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Math 8302: Topology & Manifold

Homework 5

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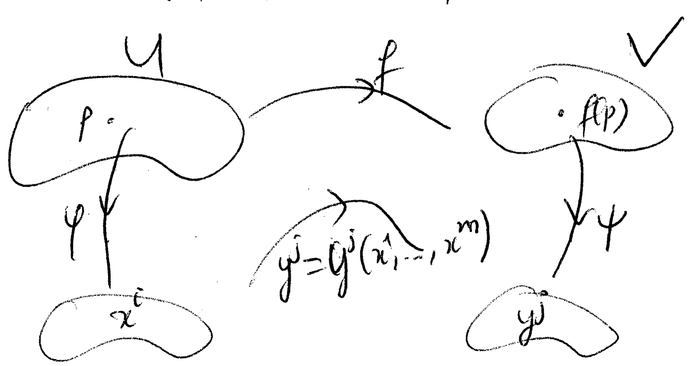
① Let $f: M \to N$ be a map of smooth manifolds. X, Y be vector ~~space~~fields on M, and $\bar{X}, \bar{Y}$ be vector fields on N. Suppose that $df_p(X(p)) = \bar{X}(f(p))$, $df_p(Y(p)) = \bar{Y}(f(p))$ for all $p \in M$. We'll show that $df_p([X,Y](p)) = [\bar{X}, \bar{Y}](f(p))$ for all $p \in M$.

Pick any smooth chart (U, φ) on M and (V, ψ) on N such that $f(U) \subset V$. Then we have the coordinate representations for the vector fields $X, Y, \bar{X}, \bar{Y}$ as follow.



$$X(p) = a^i(p) \frac{\partial}{\partial x^i}\Big|_p, \qquad Y(p) = b^i(p) \frac{\partial}{\partial x^i}\Big|_p,$$

$$\bar{X}(q) = \bar{a}^j(q) \frac{\partial}{\partial y^j}\Big|_q, \qquad \bar{Y}(q) = \bar{b}^j(q) \frac{\partial}{\partial y^j}\Big|_q,$$

where $a^i, b^i: U \to \mathbb{R}$ are smooth, and $\bar{a}^j, \bar{b}^j: V \to \mathbb{R}$ are smooth.

Then by the definition of df_p, we have

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$$df_p(X(p)) = df_p\left(a^i(p)\frac{\partial}{\partial x^i}\Big|_p\right) = a^i(p)\frac{\partial y^j}{\partial x^i}\Big|_{\varphi(p)}\frac{\partial}{\partial y^j}\Big|_{f(p)}$$

Thus, $df_p(X(p)) = \bar{X}(f(p)) = \bar{a}^j(f(p))\frac{\partial}{\partial y^j}\Big|_{f(p)}$ if and only if

$$\bar{a}^j(f(p)) = a^i(p)\frac{\partial y^j}{\partial x^i}\Big|_{\varphi(p)} \qquad (1)$$

Similarly, by replacing vector field X by vector field Y we get

$$\bar{b}^j(f(p)) = b^i(p)\frac{\partial y^j}{\partial x^i}\Big|_{\varphi(p)} \qquad (2)$$

By the definition of Lie bracket, we have

$$[X,Y](p) = \left(a^i(p)\frac{\partial b^j}{\partial x^i}(p) - b^i(p)\frac{\partial a^j}{\partial x^i}(p)\right)\frac{\partial}{\partial x^j}\Big|_p$$

Now we apply df_p to both sides. Using the linearity of df_p and the fact that $df_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial y^k}{\partial x^j}\Big|_{\varphi(p)}\frac{\partial}{\partial y^k}\Big|_{f(p)}$, we get

$$df_p([X,Y](p)) = \left(a^i(p)\frac{\partial b^j}{\partial x^i}(p) - b^i(p)\frac{\partial a^j}{\partial x^i}(p)\right)\frac{\partial y^k}{\partial x^j}\Big|_{\varphi(p)}\frac{\partial}{\partial y^k}\Big|_{f(p)} \qquad (3)$$

By the definition of Lie bracket, we have

$$[\bar{X},\bar{Y}](q) = \left(\bar{a}^i(q)\frac{\partial \bar{b}^k}{\partial y^i}(q) - \bar{b}^i(q)\frac{\partial \bar{a}^k}{\partial y^i}(q)\right)\frac{\partial}{\partial y^k}\Big|_q$$

Then $[\bar{X},\bar{Y}](f(p)) = \left(\bar{a}^i(f(p))\frac{\partial \bar{b}^k}{\partial y^i}(f(p)) - \bar{b}^i(f(p))\frac{\partial \bar{a}^k}{\partial y^i}(f(p))\right)\frac{\partial}{\partial y^k}\Big|_{f(p)} \qquad (4)$

Comparing (3) and (4), we are supposed to show that

$$\left(a^i(p)\frac{\partial b^j}{\partial x^i}(p) - b^i(p)\frac{\partial a^j}{\partial x^i}(p)\right)\frac{\partial y^k}{\partial x^j}\Big|_{\varphi(p)} = \bar{a}^i(f(p))\frac{\partial \bar{b}^k}{\partial y^i}(f(p)) - \bar{b}^i(f(p))\frac{\partial \bar{a}^k}{\partial y^i}(f(p)) \quad (5)$$

we have

$$\bar{a}^i(f(p))\frac{\partial \bar{b}^k}{\partial y^i}(f(p)) \overset{(1)}{=\!=} a^l(p)\frac{\partial y^i}{\partial x^l}\Big|_{\varphi(p)}\frac{\partial \bar{b}^k}{\partial y^i}(f(p))$$

$$\overset{\text{chain rule}}{=\!=} a^l(p)\frac{\partial}{\partial x^l}\left(\bar{b}^k(f(p))\right)$$

$$\overset{(2)}{=\!=} a^l(p)\frac{\partial}{\partial x^l}\left(b^s(p)\frac{\partial y^k}{\partial x^s}\Big|_{\varphi(p)}\right)$$

$$\overset{\text{product rule}}{=\!=} a^l(p)\,b^s(p)\frac{\partial^2 y^k}{\partial x^l\partial x^s}\Big|_{\varphi(p)} + a^l(p)\frac{\partial b^s}{\partial x^l}(p)\frac{\partial y^k}{\partial x^s}\Big|_{\varphi(p)} \quad (6)$$

Similarly, by simply switching a and b, we get

$$\bar{b}^i(f(p))\frac{\partial \bar{a}^k}{\partial y^i}(f(p)) = \underbrace{b^l(p)\,a^s(p)\frac{\partial^2 y^k}{\partial x^l\partial x^s}\Big|_{\varphi(p)}}_{} + b^l(p)\frac{\partial a^s}{\partial x^l}(p)\frac{\partial y^k}{\partial x^s}\Big|_{\varphi(p)} \quad (7)$$

$$= a^l(p)\,b^s(p)\frac{\partial^2 y^k}{\partial x^s\partial x^l}\Big|_{\varphi(p)}$$

Subtracting (7) from (6), we get

$$\text{RHS}(5) = a^l(p)\frac{\partial b^s}{\partial x^l}(p)\frac{\partial y^k}{\partial x^s}\Big|_{\varphi(p)} - b^l(p)\frac{\partial a^s}{\partial x^l}(p)\frac{\partial y^k}{\partial x^s}\Big|_{\varphi(p)}$$

$$= \left(a^l(p)\frac{\partial b^s}{\partial x^l}(p) - b^l(p)\frac{\partial a^s}{\partial x^l}(p)\right)\frac{\partial y^k}{\partial x^s}\Big|_{\varphi(p)}$$

$$= \text{LHS}(5).$$

② Let $f: M \to N$ be a smooth map, X be a vector field on M and $c: (a,b) \to M$ a flow line for X. Suppose X' is a vector field on N such that f carries X to X'. We'll show that $f\circ c: (a,b) \to N$ is also a flow line for X'.

$$(a,b) \xrightarrow{c} M \xrightarrow{f} N$$

First we notice that $f \circ c$ is also a smooth map from (a,b) to N. Since c is a flow line for X, $dc_{t_0} = X(c(t_0))$ for all $t_0 \in (a,b)$.

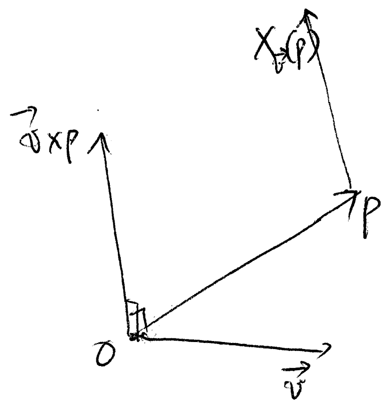
Since f carries X to X', $df_p(X(p)) = X'(f(p))$ for all $p \in M$. To show that $f\circ c$ is a flow line for N, we'll show that $d(f\circ c)_{t_0} = X'(f\circ c(t_0))$ for all $t_0 \in (a,b)$. We have

$$\begin{aligned} d(f\circ c)_{t_0} &= df_{c(t_0)} \circ dc_{t_0} \quad \text{(taking derivative is a functor)} \\ &= df_{c(t_0)}(X(c(t_0))) \quad (c \text{ is a flow line for } X) \\ &= df_p(X(p)) \quad \text{with } p = c(t_0) \\ &= X'(f(p)) \\ &= X'(f\circ c(t_0)), \end{aligned}$$

which completes the proof.

③ Consider $\mathbb{R}^3$ with the standard smooth structure, and a vector $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k} \in \mathbb{R}^3$.

Each point p in $\mathbb{R}^3$ associates with a vector $\vec{Op} = p_1\vec{i} + p_2\vec{j} + p_3\vec{k}$ where O is the origin of the chosen Cartesian coordinate system on $\mathbb{R}^3$.



We have

$$\vec{v}\times p = \begin{vmatrix} v_2 & v_3 \\ p_2 & p_3 \end{vmatrix}\vec{i} - \begin{vmatrix} v_1 & v_3 \\ p_1 & p_3 \end{vmatrix}\vec{j} + \begin{vmatrix} v_1 & v_2 \\ p_1 & p_2 \end{vmatrix}\vec{k}$$

We define the following map $X_{\vec{v}} : \mathbb{R}^3 \to T\mathbb{R}^3$,

$$X_{\vec{v}}(p) = \begin{vmatrix} v_2 & v_3 \\ p_2 & p_3 \end{vmatrix} \cdot \frac{\partial}{\partial x}\Big|_p - \begin{vmatrix} v_1 & v_3 \\ p_1 & p_3 \end{vmatrix} \cdot \frac{\partial}{\partial y}\Big|_p$$

$$+ \begin{vmatrix} v_1 & v_2 \\ p_1 & p_2 \end{vmatrix} \cdot \frac{\partial}{\partial z}\Big|_p$$

Then $X_{\vec{v}}(p) = a^1(p)\frac{\partial}{\partial x}\Big|_p + a^2(p)\frac{\partial}{\partial y}\Big|_p + a^3(p)\frac{\partial}{\partial z}\Big|_p$ where $a^1, a^2, a^3 : \mathbb{R}^3 \to \mathbb{R}$,

$a^1(p) = v_2 p_3 - v_3 p_2$,

$a^2(p) = v_3 p_1 - v_1 p_3$,

$a^3(p) = v_1 p_2 - v_2 p_1$.

Since these a^i's are linear, they are smooth maps. Thus $X_{\vec{v}}$ is a smooth vector field on $\mathbb{R}^3$. Now let $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k} \in \mathbb{R}^3$ be another vector. Then $X_{\vec{w}}(p) = b^1(p)\frac{\partial}{\partial x}\Big|_p + b^2(p)\frac{\partial}{\partial y}\Big|_p + b^3(p)\frac{\partial}{\partial z}\Big|_p$ where $b^1, b^2, b^3 : \mathbb{R}^3 \to \mathbb{R}$,

$b^1(p) = w_2 p_3 - w_3 p_2$,

$b^2(p) = w_3 p_1 - w_1 p_3$,

$b^3(p) = w_1 p_2 - w_2 p_1$.

The Lie bracket of $X_{\vec{v}}$ and $X_{\vec{w}}$ is $[X_{\vec{v}}, X_{\vec{w}}](p) = A\frac{\partial}{\partial x}\Big|_p + B\frac{\partial}{\partial y}\Big|_p + C\frac{\partial}{\partial z}\Big|_p$,

where
$$A = a^1(p)\frac{\partial b^1}{\partial x}(p) + a^2(p)\frac{\partial b^1}{\partial y}(p) + a^3(p)\frac{\partial b^1}{\partial z}(p)$$
$$- b^1(p)\frac{\partial a^1}{\partial x}(p) - b^2(p)\frac{\partial a^1}{\partial y}(p) - b^3(p)\frac{\partial a^1}{\partial z}(p),$$

$$B = a^1(p)\frac{\partial b^2}{\partial x}(p) + a^2(p)\frac{\partial b^2}{\partial y}(p) + a^3(p)\frac{\partial b^2}{\partial z}(p)$$
$$- b^1(p)\frac{\partial a^2}{\partial x}(p) - b^2(p)\frac{\partial a^2(p)}{\partial y} - b^3(p)\frac{\partial a^2(p)}{\partial z},$$

$$C = a^1(p)\frac{\partial b^3}{\partial x}(p) + a^2(p)\frac{\partial b^3}{\partial y}(p) + a^3(p)\frac{\partial b^3}{\partial z}(p)$$
$$- b^1(p)\frac{\partial a^3}{\partial x}(p) - b^2(p)\frac{\partial a^3}{\partial y}(p) - b^3(p)\frac{\partial a^3}{\partial z}(p).$$

We get
$$A = (v_3p_1 - v_1p_3)(-w_3) + (v_1p_2 - v_2p_1)w_2 - (w_3p_1 - w_1p_3)(-v_3)$$
$$- (w_1p_2 - w_2p_1)v_2$$
$$= v_1w_3p_3 + v_1w_2p_2 - v_3w_1p_3 - v_2w_1p_2$$
$$= (v_1w_3 - v_3w_1)p_3 + (v_1w_2 - v_2w_1)p_2$$
$$= \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} p_3 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} p_2$$

Similarly
$$B = (v_2p_3 - v_3p_2)w_3 + (v_1p_2 - v_2p_1)(-w_1)$$
$$- (w_2p_3 - w_3p_2)v_3 - (w_1p_2 - w_2p_1)(-v_1)$$
$$= (v_2w_3 - w_2v_3)p_3 - (v_1w_2 - v_2w_1)p_1$$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} p_3 - \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} p_1 .$$

Similarly, $C = (v_2 p_3 - v_3 p_2)(-w_2) + (v_3 p_1 - v_1 p_3) w_1$

$$= (w_2 p_3 - w_3 p_2)(-v_2) + (w_3 p_1 - w_1 p_3) v_1$$

$$= v_3 w_2 p_2 + v_3 w_1 p_1 - v_2 w_3 p_2 - v_1 w_3 p_1$$

$$= (v_3 w_2 - v_2 w_3) p_2 - (v_1 w_3 - v_3 w_1) p_1$$

$$= - \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} p_2 - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} p_1$$

Therefore,

$$[X_{\vec v}, X_{\vec w}](p) = - \begin{vmatrix} \left.\frac{\partial}{\partial x}\right|_p & \left.\frac{\partial}{\partial y}\right|_p & \left.\frac{\partial}{\partial z}\right|_p \\ \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} & -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} & \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\ p_1 & p_2 & p_3 \end{vmatrix}$$

If we identify $\left.\frac{\partial}{\partial x}\right|_p \equiv \vec{i}$, $\left.\frac{\partial}{\partial y}\right|_p \equiv \vec{j}$, $\left.\frac{\partial}{\partial z}\right|_p \equiv \vec{k}$, we will have

$$[X_{\vec v}, X_{\vec w}](p) = - \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right) \times p$$

$$= -(\vec v \times \vec w) \times p .$$

(4) Consider $\mathbb{R}^4$ as a manifold with the standard smooth structure with atlas consisting of a single chart $(\mathbb{R}^4, (x_1, y_1, x_2, y_2))$.

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Consider the vector field $X: \mathbb{R}^4 \longrightarrow T\mathbb{R}^4$ given by

$$X(x_1, y_1, x_2, y_2) = -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial y_2}.$$

This is a smooth vector field because each component function is smooth as functions from $\mathbb{R}^4$ to $\mathbb{R}$. Pick any point $(\alpha, \beta, \gamma, \eta) \in \mathbb{R}^4$ and consider the flow line starting at this point. We call such a ~~fot~~ flow line $c: (a, b) \to \mathbb{R}^4$, where $c(0) = (\alpha, \beta, \gamma, \eta)$ and (a, b) is a neighborhood of 0 in $\mathbb{R}$. By definition, we have $dc_t\left(\frac{d}{dt}\right) = X(c(t))$ for all $t \in (a,b)$.

Write $c(t) = (x_1(t), y_1(t), x_2(t), y_2(t))$. We have

$$dc_t\left(\frac{d}{dt}\right) = x_1'(t) \frac{\partial}{\partial x_1}\Big|_{c(t)} + y_1'(t) \frac{\partial}{\partial y_1}\Big|_{c(t)} + x_2'(t) \frac{\partial}{\partial x_2}\Big|_{c(t)} + y_2'(t) \frac{\partial}{\partial y_2}\Big|_{c(t)},$$

$$X(c(t)) = -y_1(t) \frac{\partial}{\partial x_1}\Big|_{c(t)} + x_1(t) \frac{\partial}{\partial y_1}\Big|_{c(t)} - y_2(t) \frac{\partial}{\partial x_2}\Big|_{c(t)} + x_2(t) \frac{\partial}{\partial y_2}\Big|_{c(t)}.$$

Then we get a system of linear differential equations of order 1.

$$\begin{cases} x_1'(t) = -y_1(t) \\ y_1'(t) = x_1(t) \\ x_2'(t) = -y_2(t) \\ y_2'(t) = x_2(t) \end{cases} \qquad \forall t \in (a, b)$$

with $(x_1(0), y_1(0), x_2(0), y_2(0)) = (\alpha, \beta, \gamma, \eta)$.

The problem of find the flow line of X at $(\alpha, \beta, \gamma, \eta)$ returns to the problem of solving the following differential equations

$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases} \quad \forall t \in (a,b)$$

with some given initial-value condition. In form of matrix,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

We should diagonalize matrix A. We have

$$\det(xI_2 - A) = \begin{vmatrix} x & 1 \\ -1 & x \end{vmatrix} = x^2+1 = (x-i)(x+i)$$

Then A has two eigenvalues $\pm i$. We proceed to find the corresponding eigenvectors.

$$(-i)I_2 - A = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \sim \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$$

Solution space is $\{(x,y) : x+iy = 0\} = \{x(1,i) : x \in \mathbb{R}\}$. Thus an eigenvector corresponding to $-i$ is $v_1 = (1,i)$.

$$iI_2 - A = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \sim \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$$

Solution space is $\{(x,y) : x - iy = 0\} = \{x(1,-i) : x \in \mathbb{R}\}$. Thus an eigenvector corresponding to i is $v_2 = (1,-i)$. Put

$$P = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

Then $P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$.

We have $P^{-1}AP = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$. Put $\begin{pmatrix} u \\ v \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$.

We get $$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -iu \\ iv \end{pmatrix}.$$

Thus $\begin{cases} u' = -iu \\ v' = iv \end{cases}$ or equivalently $\begin{cases} u' + iu = 0 \\ v' - iv = 0 \end{cases}$

Then $u = u(t) = C_1 e^{-it}$, $v = v(t) = C_2 e^{it}$ for some constant $C_1, C_2 \in \mathbb{C}$. Put $C_1 = A_1 + iB_1$, $C_2 = A_2 + iB_2$ where $A_1, A_2, B_1, B_2 \in \mathbb{R}$.

We have $$u(t) = (A_1 + iB_1)(\cos t - i\sin t)$$
$$= (A_1 \cos t + B_1 \sin t) + i(-A_1 \sin t + B_1 \cos t),$$

$$v(t) = (A_2 + iB_2)(\cos t + i\sin t)$$
$$= (A_2 \cos t - B_2 \sin t) + i(A_2 \sin t + B_2 \cos t).$$

We have $$\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u+v \\ iu - iv \end{pmatrix}$$

Thus $$x = u + v = (A_1 + A_2)\cos t + (B_1 - B_2)\sin t$$
$$+ i[(-A_1 + A_2)\sin t + (B_1 + B_2)\cos t],$$

$$y = i(u-v) = -[(-A_1 - A_2)\sin t + (B_1 - B_2)\cos t]$$
$$+ i[(A_1 - A_2)\cos t + (B_1 + B_2)\sin t].$$

Since $x(0), y(0) \in \mathbb{R}$, $A_1 + A_2 + i(B_1 + B_2) \in \mathbb{R}$ and $-(B_1 - B_2) + i(A_1 - A_2) \in \mathbb{R}$. Thus $B_1 + B_2 = 0$ and $A_1 - A_2 = 0$. Put $A = A_1 = A_2 \in \mathbb{R}$ and $B = B_1 = -B_2$. We have

$$x = 2A\cos t + 2B\sin t, \qquad y = 2A\sin t - 2B\cos t$$

Renaming the constants $2A, 2B$ by simply A, B, we get

$$\begin{cases} x = A\cos t + B\sin t \\ y = -B\cos t + A\sin t \end{cases} \qquad (*)$$

~~Thus, the sys~~ Because $\cancel{\begin{vmatrix} A & B \\ -B & A \end{vmatrix} = A^2+B^2}$ $\begin{vmatrix} \cos t & \sin t \\ -\cos t & \sin t \end{vmatrix} = 1$, we can determine $A, B \in \mathbb{R}$ such that x, y in $(*)$ satisfy any given initial value condition. Thus, the system

$$\begin{cases} x_1'(t) = -y_1(t) \\ y_1'(t) = x_1(t) \\ x_2'(t) = -y_2(t) \\ y_2'(t) = x_2(t) \end{cases} \quad \text{gives solutions} \quad \begin{cases} x_1(t) = A_1\cos t + B_1\sin t \\ y_1(t) = -B_1\cos t + A_1\sin t \\ x_2(t) = A_2\cos t + B_2\sin t \\ y_2(t) = -B_2\cos t + A_2\sin t \end{cases}$$

where A_1, B_1, A_2, B_2 are uniquely determined by the initial condition $(x_1(0), y_1(0), x_2(0), y_2(0)) = (\alpha, \beta, \gamma, \eta)$. Since these solutions exist for all $t \in \mathbb{R}$, the flow line for X at any point flows infinitely.

(5) Let M be an n-dimensional smooth manifold and $X_1, \dots, X_n$ be smooth

vector fields on M. Each point in $M \times \mathbb{R}$ is written as (x,t) with $x \in M$, $t \in \mathbb{R}$. For each $i = 1, \dots, n$, there exists an open subset $U_i \subset M \times \mathbb{R}$ with $M \times \{0\} \subset U_i$ and a smooth function $\theta_i : U_i \to M$, $(x,t) \mapsto \theta_i(x,t)$ such that

- For each fixed x, $(d\theta_i)_t\left(\frac{d}{dt}\right) = X_i(\theta_i(x,t))$,
- $\theta_i(x,0) = x$.

Pick a point $p \in M$. Since $(p,0) \in U_1$, there is $\varepsilon_1 > 0$ such that $\{p\} \times (-\varepsilon_1, \varepsilon_1) \subset U_1$. We define $f_1 : (-\varepsilon_1, \varepsilon_1) \to M$ such that $f_1(t_1) = \theta_1(p, t_1)$ for all $t_1 \in (-\varepsilon_1, \varepsilon_1)$. With this definition, f_1 is smooth.

We have $f_1(0) = \theta_1(p,0) = p$. Thus $(f_1(0), 0) = (p,0) \in U_2$. Thus there exists $\varepsilon_2 < \varepsilon_1$ such that $f_1(-\varepsilon_2, \varepsilon_2) \times (-\varepsilon_2, \varepsilon_2) \subset U_2$. We define $f_2 : (-\varepsilon_2, \varepsilon_2) \times (-\varepsilon_2, \varepsilon_2) \to M$ such that $f_2(t_1, t_2) = \theta_2(f_1(t_1), t_2)$ for all $t_1, t_2 \in (-\varepsilon_2, \varepsilon_2)$. Suppose that we found $\varepsilon_k < \varepsilon_{k-1}$ such that $f_{k-1}((-\varepsilon_k, \varepsilon_k)^{k-1}) \times (-\varepsilon_k, \varepsilon_k)$ is contained in U_k, we can define $f_k : (-\varepsilon_k, \varepsilon_k)^k \to M$, with

$$f_k(t_1, \dots, t_k) = \theta_k(f_{k-1}(t_1, \dots, t_{k-1}), t_k).$$

Then f_k is smooth. We have $f_k(\vec{0}) = \theta_k(f_{k-1}(\vec{0}), 0) = \theta_k(p,0) = p$. Thus, $(f_k(\vec{0}), 0) = (p,0) \in U_{k+1}$. Thus there exists $\varepsilon_{k+1} < \varepsilon_k$ such that $f_k((-\varepsilon_{k+1}, \varepsilon_{k+1})^k) \times (-\varepsilon_{k+1}, \varepsilon_{k+1}) \subset U_{k+1}$. Then we continue the similar arguments to define $f_{k+1}, \dots, f_n$. Put $\varepsilon = \varepsilon_n$. Then we have n functions f_j with

$$f_j : (-\varepsilon,\varepsilon)^j \longrightarrow M, \quad f_j(t_1,\dots,t_j) = \theta_j(f_{j-1}(t_1,\dots,t_{j-1}), t_j).$$

Let $\pi_i : U_i \to M$ be the i'th projection map, i.e. $\pi_i(x,t) = x$, for all $(x,t) \in U_i$. Then $\pi_i(U_i)$ is an open set in M. Since $p = \pi_i(p,0) \in \pi_i(U_i)$, $p \in \bigcap_{i=1}^{n} \pi_i(U_i)$. Thus we can find a neighborhood U of p in M such that $U \subset \bigcap_{i=1}^{n} \pi_i(U_i)$ and that there exists a diffeomorphism $\varphi : U \to \mathbb{R}^n$. Because $f_j(\vec{0}) = p \in U$, we can shrink ε such that $\operatorname{Im} f_j \subset U$ for all $j = 1,\dots,n$.

$$\begin{tikzcd}
(-\varepsilon,\varepsilon)^j \arrow[r, "f_j"] \arrow[dr, "\hat{f}_j"'] & U \ni p \arrow[d, "\varphi"] \\
(t_1,\dots,t_j) & \mathbb{R}^n \ (x^1,\dots,x^n)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_i^{-1}(U) \ni (p,0) \arrow[r, "\theta_i"] \arrow[d, "\varphi\times id"'] & U \ni p \arrow[d, "\varphi"] \\
\mathbb{R}^{n+1} \ (x^1,\dots,x^n,t) \arrow[r, "\hat{\theta}_i"] & \mathbb{R}^n
\end{tikzcd}$$

The coordinate representation of f_j is $\hat{f}_j = \varphi \circ f_j$, of $\theta_i|_{\pi_i^{-1}(U)}$ is $\hat{\theta}_i = \varphi(\theta(\varphi^{-1}(x^1,\dots,x^n), t))$. Thus we can view f_j and θ_i as $\hat{f}_j$ and $\hat{\theta}_i$ and try to prove the same statement of the ~~pro~~ problem. This point of view allows us to consider U as $\mathbb{R}^n$, ~~and~~ $f_j : (-\varepsilon,\varepsilon)^j \to \mathbb{R}^n$, θ_j as a function from a neighborhood of $(p,0)$ in $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$, and $X_1,\dots,X_n$ as functions from a neighborhood of p in $\mathbb{R}^n$ to $\mathbb{R}^n$. Put $\theta_i = \theta_i(x^1,\dots,x^n,t)$. We have $\theta_i(x^1,\dots,x^n,0) = (x^1,\dots,x^n)$. Also we put $\theta_i = (\theta_i^1,\dots,\theta_i^n)$, where each $\theta_i^k = \theta_i^k(x^1,\dots,x^n,t)$ is a real-valued function. Then

$$\theta_i^k(x^1,\dots,x^n,0) = x^i$$

Consequently, $\left.\frac{\partial \theta_i^k}{\partial x^j}\right|_{t=0} = \delta_j^k$. Moreover, $\frac{\partial \theta_i}{\partial t}(x^1,\dots,x^n,t) = X_i(\theta_i(x^1,\dots,x^n,t))$.

We write $f_j = ((f_j)_1,\dots,(f_j)_n)$ where each $(f_j)_k = (f_j)_k(t_1,\dots,t_j)$ is a real-valued function on $(-\varepsilon,\varepsilon)^j$. Then the differential of f_j can be viewed as the Jacobian matrix

$$df_j \equiv \underbrace{\begin{pmatrix} \frac{\partial (f_j)_1}{\partial t_1} & \cdots & \frac{\partial (f_j)_1}{\partial t_j} \\ \vdots & \ddots & \vdots \\ \frac{\partial (f_j)_n}{\partial t_1} & \cdots & \frac{\partial (f_j)_n}{\partial t_j} \end{pmatrix}}_{n\times j}$$

We want to show that

$$(df_j)_{\vec{0}} \begin{pmatrix} \left.\frac{\partial}{\partial t_1}\right|_{\vec{0}} \\ \vdots \\ \left.\frac{\partial}{\partial t_j}\right|_{\vec{0}} \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} X_1(p) \\ \vdots \\ X_j(p) \end{pmatrix}$$

This ~~this~~ is equivalent to showing that $(df_j)_{\vec{0}}\left(\left.\frac{\partial}{\partial t_i}\right|_{\vec{0}}\right) = X_i(p)$ for all $i=1,\dots,j$.

We'll show that $\left(\left.\frac{\partial (f_j)_1}{\partial t_i}\right|_{\vec{0}},\dots,\left.\frac{\partial (f_j)_n}{\partial t_i}\right|_{\vec{0}}\right) = X_i(p) \quad \forall i=1,\dots,j$ by induction in j. For $j=1$, $f_1(t_1) = \theta_1(p,t_1)$. Then

$$\frac{\partial f_1}{\partial t_1} = \frac{\partial \theta_1}{\partial t_1}(p, t_1) = X_1(\theta_1(p, t_1))$$

Thus $\frac{\partial f_1}{\partial t_1}(0) = X_1(\theta_1(p,0)) = X_1(p)$.

Suppose that our claim holds for $j-1$, i.e. $\left.\frac{\partial f_{j-1}}{\partial t_i}\right|_{(t_1,\dots,t_{j-1})=0} = X_i(p)$ for all $i = 1, \dots, j-1$. We'll show that it also holds for j. We have

$$f_j(t_1, \dots, t_{j-1}, t_j) = \theta_j(f_{j-1}(t_1, \dots, t_{j-1}), t_j).$$

Then $\left.\frac{\partial f_j}{\partial t_j}\right|_{t_j=0} = \left.\frac{\partial}{\partial t}\right|_{t=0} \theta_j(f_{j-1}(t_1, \dots, t_{j-1}), t) = X_j(\theta_j(f_{j-1}(t_1, \dots, t_{j-1}), 0))$

Then $\left.\frac{\partial f_j}{\partial t_j}\right|_{\vec{0}} = X_j(\theta_j(\underbrace{f_{j-1}(0, \dots, 0)}_{p}, 0)) = X_j(\theta_j(p,0)) = X_j(p)$.

For each $i = 1, 2, \dots, j-1$, we write

$$f_j(t_1, \dots, t_{j-1}, t_j) = \Big(\theta_j^1(\underbrace{f_{j-1}(t_1, \dots, t_{j-1})}_{\vec{x} = ((f_{j-1})_1, \dots, (f_{j-1})_n)}, t_j), \dots, \theta_j^n(f_{j-1}(t_1, \dots, t_{j-1}), t_j)\Big)$$

Then $\frac{\partial f_j}{\partial t_i}(t_1, \dots, t_j) = \left(\frac{\partial}{\partial t_i}\left[\theta_j^1(f_{j-1}(t_1, \dots, t_{j-1}), t_j)\right], \dots, \frac{\partial}{\partial t_i}\left[\theta_j^n(f_{j-1}(t_1, \dots, t_{j-1}), t_j)\right]\right)$

$$= \left(\frac{\partial \theta_j^1}{\partial x^1}\frac{\partial (f_{j-1})_1}{\partial t_1} + \dots + \frac{\partial \theta_j^1}{\partial x^n}\frac{\partial (f_{j-1})_n}{\partial t_1}, \dots, \frac{\partial \theta_j^n}{\partial x^1}\frac{\partial (f_{j-1})_1}{\partial t_1} + \dots + \frac{\partial \theta_j^n}{\partial x^n}\frac{\partial (f_{j-1})_n}{\partial t_1}\right)$$

Because $\left.\frac{\partial \theta_j^k}{\partial x^i}\right|_{t=0} = \delta_i^k$, we have

$$\frac{\partial f_j}{\partial t_i}(t_1,\dots,t_{j-1},0) = \left(\frac{\partial (f_{j-1})_1}{\partial t_i}, \frac{\partial (f_{j-1})_2}{\partial t_i}, \dots, \frac{\partial (f_{j-1})_n}{\partial t_i}\right).$$

Thus, $$\frac{\partial f_j}{\partial t_i}(0,\dots,0,0) = \left(\left.\frac{\partial (f_{j-1})_1}{\partial t_i}\right|_{\vec{0}}, \left.\frac{\partial (f_{j-1})_2}{\partial t_i}\right|_{\vec{0}}, \dots, \left.\frac{\partial (f_{j-1})_n}{\partial t_i}\right|_{\vec{0}}\right) = X_i(p)$$

by the induction hypothesis. Therefore, our claim holds for j, and thus holds for all $1, 2, \dots, n$.