0})^{-1} = \varphi_\alpha \circ \varphi_\beta^{-1} = \varphi_\alpha \circ \varphi_{\beta^*} : \mathbb{B}^n \to \mathbb{B}^n$, where $\beta^* = -\frac{\beta}{\alpha+\beta}$. This map is smooth at 0 iff $\beta^* = \alpha^*$, which is equivalent to $\beta = \alpha$. Thus A_α and A_β are not ~~any~~ equivalent if $\alpha \neq \beta$. Thus, we obtain a family of pairwise-inequivalent atlases $(A_\alpha)_{\alpha > -1}$. This family is uncountable.