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Math 8302: Topology & Manifolds

Homework 3

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③ Let $M$ be a smooth manifold with atlas $\{(U_i, \varphi_i)\}_{i\in I}$. The tangent bundle of $M$ is the set

$$TM = \coprod_{p\in M} T_pM = \{(p,v) : p\in M, v\in T_pM\}.$$

We'll give $TM$ a natural smooth structure. Consider the following projection map $\pi: TM \to M$, $\pi(p,v) = p$. For each $p\in U_i$, we have $\varphi_i(p)\in\mathbb{R}^n$. Let $(x^1,\dots,x^n)$ be the Cartesian coordinates on $\mathbb{R}^n$. Then the set $\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ is a basis for $T_pM$. Then for each $v\in T_pM$, we have an $n$-tuple $(a^1,\dots,a^n)\in\mathbb{R}^n$ such that $v = a^i\frac{\partial}{\partial x^i}\Big|_p$. Put $V_i = \pi^{-1}(U_i)$. We can define the map $\Psi_i : V_i \to \mathbb{R}^{2n}$, $\Psi_i(p,v) = (\varphi_i(p), a^1,\dots,a^n)$.

We'll show that $\{(V_i, \Psi_i)\}_{i\in I}$ is a smooth atlas on $TM$. Applying the Smooth Manifold Construction Lemma in Lee, page 21, we need to check 5 points:

(i) For each $i\in I$, $\Psi_i(V_i)$ is an open subset of $\mathbb{R}^{2n}$,

(ii) For each $i,j\in I$, $\Psi_i(V_i\cap V_j)$ is an open subset of $\mathbb{R}^{2n}$,

(iii) If $i,j\in I$ and $V_i\cap V_j\neq\emptyset$, then $\Psi_i\circ\Psi_j^{-1}: \Psi_j(V_i\cap V_j)\to\Psi_i(V_i\cap V_j)$ is a diffeomorphism,

(iv) Countably many $V_i$'s cover $TM$,

(v) For each $\tilde{p}, \tilde{q} \in TM$ distinct, either there exists $V_i$ such that $\tilde{p}, \tilde{q} \in V_i$ or there exists $V_i$, $V_j$ disjoint with $\tilde{p} \in V_i$, $\tilde{q} \in V_j$.

Verify (i)

For any $i, j \in I$, we have by definition

$$\begin{aligned}
\psi_i(V_i \cap V_j) &= \{ \psi_i(p,v) : (p,v) \in V_i \cap V_j \} \\
&= \{ \psi_i(p,v) : (p,v) \in \pi^{-1}(U_i \cap U_j) \} \\
&= \{ \psi_i(p,v) : p \in U_i \cap U_j, v \in T_p M \} \\
&= \{ (\varphi_i(p), a^1, \dots, a^n) : p \in U_i \cap U_j, v = a^l \frac{\partial}{\partial x^l}\Big|_p ; a^1, \dots, a^n \in \mathbb{R} \} \\
&= \{ (\varphi_i(p), a^1, \dots, a^n) : p \in U_i \cap U_j, a^1, \dots, a^n \in \mathbb{R} \} \\
&= \varphi_i(U_i \cap U_j) \times \mathbb{R}^n
\end{aligned}$$

For the case $i=j$, we get $\psi_i(V_i) = \varphi_i(U_i) \times \mathbb{R}^n$. Since $\varphi_i(U_i)$ is open in $\mathbb{R}^n$, $\psi_i(V_i)$ is open in $\mathbb{R}^{2n}$.

Verify (ii)

We have showed that $\psi_i(V_i \cap V_j) = \varphi_i(U_i \cap U_j) \times \mathbb{R}^n$. Since $\varphi_i(U_i \cap U_j)$ is open in $\mathbb{R}^n$, $\psi_i(V_i \cap V_j)$ is open in $\mathbb{R}^{2n}$.

Verify (iii) Consider the map $\psi_i \circ \psi_j^{-1} : \underbrace{\psi_j(V_i \cap V_j)}_{= \varphi_j(U_i \cap U_j) \times \mathbb{R}^n} \to \underbrace{\psi_i(V_i \cap V_j)}_{= \varphi_i(U_i \cap U_j) \times \mathbb{R}^n}$.

For each $(r, a^1, \dots, a^n) \in \varphi_j(U_i \cap U_j) \times \mathbb{R}^n$, there is $p \in U_i \cap U_j$ such that $r = \varphi_j(p)$.

Then $\psi_j^{-1}(r, a^1, \dots, a^n) = (p, v)$, with $v = a^i \frac{\partial}{\partial x^i}\Big|_p \in T_pM$.

Then $\psi_i \circ \psi_j^{-1}(r, a^1, \dots, a^n) = \psi_i(p, v) = (\varphi_i(p), b^1, \dots, b^n)$ such that $v = b^i \frac{\partial}{\partial x^i}\Big|_p$.

Since $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is a basis of $T_pM$, $a^l = b^l$ for every $l$. Thus,

$\psi_i \circ \psi_j^{-1}(r, a^1, \dots, a^n) = (\varphi_i(p), a^1, \dots, a^n) = (\varphi_i \circ \varphi_j^{-1}(r), a^1, \dots, a^n)$. Thus we can rewrite

$$\psi_i \circ \psi_j^{-1} : \varphi_j(U_i \cap U_j) \times \mathbb{R}^n \longrightarrow \varphi_i(U_i \cap U_j) \times \mathbb{R}^n,$$
$$(r, a^1, \dots, a^n) \longmapsto (\varphi_i \circ \varphi_j^{-1}(r), a^1, \dots, a^n).$$

Because the map $r \in \varphi_j(U_i \cap U_j) \longmapsto \varphi_i \circ \varphi_j^{-1}(r) \in \varphi_i(U_i \cap U_j)$ is a transition map on smooth manifold $M$, it is smooth. Thus $\psi_i \circ \psi_j^{-1}$ is also smooth as a map from an open subset of $\mathbb{R}^{2n}$ to an open subset of $\mathbb{R}^{2n}$.

Verify (iv)

Because $M$ is a manifold and $\{U_i\}_{i\in I}$ is an open cover of $M$, there exists a countable subcover. Without loss of generality, we can name it $\{U_1, U_2, \dots\}$.

Then $\bigcup_{i=1}^{\infty} V_i = \bigcup_{i=1}^{\infty} \pi^{-1}(U_i) = \pi^{-1}\left(\bigcup_{i=1}^{\infty} U_i\right) = \pi^{-1}(M) = TM$. Thus $\{V_i\}_{i\in\mathbb{N}}$ is a countable ~~ope~~ cover of $TM$.

Verify (v)

Take $(p,v), (q,w) \in TM$ which are distinct. If $p = q$ then $(p,v), (q,w)$ belongs to $V_i$ where $p \in U_i$. For the case $p \neq q$, we can find a neighborhood $A$

of $p$ and a neighborhood $B$ of $q$ such that $A\cap B=\phi$. This is possible ~~is~~ because $M$ is Hausdorff. Then we can choose a chart $U_i\subset A$ which is also a neighborhood of $p$, and a chart $U_j\subset B$ which is also a neighborhood of $q$. This is possible because $\{(U_i,\varphi_i)\}_{i\in I}$ is a maximal atlas. Then $U_i\cap U_j=\phi$. Thus $V_i\cap V_j=\pi^{-1}(U_i\cap U_j)=\pi^{-1}(\phi)=\phi$, and $(p,v)\in V_i$, $(q,w)\in V_j$.

Next, let $f: M\to N$ be a smooth map. We'll show that it gives rise to a smooth map $df: TM\longrightarrow TN$. First, let's give names to coordinate charts on $M, N, TM, TN$. Let the atlas on $M$ be $\{(U_i,\varphi_i)\}_{i\in I}$, and the atlas on $N$ be $\{(U'_j,\varphi'_j)\}_{j\in J}$. Put $\pi: TM\to M$, $\pi': TN\to N$ ~~to~~ be the projection maps. Put $V_i=\pi^{-1}(U_i)$, $V'_j=\pi'^{-1}(U'_j)$. By the construction of smooth structure on $TM, TN$ as above, the coordinate maps on $TM, TN$ are respectively

$$\Psi_i: V_i\longrightarrow\mathbb{R}^{2m},\quad \Psi_i(p,v)=(\varphi_i(p),a^1,\dots,a^m),\ \text{where } v=a^\ell\frac{\partial}{\partial x^\ell}\Big|_p.$$

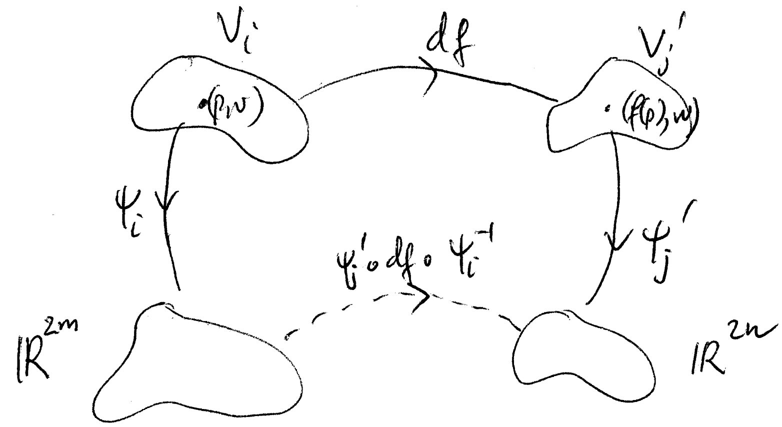
$$\Psi'_j: V'_j\longrightarrow\mathbb{R}^{2n},\quad \Psi'_j(q,w)=(\varphi'_j(q),b^1,\dots,b^n)\ \text{where } w=b^\ell\frac{\partial}{\partial x^\ell}\Big|_q$$

Here $m=\dim M$, $n=\dim N$, $(x^1,\dots,x^m)$ is the Cartesian coordinates on $\mathbb{R}^m$ and $(x^1,\dots,x^n)$ is the Cartesian coordinates on $\mathbb{R}^n$. Let $L:\mathbb{R}^m\to\mathbb{R}^n$ be any linear transformation. We can think of $L$ as an $n$-by-$m$ matrix with real coefficients.

Then we define a map $df : TM \longrightarrow TN$, such that $df(p,v) = (f(p), w)$,

where $\underbrace{v = a^i \frac{\partial}{\partial x^i}\Big|_p}_{\in T_pM}$, $\underbrace{w = b^i \frac{\partial}{\partial x^i}\Big|_{f(p)}}_{\in T_{f(p)}N}$, and $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = L \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$.

We'll show that $df$ is a smooth map. Suppose that $(p,v) \in V_i$ and $(f(p), w) \in V_j'$. We'll show that $\psi_j' \circ df \circ \psi_i^{-1} : \psi_i(V_i) \longrightarrow \psi_j'(V_j')$ is smooth.




For each $(r, a^1, \dots, a^m) \in \psi_i(V_i) \subset \mathbb{R}^{2m}$, we have $(r, a^1, \dots, a^m) = \varphi_i(U_i) \times \mathbb{R}^m$.

Thus there is $p \in U_i$ such that $r = \varphi_i(p)$. Thus $\psi_i^{-1}(r, a^1, \dots, a^m) = (p, v)$, where

$v = a^1 \frac{\partial}{\partial x^1}\Big|_p + \dots + a^m \frac{\partial}{\partial x^m}\Big|_p$. Then $df(p,v) = (f(p), w)$ where

$$w = b^1 \frac{\partial}{\partial x^1}\Big|_{f(p)} + \dots + b^n \frac{\partial}{\partial x^n}\Big|_{f(p)} \quad \text{and} \quad \begin{pmatrix} b^1 \\ \vdots \\ b^n \end{pmatrix} = L \begin{pmatrix} a^1 \\ \vdots \\ a^m \end{pmatrix}.$$

Then $\psi_j'(f(p), w) = (\varphi_j'(f(p)), b^1, \dots, b^n) = \left(\varphi_j'(f(p)), L\begin{pmatrix} a^1 \\ \vdots \\ a^m \end{pmatrix}\right)$.

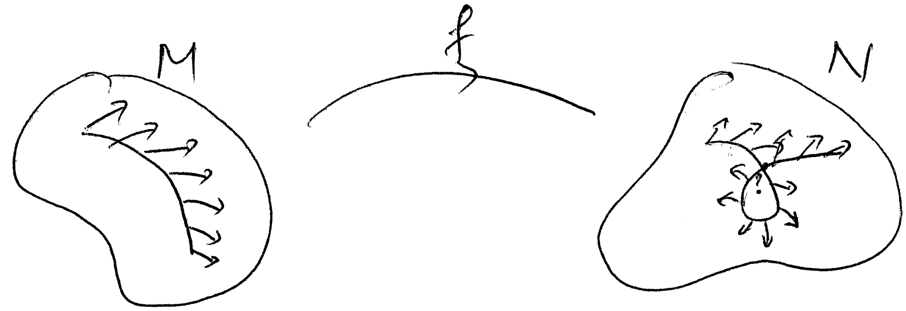
Therefore, $\psi_j' \circ df \circ \psi_i^{-1}(r, a_1, \dots, a_n) = \left(\varphi_j' \circ f \circ \varphi_i^{-1}(r), L\begin{pmatrix} a^1 \\ \vdots \\ a^m \end{pmatrix}\right)$.

Since $f$ is a smooth map, the map $r \mapsto \varphi_j' \circ f \circ \varphi_i(r)$ is smooth. Moreover, $L$ is also smooth because it is a linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$. Thus the map $\varphi_j' \circ df \circ \varphi_i$ is smooth as a map from an open subset of $\mathbb{R}^{2m}$ to an open subset of $\mathbb{R}^{2n}$.

(2) Given a map $f: M \to N$ of smooth manifolds, we'll explain why this does not push forward to a natural vector field on $N$. Let $Y: M \to TM$ be a (rough) vector field on $M$, i.e. a map such that $Y(p) \in \{p\} \times T_pM$.

$$\begin{tikzcd} M \arrow[r, "Y"] \arrow[d, "f"'] & TM \arrow[d, "df"] \\ N \arrow[r, "Y'"] & TN \end{tikzcd}$$

We have prove in problem (3) that there exists a natural smooth map $df$ from $TM$ to $TN$. Also, we showed that there can be many different such maps, depending on our choice of the linear transformation $L: \mathbb{R}^m \to \mathbb{R}^n$. A natural push forward of $Y$ on $N$ should be a map $Y': N \to TN$ that makes the above diagram commute. That is, $Y'(f(p)) = df(Y(p))$ for every $p \in M$. A problem with this definition is that $f$ may not be surjective. In such a case, we have not defined $Y'$ at every point in $N$. Also, there is no natural way to define $Y'$ on $N \setminus f(M)$. Now even if $f$ is surjective the definiton $Y'(f(p)) = df(Y(p))$ can be illegitimate because $f$ may not be injective. In such a case, there are more than one tangent vector defined at a point in $N$.




Example 1 $M = N = \mathbb{R}$ with standard structure, $f(x) = e^x$. We choose $Y: \mathbb{R} \to T\mathbb{R}$, $Y(p) = (p, f'(p))$, where $f'(p)$ can be identified with the element $f'(p)\frac{d}{dx}\Big|_p$ in $T_p\mathbb{R}$. Then $Y$ is a vector field on $\mathbb{R}$ because the first component of $Y(p)$ is $p$. Since $f$ is not surjective, $Y'$ is not defined (in a natural way everywhere).

Example 2 $M = N = \mathbb{R}$ with standard smooth structure, $f(x) = x^3 - x$. Again, we choose the vector field $Y(p) = (p, f'(p))$. We see that $f(0) = f(1) = 0$. By choosing the linear transformation $L: \mathbb{R} \to \mathbb{R}$ to be the identity map, we have $df(p, v) = (f(p), v)$ for all $p, v \in \mathbb{R}$.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "Y"] \arrow[d, "f"'] & T\mathbb{R} \arrow[d, "df"] \\ \mathbb{R} \arrow[r, "Y'"] & T\mathbb{R} \end{tikzcd}$$

If we define $Y'(f(p)) := df(Y(p))$ $\forall p \in \mathbb{R}$, then

- At $p = 0$: $Y'(0) = df(Y(0)) = df(0, f'(0)) = df(0, -1)$
$= (0, -1)$.

• At $p = 1$, $\gamma'(0) = df(\gamma(1)) = df(1, f'(1)) = df(1,2) = (9 2)$.

Thus $\gamma'(0)$ is not well-defined.

(1) Let $f: M \to N$ be a map of smooth manifolds and $w: N \to T^*N$ be a covector field. We'll show that $w$ pulls back to a natural covector field on $M$.

$$M \xrightarrow{f} N \xrightarrow{w} T^*N$$

For each $p \in M$, we have a push forward $T_pM \xrightarrow{df_p} T_{f(p)}N$, which is called the derivative of $f$ at $p$. Because $w_{f(p)} \in T^*_{f(p)}N$, we have the following composition

$$\begin{tikzcd} T_pM \arrow[r, "df_p"] \arrow[rr, bend right, "w_{f(p)} \circ df_p"'] & T_{f(p)}N \arrow[r, "w_{f(p)}"] & \mathbb{R} \end{tikzcd}.$$

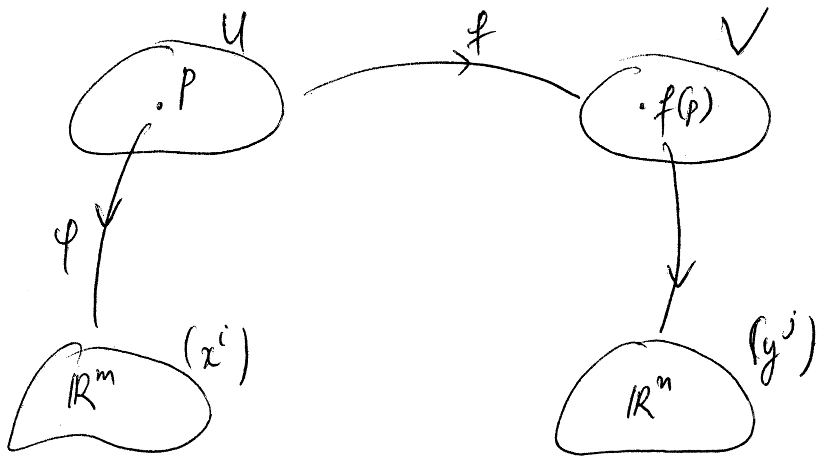
Because $w_{f(p)}$ and $df_p$ are both linear and continuous, $w_{f(p)} \circ df_p \in T_p^*M$.

Thus we get a covector field $Y: M \to T^*M$ defined by

$$Y_p = w_{f(p)} \circ df_p .$$

In case $w$ is a smooth covector field, we'll show that $Y$ is also a smooth covector field. For each $p_0 \in M$, we'll show that $Y$ is smooth at $p_0$. Let ~~$(U, \varphi)$ be a coordinate chart containing $p_0$~~ $(V, \psi)$ be a coordinate chart containing $f(p_0)$ in $N$, and $(U, \varphi)$ be a coordinate chart containing $p_0$ in

M such that $U \subset f^{-1}(V)$.




For each $p \in U$, we know that $\{dx^1|_p, \dots, dx^m|_p\}$ is a basis of $T_p^*U$. Thus,

$$Y_p = \underbrace{Y_k(p)}_{k\text{'th component function}} \, dx^k|_p$$

To show that $Y_p$ is smooth in $U$, we need to show that each component function $Y_i : U \to \mathbb{R}$ is smooth. We have

$$Y_i(p) = \underbrace{Y_i(p)\, dx^i|_p \left( \frac{\partial}{\partial x^i}\Big|_p \right)}_{\text{no sum}} = Y_p \left( \frac{\partial}{\partial x^i}\Big|_p \right)$$

$$= \omega_{f(p)} \circ df_p \left( \frac{\partial}{\partial x^i}\Big|_p \right) \qquad (1)$$

Let $(J_i^j)$ be the Jacobian matrix of the transformation $(x^i)$ to $(y^j)$.

$$J_i^j(p) = \frac{\partial y^j}{\partial x^i}(\varphi(p))$$

Then each function $J_i^j : U \to \mathbb{R}$ is smooth. By the definition of the derivative of $f$ at $p$, we get $\quad df_p \left( \frac{\partial}{\partial x^i}\Big|_p \right) = J_i^j(p) \frac{\partial}{\partial y^j}\Big|_{f(p)}. \qquad (2)$

Now substituting (2) into (1), we get

$$Y_i(p) = \omega_{f(p)}\left( J_i^j(p) \frac{\partial}{\partial y^j}\Big|_{f(p)} \right) = J_i^j(p)\, \omega_{f(p)}\left( \frac{\partial}{\partial y^j}\Big|_{f(p)} \right). \qquad (3)$$

For each $q \in V$, we know that $\{dy^1|_q, \dots, dy^n|_q\}$ is a basis of $T_q^* N$.

Thus $\omega_q = \underbrace{\omega_k(q)}_{k\text{th component function}} dy^k|_q$. Since $\omega_q$ is smooth, $\omega_k : V \to \mathbb{R}$ is also smooth.

Then $\omega_{f(p)} = \omega_k(f(p))\, dy^k|_{f(p)}$. Then

$$\omega_{f(p)}\left( \frac{\partial}{\partial y^j}\Big|_{f(p)} \right) = \omega_k(f(p)) \underbrace{dy^k|_{f(p)}\left( \frac{\partial}{\partial y^j}\Big|_{f(p)} \right)}_{\delta_j^k} = \omega_j(f(p)). \qquad (4)$$

Substituting (4) into (3), we get $Y_i(p) = J_i^j(p)\, \omega_j(f(p))$.

Because $J_i^j : \varphi(U) \to \psi(V)$, $\omega_j : V \to \mathbb{R}$, $f : M \to N$ are all smooth, $Y_i$ is smooth on $U$. In particular, $Y_i$ is smooth at $p_0$.

4 Let $M$ be a smooth manifold, $Y$ be a topological space and $p : Y \to M$ be a covering map. We'll show that $Y$ can be given a ~~stru~~ smooth structure so that $p$ is a smooth map. Let $\{Y_i\}_{i \in I}$ be the set of path-connected components of $Y$. For each $i \in I$, we define $p_i : Y_i \to M$, $p_i(y) = p(y)$. First we'll show that $p_i$ is a covering map. For each $x \in M$, there exists an open neighborhood of $x$ in $U$, which can be assumed to be path-connected because $M$ is a manifold, such that

there is a discrete space $F$ and a homeomorphism $\varphi : p^{-1}(U) \to U \times F$ such that

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "p"] \arrow[d, "\varphi"'] & U \\ U \times F \arrow[ur, "\pi"'] & \end{tikzcd}$$

commutes

For each $\alpha \in F$, $U \times \{\alpha\}$ is open and path-connected subspace of $U \times F$. Thus $\varphi^{-1}(U \times \{\alpha\})$ is also path-connected. Thus $\varphi^{-1}(U \times \{\alpha\})$ is contained in exactly one of the path-connected component of $Y$, called $Y_{i_\alpha}$. With $i \in I$ fixed, we define

$$F' = \{\alpha \in F : i_\alpha = i\}$$

We'll show that $\varphi^{-1}(U \times F') = \{y \in Y_i : \pi(\varphi(y)) \in U\}$. (1)

"$\subset$": Take $y \in \varphi^{-1}(U \times F')$. Then $\varphi(y) \in U \times F'$. Then there is $\alpha \in F'$ such that $\varphi(y) \in U \times \{\alpha\}$. Thus $y \in \varphi^{-1}(U \times \{\alpha\})$. Because $\alpha \in F'$, $y \in \varphi^{-1}(U \times \{\alpha\}) \subset Y_i$. Moreover, $\pi(\varphi(y)) \in \pi(U \times \{\alpha\}) \subset U$.

"$\supset$": Take $y \in Y_i$ such that $\pi(\varphi(y)) \in U$. Put $\varphi(y) = (z, \alpha)$ for some $z \in U$, $\alpha \in F$. Then $y \in \varphi^{-1}(U \times \{\alpha\})$. ~~Thus~~ Since $\varphi^{-1}(U \times \{\alpha\})$ is path-connected in $Y$ and $\varphi^{-1}(U \times \{\alpha\}) \cap Y_i \ni y$, we have $\varphi^{-1}(U \times \{\alpha\}) \subset Y_i$. Thus $\alpha \in F'$. Thus $y \in \varphi^{-1}(U \times F')$.

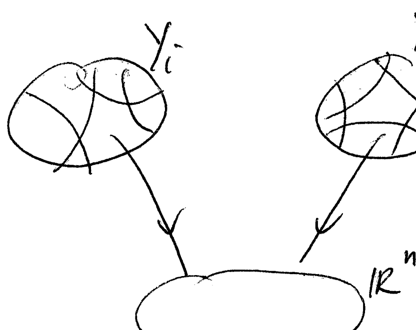
The identity (1) shows that
$$\begin{aligned} \varphi^{-1}(U \times F') &= \{y \in Y_i : p(y) \in U\} \\ &= \{y \in Y_i : p_i(y) \in U\} \\ &= p_i^{-1}(U) \end{aligned}$$

Thus $\varphi : p_i^{-1}(U) \to U \times F'$ is a homeomorphism.

Then we have the commutative diagram

$$\begin{tikzcd} p_i^{-1}(U) \arrow[r, "p_i"] \arrow[d, "\varphi", "\wr"'] & U \\ U\times F' \arrow[ur, "\pi"'] & \end{tikzcd}$$

Thus $p_i : Y_i \to U$ is a covering map. Suppose that we can give each $Y_i$ a smooth structure such that $p_i$ is a smooth map. Then we can introduce an atlas on $Y$ as a union of the atlases on each $Y_i$. The compatibility of the coordinate charts on $Y$ is simply the compatibility of coordinate charts on each component $Y_i$ because $Y_i \cap Y_j = \phi$ if $i \neq j$.




Note that the $2^{nd}$ countability of the topology on $Y$ is guaranteed only if $Y$ has a countable number of path-connected components.

To show that $p$ is smooth on this new atlas, we only need to show that each $Y_i$ is actually ~~a connected component of $Y$ (as opposed to path-connected component). To do so, it suffices to show that $Y$ is locally connected.~~ an open subset of $Y$. It suffices to show that $Y$ is locally connected. Take $y \in Y$ and put $x = p(y) \in M$. Then there is an open neighborhood $U$ of $x$ in $M$ such that we have a commutative diagram

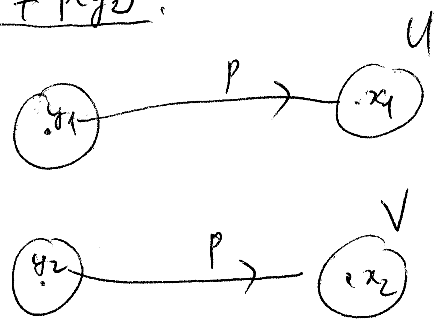
$$\begin{tikzcd} p^{-1}(U) \arrow[r, "p"] \arrow[d, "\varphi", "\wr"'] & U \\ U\times F \arrow[ur, "\pi"'] & \end{tikzcd}$$

Moreover, by shrinking $U$ if necessary, we can assume $U$ is path-connected, and $U \simeq \mathbb{R}^n$.

Let $\alpha \in F$ such that $\varphi(y) \in U \times \{\alpha\}$. Since $U \times \{\alpha\}$ is path-connected, $V = \varphi^{-1}(U \times \{\alpha\})$ is path connected and containing $y$. Since $U \times \{\alpha\}$ is open in $U \times F$, $V$ is open in $Y$. We have a homeomorphism $V \xrightarrow{\varphi} U \times \{\alpha\}$. Since $U \simeq \mathbb{R}^n$, we get $V \simeq \mathbb{R}^n$. Thus $Y$ is locally Euclidean, and hence locally connected.

Therefore, from the beginning, we can assume $Y$ is path-connected. Also we proved that $Y$ is locally looks like $\mathbb{R}^n$. Next we'll show that $Y$ is Hausdorff. Let $y_1 \neq y_2$ be two points in $Y$. We consider 2 cases.

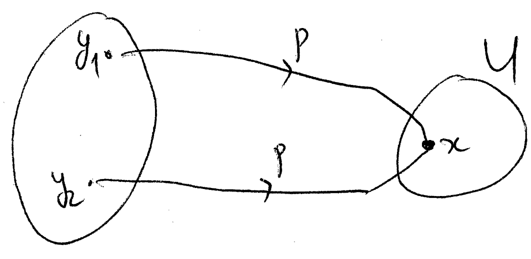
- $\underline{p(y_1) \neq p(y_2)}$:




Put $x_1 = p(y_1)$ and $x_2 = p(y_2)$. Since $M$ is Hausdorff, there exist open neighborhoods $U$ of $x_1$ and $V$ of $x_2$ that are disjoint.

Then $p^{-1}(U)$ is an open neighborhood of $y_1$ and $p^{-1}(V)$ is an open neighborhood of $y_2$, and $p^{-1}(U) \cap p^{-1}(V) = p^{-1}(\underbrace{U \cap V}_{=\emptyset}) = \emptyset$.

- $\underline{p(y_1) = p(y_2)}$:




Put $x = p(y_1) = p(y_2)$. There exists an open neighborhood $U$ of $x$ in $M$ such that we have the following commutative diagram:

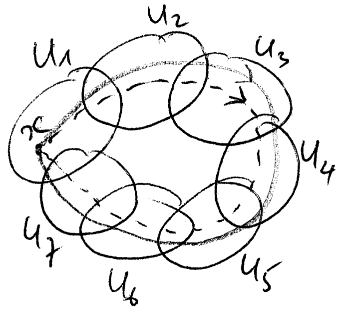
$$\begin{tikzcd} p^{-1}(U) \arrow[r, "p"] \arrow[d, "\varphi", "\cong"'] & U \\ U \times F \arrow[ur, "\pi"'] & \end{tikzcd}$$

We put $\varphi(y_1) = (z_1, \alpha)$, $\varphi(y_2) = (z_2, \beta)$. Suppose by contradiction that $\alpha = \beta$. Then we know that $x = p(y_1) = \pi(\varphi(y_1)) = z_1$,

$$x = p(y_2) = \pi(\varphi(y_2)) = z_2.$$

Then $\varphi(y_1) = \varphi(y_2)$, which leads to $y_1 = y_2$ since $\varphi$ is a homeomorphism. This is a contradiction. Thus $\alpha \neq \beta$. Then $y_1 \in \underbrace{\varphi^{-1}(U \times \{\alpha\})}_{V_1}$ and $y_2 \in \underbrace{\varphi^{-1}(U \times \{\beta\})}_{V_2}$. Then $V_1 \ni y_1$ and $V_2 \ni y_2$ are open in $Y$ and disjoint. Therefore, $Y$ is Hausdorff.

Next, we'll show that $Y$ is second countable. First, at any point $x \in M$, we have a fundamental group $\pi_1(M, x)$. Since $M$ is an $n$-manifold, it can be covered by a countable number of open sets ~~which~~ each of which is homeomorphic to $\mathbb{R}^n$. Each loop at $x$ runs through a finite number of these open sets due to the compactness of the loop. We can see that two loops running through the same ~~family~~ sequence of open sets are homotopic. Thus, the cardinality of $\pi_1(M, x)$




is at most the cardinality of ~~a~~ sequences of finite length of elements from a countable set. Thus $\pi_1(M, x)$ is countable.

Next, for each $x \in M$, we'll that the fiber $p^{-1}(x)$ is $\underbrace{\text{countable.}}_{\text{Fix } y \in p^{-1}(x).}$ ~~Co~~ For each loop $\gamma$ at $x$, we

know that $\gamma$ ~~is~~ can be lifted to a unique path $\tilde{\gamma}$ in $Y$ such that $\tilde{\gamma}(0)=y$. We define the map
$$\delta: \pi_1(M,x) \longrightarrow p^{-1}(x),$$
$$[\gamma] \longmapsto \tilde{\gamma}(1).$$

This map is well-defined because $p(\tilde{\gamma}(1)) = \gamma(1) = x$. Thus $\tilde{\gamma}(1) \in p^{-1}(x)$. Moreover, by the property of path lifting, if $[\gamma]=[\gamma']$ then $\tilde{\gamma}(1)=\tilde{\gamma}'(1)$. Now we'll show that $\delta$ is surjective. For each $z \in p^{-1}(x)$, there is a path in $Y$ from $y$ to $z$ because $Y$ is path-connected. We call this path $\eta$. Put $\gamma = f\circ\eta$. Then $\gamma(0)=f(\eta(0))=f(y)=x$, $\gamma(1)=f(\eta(1))=f(z)=x$. Thus $\gamma$ is a loop at $x$ in $M$. Because $\gamma = p\circ\eta$ and $\eta(0)=y$, $\eta$ is the lift of $\gamma$ at $y$. Thus, $\delta([\gamma]) = \eta(1) = z$. Thus $\delta$ is surjective. Since $\pi_1(M,x)$ is ~~ca~~ countable, so is $p^{-1}(x)$.

We will call an open subset $U$ of $M$ evenly covered if there exists a discrete space $F$ together with a homeomorphism $\varphi: p^{-1}(U) \to U\times F$ so that the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "p"] \arrow[d, "\varphi"'] & U \\ U\times F \arrow[ur, "\pi"'] & \end{tikzcd}$$

Put $\mathcal{B} = \{U\subset M : U \text{ is open, connected evenly covered}\}$.

Since $p$ is a covering map, every point of $M$ has an evenly covered open neighborhood. Thus $\mathcal{B}$ is a covering of $M$. Moreover, we know that any ~~evenly covered~~ open subset of an evenly cover open subset of $M$ is also evenly covered. Thus, for each open set $V$ of $M$, and for each $x \in V$, we can find an open, evenly ~~coverd~~ covered neighborhood of $x$ that is contained in $V$. In other words, $V$ is a union of open, evenly covered subsets. Thus $\mathcal{B}$ is a topological basis of $M$.

Since $M$ is second countable, we can extract a countable covered from $\mathcal{B}$, namely $\mathcal{C} = \{U_i\}_{i \in \mathbb{N}}$. As a space, $U_i$ is second countable. Thus, each $U_i$ has ~~an~~ a countable topological basis $\mathcal{B}_i$. For each $U$ open in $M$, we have $U = \bigcup_{i=1}^{\infty} (U \cap U_i)$. Each $U \cap U_i$ is a union of members in $\mathcal{B}_i$. Thus $\bigcup_{i=1}^{\infty} \mathcal{B}_i$ gives us a basis of $M$. This is a countable union of countable sets. Thus $\bigcup_{i=1}^{\infty} \mathcal{B}_i$ is countable. By replacing $\mathcal{C}$ by $\bigcup_{i=1}^{\infty} \mathcal{B}_i$, we could have assumed that $\mathcal{C} = \{U_i\}_{i \in \mathbb{N}}$ ~~a which~~ is also a topological basis of $M$. Thus we obtain a countable basis of $M$ consisting of evenly covered open subset of $M$.

For each $i$, we have a commutative diagram

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "p"] \arrow[d, "\varphi_i \cong"'] & U_i \\ U_i \times F_i \arrow[ur, "\pi"'] & \end{tikzcd}$$

Fix $x \in U_i$, we get a diagram
$$\begin{tikzcd} p^{-1}(x) \arrow[r, "p"] \arrow[d, "\varphi_i \cong"'] & \{x\} \\ \{x\} \times F_i \arrow[ur, "\pi"'] & \end{tikzcd}$$

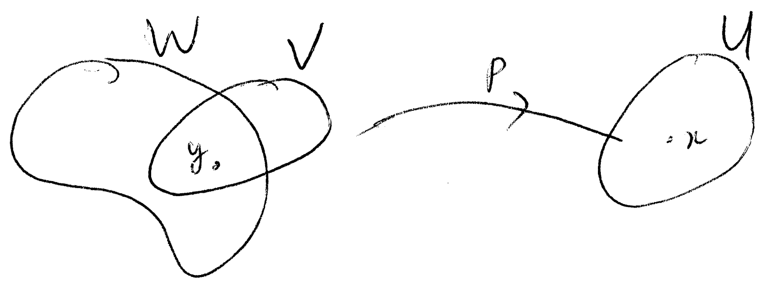
Thus the cardinality of $F_i$ is equal to that of $p^{-1}(x)$, which is countable. Thus, $p^{-1}(U_i)$ has countably many connected components, which are named $V_{ij}$.

Techniquely, the index $j$ can run through a ~~fininite~~ finite or infinite (countable) index set. However, by allowing $V_{ij}$ to repeat itself, we can assume that $j$ runs through $\mathbb{N}$. What we need is that $p|_{V_{ij}} : V_{ij} \to U_i$ is a homeomorphism. Now with that definition we have

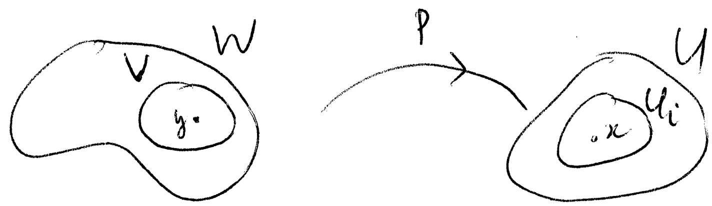
$$p^{-1}(U_i) = \bigcup_{j=1}^{\infty} V_{ij}.$$

Put ~~R~~ $\mathcal{D} = \{V_{ij} \mid i, j \in \mathbb{N}\}$. Then $\mathcal{D}$ is countable. We'll show that $\mathcal{D}$ is a basis of $Y$. Let $W$ be any open subset of $Y$ and $y \in W$. We'll show that there exists $V_{ij}$ such that $y \in V_{ij} \subset W$. If we can prove that, we will finish the proof that $Y$ is second countable.

Put $x = p(y)$. Because $p$ is a covering map, there exists $U \in \mathcal{B}$ such that $x \in U$. Because $y \in p^{-1}(U)$, there exists a connected component $V$ of $p^{-1}(U)$ that contains $y$. We have $p|_V : V \to U$ is a homeomorphism. Thus




$p|_{V \cap W} : V \cap W \to p(V \cap W)$ is also a homeomorphism. By replacing $V$ with $V \cap W$, $U$ with $p(V \cap W)$, we can assume that $V \subset W$. Now we know that $\mathcal{C} = \{U_i\}_{i \in \mathbb{N}}$ is a topological basis of $M$. Thus $U$ is a union of some of these $U_i$'s. Thus there exists $i \in \mathbb{N}$ such that $x \in U_i$.




~~We'll show that $(p|_V)^{-1}(U_i) = \bigcup_{j : V_{ij} \subset V} V_{ij}$.~~

~~"$\subset$" take $z \in (p|_V)^{-1}(U_i)$. Then $z \in V$ and $p(z) \in U_i$. Thus $z \in V$ and $z \in p^{-1}(U_i) = \bigcup_{j=1}^{\infty} V_{ij}$. Thus~~

Put $V' := (p|_V)^{-1}(U_i)$. Then $V' \subset V$ and $p|_{V'} : V' \to U_i$ is a homeomorphism. Since $U_i$ is connected, $V'$ is also connected. Thus $V'$ is a connected subspace of $p^{-1}(U_i)$. Thus $V'$ is contained in some connected component $V_{ij}$. Suppose by contradiction that $V' \subsetneq V_{ij}$. Then there is $z \in V_{ij} \setminus V'$. Put $p(z) = \tilde{x} \in U_i$. Then there is $z' \in V'$ such that $p(z') = \tilde{x}$. Then $p(z) = p(z')$. Because $p|_{V_{ij}} : V_{ij} \to U_i$ is a homeomorphism, $z = z' \in V'$, this is a contradiction. Thus $V' = V_{ij}$. Thus $y \in V' = V_{ij} \subset V \subset W$.

Until now, we have showed that $Y$ is a manifold of the same dimension as $M$. We will introduce an atlas on $Y$ so that the covering map $p : Y \to M$ becomes smooth. We know that $\mathcal{C} = \{U_i\}_{i \in \mathbb{N}}$ is a topological basis of $M$ consisting of evenly covered, open subset of $M$. In the definition of $\mathcal{B}$ on page 15, we could have include one more constraint, namely, $U$ lies in some coordinate chart. If we had done so, now we would have that each $U_i$ lies in some coordinate chart. Now we should assume so. Then there is a homeomorphism $\varphi_i : U_i \to W_i$ from $U_i$ to an open subset of $\mathbb{R}^n$, and $(U_i, \varphi_i)$ is a coordinate chart of $M$. We also define $\mathcal{D} = \{ V_{ij} : i, j \in \mathbb{N} \}$ where $V_{ij}$ is a connected component of $p^{-1}(U_i)$. We showed that $V_{ij} \xrightarrow{p|_{V_{ij}}} U_i$ is a homeomorphism and $\mathcal{D}$ is a topological basis of $Y$.

$$\begin{tikzcd} V_{ij} \arrow[r, "p|_{V_{ij}}", "\sim"'] \arrow[dr, dashed, "\psi_{ij}"'] & U_i \arrow[d, "\varphi_i"] \\ & W_i \end{tikzcd}$$

Define $\psi_{ij} = \varphi_i \circ (p|_{V_{ij}})$. Then $\psi_{ij}$ is a homeomorphism.

Thus $(V_{ij}, \psi_{ij})$ is a coordinate chart on $Y$. Since $\mathcal{D}$ is a basis of $Y$, the family covers $Y$. We'll check the compatibility. Let $i, j, k, l \in \mathbb{N}$.

$$\psi_{ij} \circ \psi_{kl}^{-1} : \psi_{kl}(V_{ij} \cap V_{kl}) \longrightarrow \psi_{ij}(V_{ij} \cap V_{kl})$$

For every $x \in \psi_{kl}(V_{ij} \cap V_{kl})$, we have

$$\psi_{ij} \circ \psi_{kl}^{-1}(x) = \psi_{ij}\left( (\varphi_k \circ p|_{V_{kl}})^{-1} \right)(x) = \psi_{ij}\left( (p|_{V_{kl}})^{-1} \varphi_k^{-1}(x) \right)$$

$$= \varphi_i \circ p|_{V_{ij}} \left( (p|_{V_{kl}})^{-1} \varphi_k^{-1}(x) \right)$$

$$= \varphi_i \circ \varphi_k^{-1}(x)$$

Since $\varphi_i \circ \varphi_k^{-1}$ is smooth at $x$ (and note that $\psi_{kl}(V_{ij} \cap V_{kl})$ is open in $Y$ since

$\psi_{kl}$ is a homeomorphism), $\psi_{ij} \circ \psi_{kl}^{-1}$ is smooth at $x$. Thus the compatibility is now ~~eve~~ verified. Therefore $\{(V_{ij}, \psi_{ij})\}_{i,j \in \mathbb{N}}$ is an (smooth) atlas on $Y$. Now we'll show that $p$ is a smooth map.

Let $x \in Y$. There is a chart $(V_{ij}, \psi_{ij})$ containing $x$. Then $U_i = p(V_{ij})$ is a chart in $M$ containing $p(x)$. By definition, to show that $p$ is smooth at $p$, we only need to show that $\varphi_i \circ p \circ \psi_{ij}^{-1} : W_i \to W_i$ is smooth at $\psi_{ij}(x)$.

$$\begin{tikzcd} V_{ij} \arrow[r, "p"] \arrow[dr, "\psi_{ij}"'] & U_i \arrow[d, "\varphi_i"] \\ & W_i \subset \mathbb{R}^n \end{tikzcd}$$

For any $z \in W_i$, we have $\psi_{ij}^{-1}(z) = (p|_{V_{ij}})^{-1} \circ \varphi_i^{-1}(z)$. Thus

$$\varphi_i \circ p \circ \psi_{ij}^{-1}(z) = \varphi_i \circ \underbrace{p \circ (p|_{V_{ij}})^{-1}}_{id} \circ \varphi_i^{-1}(z) = \varphi_i(\varphi_i^{-1}(z)) = z.$$

Thus $\varphi_i \circ p \circ \psi_{ij}^{-1} = id|_{W_i}$, which is smooth at $x$. Therefore, $p$ is a smooth map.

(5) Let $M$ be an smooth $m$-manifold and $f : M \to \mathbb{R}^n \setminus \{0\}$ be a smooth function. We define the function $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$, with

$$g(y^1, \dots, y^n) = \frac{(y^1, \dots, y^n)}{\sqrt{(y^1)^2 + \dots + (y^n)^2}},$$

where $(y^1, \dots, y^n)$ are Cartesian coordinates of a point in $\mathbb{R}^n \setminus \{0\}$. We'll give necessary and sufficient conditions for the smooth map $g \circ f : M \to S^{n-1}$ to be either ~~su~~ a submersion, an ~~i~~ immersion or a local diffeomorphism.

Let $(y^1, \dots, y^n)$ be Cartesian coordinates in $\mathbb{R}^n$. We denote, for each $i = 1, 2, \dots, n$,

$$V_i^+ = \{(y^1, \dots, y^n) \in \mathbb{R}^n : y^i > 0\},$$

$$V_i^- = \{(y^1, \dots, y^n) \in \mathbb{R}^n : y^i < 0\}.$$

Then $\mathbb{R}^n \setminus \{0\} = \bigcup_{i=1}^n V_i^\pm$. In $V_n^+$, we have a spherical coordinate system $(r, \theta^1, \dots, \theta^n)$ with $r > 0$, $0 < \theta^1, \dots, \theta^{n-1} < \pi$ whose relation with the Cartesian system is given by

$$(*) \quad \begin{cases} y^1 = r\cos\theta^1 \\ y^2 = r\sin\theta^1 \cos\theta^2 \\ \vdots \\ y^{n-1} = r\sin\theta^1 \cdots \sin\theta^{n-2} \cos\theta^{n-1} \\ y^n = r\sin\theta^1 \cdots \sin\theta^{n-2} \sin\theta^{n-1} \end{cases}$$

The backward transformation is a bit more complicated: for each point $y \in V_n^+$ ~~$y \in \mathbb{R}^n$~~ whose Cartesian coordinates are $(y^1, \dots, y^n)$, we put $r = \sqrt{(y^1)^2 + \dots + (y^n)^2} > 0$. Then

$$\left(\frac{y^1}{r}\right)^2 + \left(\sqrt{\left(\frac{y^2}{r}\right)^2 + \dots + \left(\frac{y^n}{r}\right)^2}\right)^2 = 1$$

Thus there exists a unique $\theta^1 \in (0, \pi)$ such that

$$\frac{y^1}{r} = \cos\theta^1 \quad \text{and} \quad \sqrt{\left(\frac{y^2}{r}\right)^2 + \dots + \left(\frac{y^n}{r}\right)^2} = \sin\theta^1$$

(note that $\theta^1$ cannot be $0$ or $\pi$ because $y^n > 0$). Then

$$\left(\frac{y^2}{r}\right)^2 + \left(\sqrt{\left(\frac{y^3}{r}\right)^2 + \dots + \left(\frac{y^n}{r}\right)^2}\right)^2 = \sin^2\theta^1$$

Then there exists a unique $\theta^2 \in (0, \pi)$ such that

$$\frac{y^2}{r \sin\theta^1} = \cos\theta^2 \quad \text{and} \quad \sqrt{\left(\frac{y^3}{r}\right)^2 + \cdots + \left(\frac{y^n}{r}\right)^2}\,\frac{1}{\sin\theta^1} = \sin\theta^2$$

We continue doing so to get the definition of $\theta^3, \theta^4, \dots, \theta^{n-2}, \theta^{n-1}$.

In $V_n^-$, we also have a spherical coordinate system, namely $(r, \theta^1, \dots, \theta^n)$ satisfying $(*)$ but this time $r > 0$, $0 < \theta^1, \dots, \theta^{n-2} < \pi$, $-\pi < \theta^{n-1} < 0$.

In $V_i^+$, we label $y^1, \dots, y^n$ as $\bar{y}^1, \dots, \bar{y}^n$ with $\underbrace{y^1}_{\bar{y}^1}, \dots, \underbrace{y^{i-1}}_{\bar{y}^{i+1}}, \underbrace{y^{i+1}}_{\bar{y}^i}, \dots, \underbrace{y^n}_{\bar{y}^{n-1}}, \underbrace{y^i}_{\bar{y}^n}$

Then we get a spherical coordinate system $(\bar{r}, \bar{\theta}^1, \dots, \bar{\theta}^n)$ with

$$\begin{cases} \bar{y}^1 = \bar{r}\cos\bar{\theta}^1, \\ \bar{y}^2 = \bar{r}\sin\bar{\theta}^1\cos\bar{\theta}^2, \\ \quad \cdots \\ \bar{y}^{n-1} = \bar{r}\sin\bar{\theta}^1 \cdots \sin\bar{\theta}^{n-2}\cos\bar{\theta}^{n-1} \\ \bar{y}^n = \bar{r}\sin\bar{\theta}^1 \cdots \sin\bar{\theta}^{n-2}\sin\bar{\theta}^{n-1}. \end{cases} \qquad \text{where } \bar{r} > 0,\ 0 < \bar{\theta}^1, \dots, \bar{\theta}^{n-1} < \pi.$$

In $V_i^-$, we also have a spherical coordinate system $(\bar{r}, \bar{\theta}^1, \dots, \bar{\theta}^n)$ satisfying the same identities at above, but this time $\bar{r} > 0$, $0 < \bar{\theta}^1, \dots, \bar{\theta}^{n-2} < \pi$, $-\pi < \bar{\theta}^{n-1} < 0$.

From now on, we only use spherical coordinates in $V_i^\pm$ instead of the usual Cartesian coordinates. Up to now, we know that $\{(V_i^\pm, id_{V_i^\pm})\}_{i=1,\dots,n}$ is an atlas on $\mathbb{R}^n \setminus \{0\}$.

Put $W_i^\pm = V_i^\pm \cap S^{n-1}$. Then $W_i^\pm$ is open in $S^{n-1}$ and $\bigcup_{i=1}^n W_i^\pm = S^{n-1}$.

For each $i = 1, \dots, n$, we define a map $\psi_i^+ : W_i^+ \to (0, \pi)^{n-1} \subset \mathbb{R}^{n-1}$,

$$\psi_i^+(\underbrace{1, \theta^1, \dots, \theta^{n-1}}_{\text{spherical coords}}) = (\underbrace{\theta^1, \dots, \theta^{n-1}}_{\text{Cartesian coords}}).$$

Also, we define a map $\psi_i^- : W_i^- \to (0,\pi) \times \cdots \times \underbrace{(-\pi, 0)}_{i} \times \cdots \times (0,\pi) \subset \mathbb{R}^{n-1}$,

$$\psi_i^-(\underbrace{1, \theta^1, \dots, \theta^{n-1}}_{\text{spherical coords}}) = (\underbrace{\theta^1, \dots, \theta^{n-1}}_{\text{Cartesian coords}}).$$

Then each $\psi_i^\pm$ is a homeomorphism.

We'll show that $\{(W_i^\pm, \psi_i^\pm)\}_{i=1,\dots,n}$ is an atlas on $S^{n-1}$. To do so we only have to check the compatibility of any two charts. By the definition of $\psi_i^\pm$, we see that $\psi_i^+$ and $\psi_j^-$ have disjoint codomains. Also, $\psi_i^-$ and $\psi_j^-$ have disjoint codomains if $i \neq j$. Thus we only have to check the compatibility of $\psi_i^+$ and $\psi_j^+$. We have

$$(\theta^1, \dots, \theta^{n-1}) \in (0,\pi)^{n-1} \xmapsto{(\psi_j^+)^{-1}} \begin{cases} \bar{y}^1 = \cos\bar{\theta}^1 \\ \bar{y}^2 = \sin\bar{\theta}^1 \cos\bar{\theta}^2 \\ \vdots \\ \bar{y} = \sin\bar{\theta}^1 \cdots \sin\bar{\theta}^{n-1} \end{cases} \xmapsto{\psi_i^+} (\theta^1, \dots, \theta^{n-1}) \in (0,\pi)^{n-1}.$$

Thus $(\psi_i^+) \circ (\psi_j^+)^{-1} = id$, which is smooth. Therefore $\{(W_i^\pm, \psi_i^\pm)\}_{i=1,\dots,n}$ is an atlas on $S^{n-1}$.

Now we take $p \in M$ and put $q = f(p) \in \mathbb{R}^{n}\setminus\{0\}$. Then $q \in V_i^\pm$ for some $i = 1, \dots, n$. In $V_i^\pm$, we have

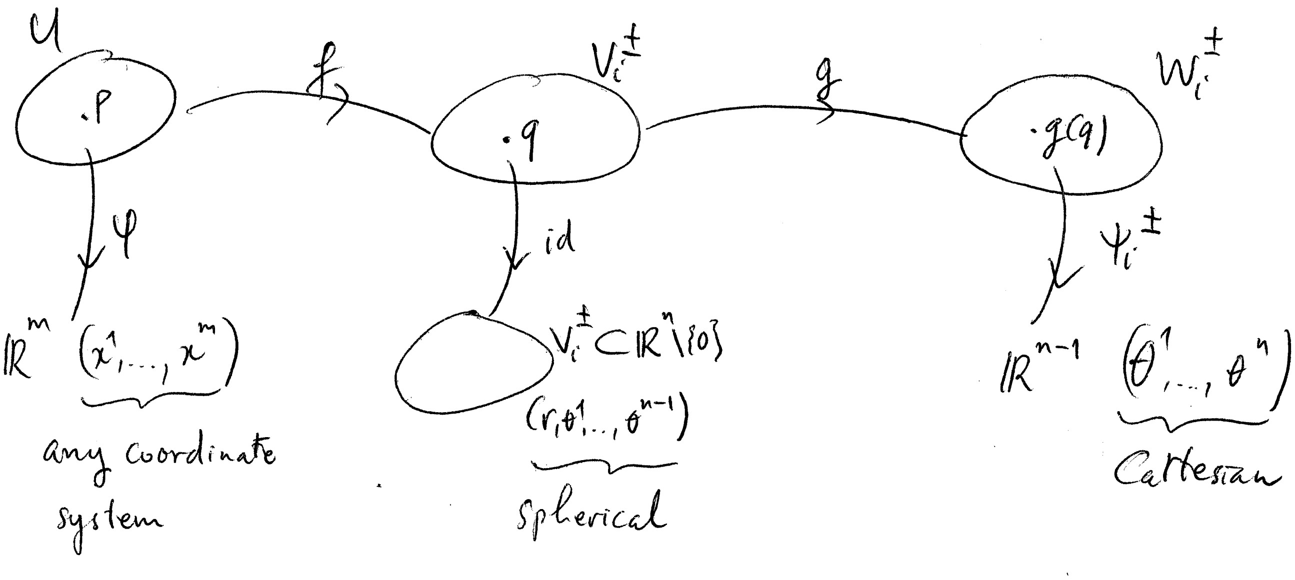
$$\underbrace{g(r, \theta^1, \dots, \theta^{n-1})}_{\substack{\text{spherical coords} \\ \text{in } V_i^{\pm}}} = \underbrace{(1, \theta^1, \dots, \theta^{n-1})}_{\text{spherical coords in } V_i^{\pm}}$$

$$\Psi_i^{\pm}\underbrace{(1, \theta^1, \dots, \theta^{n-1})}_{\text{spherical in } V_i^{\pm}} = \underbrace{(\theta^1, \dots, \theta^{n-1})}_{\text{Cartesian in } \mathbb{R}^{n-1}}$$

Thus, $\Psi_i^{\pm} \circ g \underbrace{(r, \theta^1, \dots, \theta^{n-1})}_{\text{spherical in } V_i^{\pm}} = \underbrace{(\theta^1, \dots, \theta^{n-1})}_{\text{Cartesian in } \mathbb{R}^{n-1}}$.

Denote by $J_q(g)$ the Jacobian matrix of $g$ at $q$. We have

$$J_q(g) = \begin{pmatrix} \left.\frac{\partial \theta^1}{\partial r}\right|_q & \left.\frac{\partial \theta^1}{\partial \theta^1}\right|_q & \cdots & \left.\frac{\partial \theta^1}{\partial \theta^{n-1}}\right|_q \\ \left.\frac{\partial \theta^2}{\partial r}\right|_q & \left.\frac{\partial \theta^2}{\partial \theta^1}\right|_q & \cdots & \left.\frac{\partial \theta^2}{\partial \theta^{n-1}}\right|_q \\ \vdots & & & \\ \left.\frac{\partial \theta^{n-1}}{\partial r}\right|_q & \left.\frac{\partial \theta^{n-1}}{\partial \theta^1}\right|_q & \cdots & \left.\frac{\partial \theta^{n-1}}{\partial \theta^{n-1}}\right|_q \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \\ \vdots & \boxed{I_{n-1}} \\ 0 & \end{pmatrix}}_{(n-1) \times n}$$



Let $(U,\varphi)$ be any coordinate chart on $M$ containing $p$, and $(x^1,\dots,x^m)$ be any coordinate system in $\varphi(U)\subset\mathbb{R}^m$. Moreover, we can choose $(U,\varphi)$ such that $f(U)\subset V_i^\pm$. Then the map $U \xrightarrow{f} V_i^\pm \xrightarrow{g} W_i^\pm$ pushes forward

$$\underbrace{T_p(U)}_{\text{basis } \left\{\frac{\partial}{\partial x^1}\big|_p,\dots,\frac{\partial}{\partial x^m}\big|_p\right\}} \xrightarrow{df_p} \underbrace{T_q(V_i^\pm)}_{\text{basis } \left\{\frac{\partial}{\partial r}\big|_q,\frac{\partial}{\partial \theta^1}\big|_q,\dots,\frac{\partial}{\partial \theta^{n-1}}\big|_q\right\}} \xrightarrow{dg_q} \underbrace{T_{g(q)}(W_i^\pm)}_{\text{basis } \left\{\frac{\partial}{\partial \theta^1}\big|_{g(q)},\dots,\frac{\partial}{\partial \theta^{n-1}}\big|_{g(q)}\right\}}$$

We have

$$df_p\begin{pmatrix}\frac{\partial}{\partial x^1}\big|_p\\ \vdots\\ \frac{\partial}{\partial x^m}\big|_p\end{pmatrix} = J_p(f)^T\begin{pmatrix}\frac{\partial}{\partial r}\big|_q\\ \frac{\partial}{\partial \theta^1}\big|_q\\ \vdots\\ \frac{\partial}{\partial \theta^{n-1}}\big|_q\end{pmatrix} \qquad (**)$$

where $J_p(f)$ is the Jacobian matrix of $f$ at $p$:

$$J_p(f)=\begin{pmatrix}\frac{\partial f_1}{\partial x^1} & \cdots & \frac{\partial f_1}{\partial x^m}\\ \vdots & \ddots & \vdots\\ \frac{\partial f_n}{\partial x^1} & \cdots & \frac{\partial f_n}{\partial x^m}\end{pmatrix} \quad \text{and} \quad f=\underbrace{(f_1,\dots,f_n)}_{\text{spherical coord. in } V_i^\pm}$$

Also, we have

$$dg_q\begin{pmatrix}\frac{\partial}{\partial r}\big|_q\\ \frac{\partial}{\partial \theta^1}\big|_q\\ \vdots\\ \frac{\partial}{\partial \theta^{n-1}}\big|_q\end{pmatrix} = J_q(g)^T\begin{pmatrix}\frac{\partial}{\partial \theta^1}\big|_{g(q)}\\ \vdots\\ \frac{\partial}{\partial \theta^{n-1}}\big|_{g(q)}\end{pmatrix} \qquad (***)$$

From (**) and (***) we get

$$dg_q \circ df_p \begin{pmatrix} \left.\frac{\partial}{\partial x^1}\right|_p \\ \vdots \\ \left.\frac{\partial}{\partial x^m}\right|_p \end{pmatrix} = J_p(f)^T J_q(g)^T \begin{pmatrix} \left.\frac{\partial}{\partial \theta^1}\right|_{g(q)} \\ \vdots \\ \left.\frac{\partial}{\partial \theta^{n-1}}\right|_{g(q)} \end{pmatrix}$$

Thus $\operatorname{rank}(dg_q \circ df_p) = \operatorname{rank}\left(J_p(f)^T J_q(g)^T\right)$

$$= \operatorname{rank}\left(J_q(g) \cdot J_p(f)\right)$$

Since the pushing forward is a functor, we have $\cancel{df_p \circ dg_q = dg \circ f}$ $dg_q \circ df_p = d(g \circ f)_p$. Thus $\operatorname{rank}(d(g\circ f)_p) = \operatorname{rank}(J_q(g) J_p(f))$.

Because $J_q(g)$ consists of a zero column on the left and the unit matrix $I_{n-1}$ on the right, $J_q(g)\, J_p(f)$ is simply taking the last $(n-1)$-rows of $J_p(f)$.

Put

$$A_p(f) = \begin{pmatrix} \left.\frac{\partial f_2}{\partial x^1}\right|_p & \cdots & \left.\frac{\partial f_2}{\partial x^m}\right|_p \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial f_n}{\partial x^1}\right|_p & \cdots & \left.\frac{\partial f_n}{\partial x^m}\right|_p \end{pmatrix}$$

Then $\underbrace{J_q(g)}_{(n-1)\times n} \underbrace{J_p(f)}_{n \times m} = \underbrace{A_p(f)}_{(n-1)\times m}$. Then we have the following conclusions:

- $g \circ f$ is a submersion at $p \iff \begin{cases} \operatorname{rank} A_p(f) = n-1, \\ m \geq n-1. \end{cases}$

Thus $g \circ f$ is a submersion iff $m \geq n-1$ and $\operatorname{rank} A_p(f) = n-1$ for all $p \in M$. The coordinates in charts of $M$ can be chosen arbitrarily, namely $(x^1, \dots, x^m)$ is not necessarily the Cartesian coordinates.

- $g \circ f$ is an immersion at $p \iff \begin{cases} \operatorname{rank} A_p(f) = m, \\ m \leq n-1. \end{cases}$

Thus $g \circ f$ is an immersion iff $m \leq n-1$ and $\operatorname{rank} A_p(f) = m$ for all $p \in M$.

- $g \circ f$ is a local diffeomorphism at p $\iff \begin{cases} \operatorname{rank} A_p(f) = m \\ m = n-1 \end{cases}$.

Thus $g \circ f$ is a local diffeomorphism iff $m = n-1$ and $\operatorname{rank} A_p(f) = m$ for all $p \in M$.