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Math 8302: Topology and Manifolds

Homework 2

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① We'll give a definition of the complex Grassmannian $Gr_{\mathbb{C}}(n,m)$ of n-dimensional subspaces of $\mathbb{C}^m$.

Denote by $Gr_{\mathbb{C}}(n,m)$ the set of all n-dimensional subspaces of $\mathbb{C}^m$. If $n=0$ or $n=m$ then $Gr_{\mathbb{C}}(0,m)=\{0\}$ and $Gr_{\mathbb{C}}(m,m)=\{\mathbb{C}^m\}$ respectively. The topology on each of them is simply the unique topology $\{\phi, \{*\}\}$. The (maximal) atlas is $\{(\{*\}, \varphi: * \to \mathbb{C}^0=\{0\})\}$. In such a case, $Gr_{\mathbb{C}}(n,m)$ is a smooth 0-manifold.

Now we consider $1 \le n \le m-1$. ~~From the topological point of view, we can identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and $\mathbb{R}^{2m}$ with $\mathbb{C}^m$.~~ For each n-combination τ of the set $\{1,2,\dots,m\}$, we put

$$W_\tau = \{(x_1,\dots,x_m)\in\mathbb{C}^m:\ x_i = 0 \text{ for all indices } i\notin\tau\},$$

$$W'_\tau = \{(x_1,\dots,x_m)\in\mathbb{C}^m:\ x_i = 0 \text{ for all indices } i\in\tau\}.$$

If we put $\tau^c = \{1,2,\dots,m\}\setminus\tau$ then $W'_\tau = W_{\tau^c}$. We see that $\mathbb{C}^m = W_\tau \oplus W'_\tau$.

Also, we put $\pi_\tau: \mathbb{C}^m \to W_\tau$,

$$\pi_\tau(x_1,\dots,x_m) = (y_1,\dots,y_m), \text{ with } y_i = \begin{cases} x_i & \text{if } i\in\tau, \\ 0 & \text{otherwise.}\end{cases}$$

Put $U_\tau = \{V\in Gr_{\mathbb{C}}(n,m) \mid \pi_\tau|_V: V\to W_\tau \text{ is bijective}\}$. Let $\mathcal{L}(W_\tau, W'_\tau)$

be the vector space over $\mathbb{C}$ of linear maps from W_τ to W_τ'. For each element $V \in \mathcal{U}_\tau$, each $x \in V$ is written in a unique way

$$x = \underbrace{\pi_\tau(x)}_{\in W_\tau} + \underbrace{\pi_{\tau^c}(x)}_{W_\tau'}$$

To V we associate the function $f: W_\tau \to W_\tau'$, $f(\pi_\tau(x)) = \pi_{\tau^c}(x)$.

$$\begin{tikzcd} V \arrow[r, "\pi_\tau"] \arrow[dr, "\pi_{\tau^c}"'] & W_\tau \arrow[d, "f"] \arrow[dl, phantom, "2"] \\ & W_\tau' \end{tikzcd}$$

Then f is well-defined because $\pi_\tau|_V$ is bijective. Then f becomes naturally a linear map. Thus, we obtain a map $\rho_\tau : \mathcal{U}_\tau \to \mathcal{L}(W_\tau, W_\tau')$,

$$V \mapsto f$$

We'll show that ρ_τ is bijective. For each $f \in \mathcal{L}(W_\tau, W_\tau')$, we have $\rho_\tau(V) = f$ only if $f(\pi_\tau(x)) = \pi_{\tau^c}(x)$ $\forall x \in V$. Thus $\underbrace{\{x = y + f(y) : y \in W_\tau\}}_{V'}$ is contained in V. Moreover, V' is an n-dimensional subspace of $\mathbb{C}^m$ because the linear map $W_\tau \to V'$, $y \mapsto y + f(y)$ is bijective. Thus V, if exists, must be V'. By the definition of V', we can see that $\rho_\tau(V') = f$. Therefore, ρ_τ is bijective. We know that each $f \in \mathcal{L}(W_\tau, W_\tau')$ associates one-to-one with its matrix representation. Since W_τ is n-dimensional, and W_τ' is $(m-n)$-dimensional, the representation matrix is of size $(m-n) \times n$. Thus there is a natural bijective linear map from $\mathcal{L}(W_\tau, W_\tau')$ to $\mathbb{C}^{(m-n)n}$.

$$\begin{tikzcd} \mathcal{U}_\tau \arrow[r, "\rho_\tau"] \arrow[dr, dashed] & \mathcal{L}(W_\tau, W_\tau') \arrow[d] \\ & \mathbb{C}^{(m-n)n} \end{tikzcd}$$

Then we have a bijection $\varphi_\tau : U_\tau \to \mathbb{C}^{(m-n)n}$. Denote by S the set of all n-combinations of $\{1, 2, \dots, m\}$. Then we have a ~~man~~ family $\{(U_\tau, \varphi_\tau)\}_{\tau \in S}$. This family has $C_m^n = \dfrac{m!}{n!(m-n)!}$ members. The topology on $Gr_{\mathbb{C}}(n,m)$ is now defined as the topology generated by the maps $\{\varphi_\tau\}_{\tau \in S}$. This is the coarsest topology on $Gr_{\mathbb{C}}(n,m)$ that makes these maps continuous. Then φ_τ turns out to be a homeomorphism. Then $Gr_{\mathbb{C}}(n,m)$ becomes a $(m-n)n$-dimensional complex manifold. Then we can check that $\{(U_\tau, \varphi_\tau)\}_{\tau \in S}$ is a smooth complex structure on $Gr_{\mathbb{C}}(n,m)$, which is called that Grassmannian atlas on $Gr_{\mathbb{C}}(n,m)$.

- First we will show that $\bigcup_{\tau \in S} U_\tau = Gr_{\mathbb{C}}(n,m)$.

We already have $\bigcup_{\tau \in S} U_\tau \subset Gr_{\mathbb{C}}(n,m)$. Now let $V \in Gr_{\mathbb{C}}(n,m)$, we'll show that V belongs to some U_τ. Let $\{v_1, \dots, v_n\}$ be a basis of V. Each v_i is a vector in $\mathbb{C}^m$. Thus we have an $n \times m$ matrix

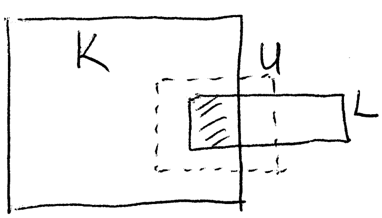
$$\begin{array}{cc} \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & & \\ v_{n1} & v_{n2} & \cdots & v_{nm} \end{pmatrix} & \begin{array}{c} \leftarrow v_1 \\ \leftarrow v_2 \\ \\ \leftarrow v_n \end{array} \end{array}$$

Because $\{v_1, \dots, v_n\}$ is linearly independent, the above matrix has rank n. After doing row reduction, we'll get a row-echelon matrix with exactly

n columns of the form $\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j \qquad (1 \leq j \leq n)$

Let τ be the set of indices of these n columns in $\{1, 2, \dots, m\}$. Then $\tau \in S$. Moreover, the projection of V on each elementary vector e_i, where $i \in \tau$, is not zero. Thus the projection of V on $\mathbb{C} e_i$ is the whole $\mathbb{C} e_i$. Thus the projection of V on $W_\tau = \bigoplus_{i \in \tau} \mathbb{C} e_i$ is the whole W_τ. Since V and W_τ are both n-dimensional vector spaces, this projection map is bijective. Thus $V \in U_\tau$.

② Let M be a manifold and K, L be closed sub set of M. Let $f, g : M \to \mathbb{R}$ be smooth maps such that $f = g$ on a open subset $U \supset (K \cap L)$.



We have
$$\begin{aligned} U \cup (M \setminus K) \cup (M \setminus L) &= U \cup (M \setminus (K \cap L)) \\ &\supset (K \cap L) \cup (M \setminus (K \cap L)) \\ &= M. \end{aligned}$$

Thus $\{U, M \setminus K, M \setminus L\}$ is an open cover of M. Let $\{\varphi_1, \varphi_2, \varphi_3\}$ be ~~the~~ a smooth partition of unity of this cover, such that $\operatorname{supp} \varphi_1 \subset U$, $\operatorname{supp} \varphi_2 \subset M \setminus K$, $\operatorname{supp} \varphi_3 \subset M \setminus L$. We have $\varphi_1(x) + \varphi_2(x) + \varphi_3(x) = 1$ for all $x \in M$. Put $h : M \to \mathbb{R}$, $h = f\varphi_3 + g\varphi_2 + f\varphi_1$. Then h is smooth. We'll show that $h|_K = f|_K$ and $h|_L = g|_L$.

- For $x \in K \setminus U$: $\varphi_2(x) = \varphi_1(x) = 0$. Thus $h(x) = f(x)\underbrace{\varphi_3(x)}_{1} = f(x)$.
- For $x \in K \cap U$: $\varphi_2(x) = 0$. Then $h(x) = f(x)\varphi_3(x) + f(x)\varphi_1(x)$

$$= f(x)\big(\underbrace{\varphi_3(x) + \varphi_1(x)}_{1}\big)$$

$$= f(x)$$

Therefore $h(x) = f(x)$ for all $x \in K$.

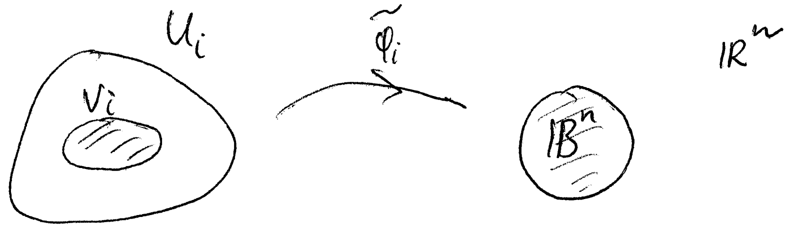
- For $x \in L \setminus U$: $\varphi_3(x) = \varphi_1(x) = 0$. Thus $h(x) = g(x)\underbrace{\varphi_2(x)}_{1} = g(x)$.
- For $x \in L \cap U$: $\varphi_3(x) = 0$. Thus $h(x) = g(x)\varphi_2(x) + \underbrace{f(x)}_{= g(x) \text{ because } x \in U}\varphi_1(x)$.

Then $h(x) = g(x)\big(\underbrace{\varphi_2(x) + \varphi_1(x)}_{1}\big) = g(x)$.

Therefore $h(x) = g(x)$ for all $x \in L$.

(3) Let M be an n-dimensional compact manifold, with a ~~stru~~ smooth structure on it. By restricting the coordinate charts, we can say that each point $x \in M$ has an open neighborhood that is diffeomorphic to the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$. Since $\mathbb{B}^n$ is diffeomorphic to $\mathbb{R}^n$ by the map $x \mapsto \frac{x}{1-|x|^2}$, each point $x \in M$ has an open neighborhood U_x that is diffeomorphic to $\mathbb{R}^n$ under the map $\varphi_x : U_x \to \mathbb{R}^n$. Put $V_x = \varphi_x^{-1}(\mathbb{B}^n)$. We could have assume that $\varphi_x(x) = 0$. Then V_x is an open neighborhood of x. Then $\{V_x\}_{x \in M}$ is an open cover of M. Since M is compact, there exists an open subcover that is finite. We call that subcover $V_1, V_2, \ldots, V_k$, which corresponds to

$U_1, \dots, U_k$ and the maps $\varphi_i : U_i \to \mathbb{R}^n$, where $1 \le i \le k$. We have $V_i \subset \overline{V_i} = \varphi_i^{-1}(\overline{\mathbb{B}^n}) \subset U_i$. Thus there is a smooth extension of φ_i, namely $\tilde{\varphi}_i$, such that $\tilde{\varphi}_i : M \to \mathbb{R}^n$ and $\tilde{\varphi}_i|_{\overline{V_i}} = \varphi_i$ and $\operatorname{supp} \tilde{\varphi}_i \subset U_i$.



Put $\vec{\omega} = (0, 0 \dots, 0, 3) \in \mathbb{R}^n$ and $D_i = \mathbb{B}^n + (i-1)\vec{\omega}$ for every $1 \le i \le k$. Then each D_i is just a translation of $\mathbb{B}^n$ such that $D_i \cap D_j = \phi$ if $i \ne j$. For each $i = 1, \dots, k$, we define a map $\psi_i : M \to \mathbb{R}^n$, $\psi_i(x) = \tilde{\varphi}_i(x) + (i-1)\vec{\omega}$. Then ψ_i is still a smooth map. Moreover, for each $x \in V_i$ we have

$$\psi_i(x) = \tilde{\varphi}_i(x) + (i-1)\vec{\omega} = \varphi_i(x) + (i-1)\vec{\omega} \in \mathbb{B}^n + (i-1)\vec{\omega} = D_i$$

Thus $\psi_i(V_i) \subset D_i$ and ψ_i is injective on V_i.

Now we define the map $f : M \to \mathbb{R}^{nk}$,

$$f(x) = (\psi_1(x), \dots, \psi_k(x)).$$

Then f is smooth because each component ψ_i is smooth. Suppose that $f(x) = f(y)$, ~~Then~~ for some $x, y \in M$. Since $\{V_1, \dots, V_k\}$ is an open cover of M, $x \in V_i$ and $y \in V_j$ for some $i, j = 1, \dots, k$. We have $\psi_i(x) = \psi_i(y)$ because $f(x) = f(y)$. ~~Moreover,~~ We notice that $\tilde{\varphi}_i$ could be chosen as $\varphi_i f_i$ where f_i is a bump function of $\overline{V_i}$ and supported by U_i. Because $0 \le f_i(x) \le 1$

as a matter of fact, other choices may fail.

for all $z \in M$, we have $|\tilde{\varphi}_i(z)| = |\varphi_i(z) f_i(z)| \leq |\varphi_i(z)| < 1$ for all $z \in U_i$. Therefore $|\tilde{\varphi}_i(z)| < 1$ for all $z \in M$. Thus $\tilde{\varphi}_i(M) \subset \mathbb{B}^n$.

Therefore $\psi_i(M) \subset \mathbb{B}^n + (i-1)\vec{w} \subset D_i$. Return to the problem, we have $\psi_i(x) = \psi_i(y)$ and $\psi_i(x) \in D_i$, $\psi_i(y) \in D_j$. Thus $\psi_i(y) \in D_i \cap D_j$.

Thus $i=j$. Thus $x, y \in V_i$. We know that ψ_i is injective on V_i.

Therefore $x=y$. Thus f is injective.

④ Consider the point $p = (3,4,1)$ in Cartesian coordinates on $\mathbb{R}^3$. Denote by (x,y,z) the Cartesian coordinates and (r,θ,z) the cylindrical coordinates.

A generic tangent vector at p in Cartesian coordinates is

$$a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} : C^\infty(\mathbb{R}^3) \to \mathbb{R}$$

$$f \mapsto a\frac{\partial f}{\partial x}\Big|_p + b\frac{\partial f}{\partial y}\Big|_p + c\frac{\partial f}{\partial z}\Big|_p,$$

where $a, b, c \in \mathbb{R}$. To translate this tangent vector into cylindrical coordinates, we'll use the chain rule to find $a', b', c' \in \mathbb{R}$ such that

$$a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} = a'\frac{\partial}{\partial r} + b'\frac{\partial}{\partial \theta} + c'\frac{\partial}{\partial z}.$$

We have

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \underbrace{\frac{\partial z}{\partial x}}_{0}\frac{\partial}{\partial z},$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \underbrace{\frac{\partial z}{\partial y}}_{0}\frac{\partial}{\partial z},$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}.$$

Then $a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} = \left(a\frac{\partial r}{\partial x} + b\frac{\partial r}{\partial y}\right)\frac{\partial}{\partial r} + \left(a\frac{\partial \theta}{\partial x} + b\frac{\partial \theta}{\partial y}\right)\frac{\partial}{\partial \theta} + c\frac{\partial}{\partial z}$ (*)

We have the Jacobian matrix

$$\begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}^{-1} = \frac{1}{r}\begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -y & x \end{bmatrix}$$

Because $(x,y,z) = (3,4,1)$, we have $\begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} 3/5 & 4/5 \\ -4 & 1 \end{bmatrix}$

Thus (*) gives $a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} = \left(\frac{3}{5}a + \frac{4}{5}b\right)\frac{\partial}{\partial r} + (-4a+b)\frac{\partial}{\partial \theta} + c\frac{\partial}{\partial z}$.

The corresponding tangent vector in cylindrical coordinate is

$$C^\infty(\mathbb{R}^3) \longrightarrow \mathbb{R}$$
$$f \longmapsto \left(\frac{3}{5}a + \frac{4}{5}b\right)\frac{\partial f}{\partial r} + (-4a+b)\frac{\partial f}{\partial \theta} + c\frac{\partial f}{\partial z}.$$

(5) Let p be a point in a smooth manifold M. We denote by $T_p(M)$ the set of all tangent vectors of M at p, i.e. the set of linear maps $D: C^\infty(M) \to \mathbb{R}$ that satisfy the product rule $D(fg) = (Df)g(p) + (Dg)f(p)$.

Then we can define an addition and scalar multiplication on $T_p(M)$ in a natural way

$$(D_1 + D_2)(f) := D_1(f) + D_2(f), \quad \forall D_1, D_2 \in T_p(M),\ f \in C^\infty(M),$$
$$(cD)f := c(Df), \quad \forall D \in T_p(M),\ f \in C^\infty(M),\ c \in \mathbb{R}.$$

With these definitions, the operators on $T_p(M)$ satisfy the axioms of modules over $\mathbb{R}$. Then $T_p(M)$ becomes a vector space over $\mathbb{R}$.

Because M is a smooth manifold, there is a chart $(U, \varphi : U \to \mathbb{R}^n)$ containing p such that $\varphi : U \to \mathbb{R}^n$ is a diffeomorphism. We define the following map $\varphi_* : T_p(U) \longrightarrow T_{\varphi(p)}(\mathbb{R}^n)$,

$$\varphi_*(f)(g) = f(g \circ \varphi) \qquad \forall\, f \in T_p(U),\ g \in C^\infty(\mathbb{R}^n)$$ ~~a tangent vector of $\mathbb{R}^n$ at $\varphi(p)$~~.

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[dr, dashed] & \mathbb{R}^n \arrow[d, "g"] \\ & \mathbb{R} \end{tikzcd}$$

First we show that φ_* is well-defined by showing that $\varphi_*(f) \in T_{\varphi(p)}(\mathbb{R}^n)$. The map $g \longmapsto f(g \circ \varphi)$ is linear because f is linear.

About the product rule,
$$\begin{aligned} f((gh)\circ\varphi) &= f((g\circ\varphi)(h\circ\varphi)) \\ &= f(g\circ\varphi)\, h\circ\varphi(p) + f(h\circ\varphi)\, g\circ\varphi(p) \\ &= f(g\circ\varphi)\, h(\varphi(p)) + f(h\circ\varphi)\, g(\varphi(p)). \end{aligned}$$

Thus $\varphi_*(f)$ satisfies the product rule. Then $\varphi_*(f) \in T_{\varphi(p)}(\mathbb{R}^n)$.

Secondly, we see that φ_* is $\mathbb{R}$-linear because

$$\varphi_*(f_1 + f_2)(g) = (f_1 + f_2)(g\circ\varphi) = f_1(g\circ\varphi) + f_2(g\circ\varphi) = \varphi_*(f_1)(g) + \varphi_*(f_2)(g),$$

$$\varphi_*(cf)(g) = (cf)(g\circ\varphi) = c\, f(g\circ\varphi) = c\, \varphi_*(f)(g).$$

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Since $U \xrightarrow{\varphi} \mathbb{R}^n$ is a diffeomorphism, there exists a smooth inverse $\mathbb{R}^n \xrightarrow{\Psi} U$. We have

$$U \xrightarrow{\varphi} \mathbb{R}^n \xrightarrow{\Psi} U,$$

$$T_p(U) \xrightarrow{\varphi_*} T_{\varphi(p)}(\mathbb{R}^n) \xrightarrow{\Psi_*} T_p(U).$$

For any $f \in T_p(U)$ and $g \in C^\infty(U)$, we have

$$\Psi_* \circ \varphi_*(f)(g) = \varphi_*(f)(g \circ \Psi) = f\big(\underbrace{(g \circ \Psi) \circ \varphi}_{id}\big) = f(g).$$

Thus $\Psi_* \circ \varphi_*(f) = f$. Since f is arbitrary, $\Psi_* \circ \varphi_* = id_{T_p(U)}$. Similarly, by using the chains $\mathbb{R}^n \xrightarrow{\Psi} U \xrightarrow{\varphi} \mathbb{R}^n$,

$$T_{\varphi(p)}(\mathbb{R}^n) \xrightarrow{\Psi_*} T_p(U) \xrightarrow{\varphi_*} T_{\varphi(p)}(\mathbb{R}^n),$$

we get $\varphi_* \circ \Psi_* = id_{T_{\varphi(p)}(\mathbb{R}^n)}$. Therefore φ_* is bijective. Therefore it's a linear isomorphism between $T_p(U)$ and $T_{\varphi(p)}(\mathbb{R}^n)$. In the lecture, we have shown that the tangent space of $\mathbb{R}^n$ at any point is an n-dimensional vector space with the basis $\left\{a^i \frac{\partial}{\partial x^i} : i = 1, \dots, n\right\}$, where $p = (a^1, \dots, a^n)$ in Cartesian coordinates. Thus $T_p(U)$ is also an n-dimensional vector space over $\mathbb{R}$. Since we can identify $T_p(M)$ with $T_p(U)$, $T_p(M)$ is an n-dimensional vector space over $\mathbb{R}$.