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Math 8302: Topology & Manifolds

Homework 6

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① Let $f: E \to M$ be a vector bundle. By definition, $f$ is a map of smooth manifolds, together with vector space structure (over $\mathbb{R}$) on each fiber $f^{-1}(p)$, $p \in M$, and an integer $k \in \mathbb{N}$ such that for each $p \in M$, there is an open neighborhood $U_p$ of $p$ in $M$ and a diffeomorphism $\varphi_p : f^{-1}(U_p) \to U_p \times \mathbb{R}^k$ that makes the following diagram commutes

$$\begin{tikzcd} f^{-1}(U_p) \arrow[rr, "\varphi_p"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowleft"] & U_p \times \mathbb{R}^k \arrow[dl, "proj"] \\ & U_p & \end{tikzcd}$$

and that induces linear maps on each fiber, i.e. $\forall q \in U_p$, $\varphi_p|_{f^{-1}(q)} : f^{-1}(q) \to \{q\} \times \mathbb{R}^k$ is linear.

Let $\mathcal{G}$ be the set of all sections of $f$. We'll show that $\mathcal{G}$ can be endowed with a vector space structure (over $\mathbb{R}$).

For any two sections $s_1, s_2 \in \mathcal{G}$, we define the map $s_1 + s_2 : M \to E$ as follow $(s_1+s_2)(p) := \underbrace{s_1(p) + s_2(p)}_{\text{addition of two vectors in } f^{-1}(p)}$, for all $p \in M$.

Then $s_1 + s_2$ is a well-defined map with $f \circ (s_1+s_2)(p) = p$ because $(s_1+s_2)(p) \in f^{-1}(p)$. To show that $s_1+s_2 \in \mathcal{G}$, we only need ~~to~~ check the

Smoothness of $s_1+s_2$. It suffices to check the smoothness in a neighborhood of an arbitrary point $p \in M$. For this purpose, we can assume that $\big(U_p\,,\ \underbrace{\psi : U_p \to \mathbb{R}^m}_{\text{diffeomorphism}}\big)$ is a smooth chart on $M$.

Note that this can be obtained by shrinking the original $U_p$. Because $f \circ s_1 = f \circ s_2 = f \circ (s_1+s_2) = id_M$, we have $s_1, s_2, s_1+s_2 : U_p \to f^{-1}(U_p)$.

Consider the compositions

$$\begin{tikzcd}
U_p \arrow[r, "{s_1, s_2, s_1+s_2}"] \arrow[rr, bend right, "{\tilde{s}_1, \tilde{s}_2, \widetilde{s_1+s_2}}"'] & f^{-1}(U_p) \arrow[r, "\varphi_p"] & U_p \times \mathbb{R}^k \arrow[r, "proj_2"] & \mathbb{R}^k
\end{tikzcd}$$

Since $\varphi_p$ is a diffeomorphism, it suffices to show that $\widetilde{s_1+s_2}$ is smooth on $U_p$.

For every $q \in U_p$, we have

$$\begin{aligned}
\tilde{s}_1(q) &= \big(proj_1\, \tilde{s}_1(q),\ proj_2\, \tilde{s}_1(q)\big) \\
&= \big(\underbrace{proj_1 \circ \varphi_p}_{f} \circ s_1(q),\ proj_2 \circ \varphi_p \circ s_1(q)\big) \\
&= \big(q,\ proj_2 \circ \tilde{s}_1(q)\big) \qquad (*)
\end{aligned}$$

Similarly, $\tilde{s}_2(q) = \big(q, proj_2 \circ \tilde{s}_2(q)\big)$. Since $s_1, s_2$ are smooth, so are $\tilde{s}_1$ and $\tilde{s}_2$. We have the coordinate representations of $\tilde{s}_1, \tilde{s}_2, \widetilde{s_1+s_2}$:

$$\begin{tikzcd}
U_p \arrow[r, "{\tilde{s}_1, \tilde{s}_2, \widetilde{s_1+s_2}}"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowright"] & U_p \times \mathbb{R}^k \arrow[d, "\psi \times id_{\mathbb{R}^k}"] \\
\mathbb{R}^m \arrow[r, dashed, "{\hat{s}_1, \hat{s}_2, \widehat{s_1+s_2}}"] & \mathbb{R}^m \times \mathbb{R}^k
\end{tikzcd}$$

We need to check the smoothness of $\widehat{s_1+s_2} : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^k$.

For each $q \in U_p$, we have

$$\begin{aligned}
\widehat{s_1+s_2}(\psi(q)) &= (\psi \times id_{\mathbb{R}^k})(\widetilde{s_1+s_2}(q)) \quad \text{(Commutative diagram)}\\
&= (\psi \times id_{\mathbb{R}^k})(\varphi_p((s_1+s_2)(q))) \quad \text{(definition of } \widetilde{s_1+s_2})\\
&= (\psi \times id_{\mathbb{R}^k})(\varphi_p(s_1(q)+s_2(q)))\\
&= (\psi \times id_{\mathbb{R}^k})(\varphi_p(s_1(q)) + \varphi_p(s_2(q))) \quad (\varphi_p|_{f^{-1}(q)} \text{ is linear})\\
&= (\psi \times id_{\mathbb{R}^k})(\tilde{s}_1(q) + \tilde{s}_2(q))\\
&\overset{(*)}{=} (\psi \times id_{\mathbb{R}^k})((q, proj_2 \circ \tilde{s}_1(q)) + (q, proj_2 \circ \tilde{s}_2(q)))\\
&= (\psi \times id_{\mathbb{R}^k})(q, \underbrace{proj_2 \circ \tilde{s}_1(q) + proj_2 \circ \tilde{s}_2(q)}_{\text{addition in } \mathbb{R}^k})\\
&= (\psi(q), proj_2 \circ \tilde{s}_1(q) + proj_2 \circ \tilde{s}_2(q))\\
&= (\psi(q), proj_2 \circ \tilde{s}_1(q)) + (\psi(q), proj_2 \circ \tilde{s}_2(q)) - (\psi(q), 0)\\
&\qquad \text{(addition in } \mathbb{R}^m \times \mathbb{R}^k)\\
&= \hat{s}_1(\psi(q)) + \hat{s}_2(\psi(q)) - (\psi(q), 0).
\end{aligned}$$

Therefore, for every $\vec{x} \in \mathbb{R}^m$, $\widehat{s_1+s_2}(\vec{x}) = \hat{s}_1(\vec{x}) + \hat{s}_2(\vec{x}) - (\vec{x}, 0)$.

Since $\hat{s}_1, \hat{s}_2 : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^k$ are smooth, so is $\widehat{s_1+s_2}$.

We have defined an addition operation on $\mathcal{G}$. This addition is commutative, associative because it was based on the addition in $\mathbb{R}^k$.

The additive identity is the zero section $0: M \to E$,

$$p \mapsto \vec{0} \in f^{-1}(p).$$

(This is smooth because on each coordinate chart it looks like $p \mapsto (p, \vec{0})$).

The additive inverse of a section $s: M \to E$ is $-s: M \to E$,

$$(-s)(p) = -s(p) \in f^{-1}(p).$$

Thus $(\mathcal{G}, +)$ is an abelian group.

We define the scalar multiplication $\mathbb{R} \times \mathcal{G} \to \mathcal{G}$ as follow:

for each $c \in \mathbb{R}$, we define $(cs)(p) := \underbrace{c \cdot s(p)}_{\text{multiplication in } \mathbb{R} \times f^{-1}(p)} \in f^{-1}(p)$

This is a well-defined map from $M$ to $E$ and $f \circ (cs) = Id_M$. We'll check the smoothness of $cs$. For each $s_1 \in \mathcal{G}$ and $c \in \mathbb{R}$, we use the same notation $\tilde{s}_1, \hat{s}_1$ as above. We have

$$\begin{aligned}
\widehat{cs_1}(\psi(q)) &= (\psi \times id_{\mathbb{R}^k})(\widetilde{cs_1}(q)) \\
&= (\psi \times id_{\mathbb{R}^k})(\varphi_p(cs_1(q))) \\
&= (\psi \times id_{\mathbb{R}^k})(c\,\varphi_p(s_1(q))) \quad (\text{since } \varphi_p|_{f^{-1}(q)} \text{ is linear}) \\
&= (\psi \times id_{\mathbb{R}^k})(c\,(q, proj_2 \circ \tilde{s}_1(q))) \\
&= (\psi \times id_{\mathbb{R}^k})(q, \underbrace{c\,proj_2 \circ \tilde{s}_1(q)}_{\text{multiplication } \mathbb{R} \times \mathbb{R}^k}) \\
&= (\psi(q), c\,proj_2 \circ \tilde{s}_1(q))
\end{aligned}$$

$$= \overbrace{c\left(\psi(q), \mathrm{proj}_k \circ \tilde{s}_1(q)\right)}^{\text{multiplication } \mathbb{R}\times(\mathbb{R}^m\times\mathbb{R}^k)} + (1-c)\left(\psi(q), 0\right)$$

Therefore, for each $\vec{x} \in \mathbb{R}^m$ we have

$$\widehat{cs_1}(\vec{x}) = c\,\hat{s}_1(\vec{x}) + (1-c)(\vec{x}, 0).$$

Since $\hat{s}_1$ is smooth, so is $\widehat{cs_1}$.

We have defined the scalar multiplication on $\mathcal{G}$. Then $\mathcal{G}$ satisfies all properties of being a module over $\mathbb{R}$ because the addition and multiplication are defined pointwise and based on the existent vector structure on $\mathbb{R}^k$. Namely we have

$$(c_1 c_2)s = c_1(c_2 s), \quad \forall c_1, c_2 \in \mathbb{R},\ \forall s \in \mathcal{G}$$

$$(c_1 + c_2)s = c_1 s + c_2 s, \quad \forall c_1, c_2 \in \mathbb{R},\ \forall s \in \mathcal{G}$$

$$c(s_1 + s_2) = cs_1 + cs_2, \quad \forall c \in \mathbb{R},\ \forall s_1, s_2 \in \mathcal{G}$$

$$1.s = s, \quad \forall s \in \mathcal{G}.$$

An example where $\mathcal{G}$ is finite dimensional but not zero: consider the case where $M$ is a singleton. For convenience, we write $M = \{0\}$. Choose $E = \mathbb{R}$ and $f: \mathbb{R} \to \{0\}$, $f(x) = 0$ $\forall x \in \mathbb{R}$. Any section $s: \{0\} \to \mathbb{R}$ of $f$ can be identified with its image. In this case $\mathcal{G}$ is particularly simple $\mathcal{G} = \{s : \{0\} \to \mathbb{R} \text{ any map}\} \overset{s \mapsto s(0)}{\simeq} \mathbb{R}$. Thus $\mathcal{G}$ is a one dimensional vector space over $\mathbb{R}$.

② Let $f: E \to M$ be a vector bundle. Let $U_1, U_2 \subset M$ be two open subsets with trivializations $g_i: f^{-1}(U_i) \to U_i \times \mathbb{R}^k$. Put $U = U_1 \cap U_2$, which is assumed to be nonempty. Put $g = g_2 \circ g_1^{-1}: U \times \mathbb{R}^k \to U \times \mathbb{R}^k$. Then $g$ is a diffeomorphism. We'll show that there exists a smooth map $A: U \to GL(k, \mathbb{R})$ such that $g(p, \vec{v}) = (p, A(p)\vec{v})$ $\forall p \in U$, $\forall \vec{v} \in \mathbb{R}^k$.

For each $p \in U$, $f^{-1}(p) \xrightarrow{g_i} \{p\} \times \mathbb{R}^k$ is a linear isomorphism. Thus the composition map $\{p\} \times \mathbb{R}^k \xrightarrow{g} \{p\} \times \mathbb{R}^k$ is also a linear ~~dif~~ isomorphism. Thus there exists $A(p) \in GL(k, \mathbb{R})$ such that $g(p, \vec{v}) = (p, A(p)\vec{v})$. Now we have defined a map $A: U \to GL(k, \mathbb{R})$. We'll show that $A$ is smooth.

Pick any $p \in U$. We'll show that $A$ is smooth at $p$. We know that $\mathbb{R}^k$ is a smooth manifold with the single chart $(\mathbb{R}^k, \mathrm{id}_{\mathbb{R}^k})$. Consider the canonical basis $(\vec{e}_1, \dots, \vec{e}_k)$ on $\mathbb{R}^k$. Because of the product smooth structure on $U \times \mathbb{R}^k$, the projection $U \times \mathbb{R}^k \to \mathbb{R}^k$ and any inclusion $U \hookrightarrow U \times \mathbb{R}^k$ are smooth. For each $i = 1, \dots, k$, we put

$$\lambda_i: U \to U \times \mathbb{R}^k, \quad \lambda(q) = (q, \vec{e}_i)$$

then $\lambda_i$ is smooth. We represent the map $A$ as

$$A(p) = \begin{pmatrix} a_{11}(p) & \cdots & a_{1k}(p) \\ \vdots & \ddots & \vdots \\ a_{k1}(p) & \cdots & a_{kk}(p) \end{pmatrix}, \text{ where } a_{ij}: U \to \mathbb{R}$$

To show that $A$ is smooth, we'll show that each $a_{ij}$ is smooth.

We have the composition map

$$\begin{array}{ccccccc} U & \xrightarrow{\lambda_i} & U \times \mathbb{R}^k & \xrightarrow{g} & U \times \mathbb{R}^k & \xrightarrow{\pi} & \mathbb{R}^k \\ q & \longmapsto & (q, \vec{e_i}) & \longmapsto & (q, A(q)\vec{e_i}) & \longmapsto & A(q)\vec{e_i} = \begin{pmatrix} a_{1i}(q) \\ a_{2i}(q) \\ \vdots \\ a_{ki}(q) \end{pmatrix}. \end{array}$$

This composite map is smooth. Thus each map $a_{1i}, a_{2i}, \dots, a_{ki}$ is smooth. Since $i = 1, \dots, k$ can be chosen arbitrarily, all entry-functions of $A$ are smooth. Thus $A$ is smooth.

(3) Let $f : E \to M$ be a vector bundle of dimension $k$. Let $(U, \varphi : U \to V \subset \mathbb{R}^n)$ be a coordinate chart of $M$ together with a trivialization $f^{-1}(U) \xrightarrow{\Psi} U \times \mathbb{R}^k$. Put $W = f^{-1}(U)$, which is open in $E$. We want to construct $(n+k)$-linearly independent vector fields on $W$.

We have a composition of diffeomorphisms

$$\begin{tikzcd} W \arrow[r, "\Psi"] \arrow[rr, bend right, "\lambda"'] & U \times \mathbb{R}^k \arrow[r, "\varphi \times Id_{\mathbb{R}^k}"] & V \times \mathbb{R}^k. \end{tikzcd}$$

Then $\lambda = (\varphi \times Id_{\mathbb{R}^k}) \circ \Psi$ is a diffeomorphism. Thus $W$ is a smooth manifold of $(n+k)$-dimensions whose atlas consists of a single chart $(W, \lambda)$. Let $(y^1, \dots, y^{n+k})$ be any coordinate system in $V \times \mathbb{R}^k \subset \mathbb{R}^n \times \mathbb{R}^k$. Then we have $(n+k)$ vector fields on $W$ obtained by taking partial derivatives

$$X_i(p) := \left.\frac{\partial}{\partial y^i}\right|_p \quad \forall p \in W, \ \forall i = 1, \dots, n+k.$$

More precisely, for each $p \in W$, we get a map $d\lambda_p : T_pW \to T_{\lambda(p)}(V \times \mathbb{R}^k)$. Since $\lambda_p$ is a diffeomorphism, $d\lambda_p$ is a linear isomorphism. We know that $T_{\lambda(p)}(V \times \mathbb{R}^k)$ is an $(n+k)$-dimensional vector space with basis $\left\{ \frac{\partial}{\partial y^1}\Big|_{\lambda(p)}, \dots, \frac{\partial}{\partial y^{n+k}}\Big|_{\lambda(p)} \right\}$. The vector field $X_i$ that we defined is actually

$$X_i(p) = (d\lambda_p)^{-1}\left( \frac{\partial}{\partial y^i}\Big|_{\lambda(p)} \right) \qquad \forall p \in W.$$

thus $\{ X_1(p), \dots, X_{n+k}(p) \}$ are linearly independent for every $p \in W$. The component functions of $X_i$ are either constant 1 or constant 0, which are smooth as functions from $W$ to $\mathbb{R}$. Thus $X_i$ is smooth.

(4) Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. Put

$$\mathcal{U} = \{ V \in Gr(k,m) : V \cap \ker T = 0 \}$$

(a) We'll show that $\mathcal{U}$ is open in $Gr(k,m)$.

Put $r = \operatorname{rank} T \leqslant \min\{m,n\}$. By a result in linear algebra, there exist a basis $(u_1, \dots, u_m)$ of $\mathbb{R}^m$ and a basis $(v_1, \dots, v_n)$ of $\mathbb{R}^n$ such that

$$T(u_i) = \begin{cases} v_i, & 1 \leq i \leq r, \\ 0, & r < i \leq n. \end{cases}$$

We'll use these bases to represent vectors and points in $\mathbb{R}^m$ and $\mathbb{R}^n$. Each vector $\vec{x} \in \mathbb{R}^m$ is represented as a tuple $(x_1, \dots, x_m)$ such that $\vec{x} = \sum_{i=1}^m x_i u_i$. Each vector $\vec{y} \in \mathbb{R}^n$ is represented as a tuple $(y_1, \dots, y_n)$ such that $\vec{y} = \sum_{j=1}^n y_j v_j$.

We can construct the smooth structure on $Gr(k,m)$ as follow:

For each $k$-combination $\tau$ of $\{1,\dots,m\}$, we put

$$W_\tau = \{(x_1,\dots,x_m) \in \mathbb{R}^m : x_i = 0 \quad \forall i \notin \tau\},$$

$$W'_\tau = \{(x_1,\dots,x_m) \in \mathbb{R}^m : x_i = 0 \quad \forall i \in \tau\}.$$

Then $W_\tau$ and $W'_\tau$ are subspaces of $\mathbb{R}^m$ with $\dim W_\tau = k$ and $\dim W'_\tau = m-k$. $W_\tau$ has a basis $(u_i : i \in \tau)$, and $W'_\tau$ has a basis $(u_i : i \notin \tau)$. Let $\pi_\tau : \mathbb{R}^m \to W_\tau$ be the projection onto $W_\tau$.

$$\pi_\tau(x_1,\dots,x_m) = (\tilde{x}_1,\dots,\tilde{x}_m) \text{ with } \tilde{x}_i = \begin{cases} x_i, & i \in \tau, \\ 0, & i \notin \tau. \end{cases}$$

$$\mathcal{U}_\tau := \{V \in Gr(k,m) : \pi_\tau|_V : V \to W_\tau \text{ is bijective}\}.$$

Let $\mathcal{L}(W_\tau, W'_\tau)$ be the set of all linear maps from $W_\tau$ to $W'_\tau$. Each $V \in \mathcal{U}_\tau$ associates with a map $f_V \in \mathcal{L}(W_\tau, W'_\tau)$ as follow: each $x \in W_\tau$ corresponds to a unique $y \in V$ such that $\pi_\tau(y) = x$; then $f_V(x) := y - x \in W'_\tau$.

Then we get a bijection $V \in \mathcal{U}_\tau \longmapsto f_V \in \mathcal{L}(W_\tau, W'_\tau)$. Moreover,

$$\mathcal{L}(W_\tau, W'_\tau) \simeq M_{(m-k)\times k}(\mathbb{R}) \simeq \mathbb{R}^{(m-k)k}.$$

representation matrix in

bases $(u_1,\dots,u_m)$ and $(v_1,\dots,v_n)$

Thus we get a bijection $\varphi_\tau : \mathcal{U}_\tau \to \mathbb{R}^{(m-k)k}$. Then we declare that $\{(\mathcal{U}_\tau, \varphi_\tau) : \tau \text{ is a } k\text{-combination of } \{1,\dots,m\}\}$ is a smooth atlas of $Gr(k,m)$.

Return to the problem. For each $V \in \mathcal{U}$, we have

$$k = \dim V = \underbrace{\dim T(V)}_{\leq r} + \underbrace{\dim (V \cap \ker T)}_{0} \leq r$$

If $k > r$ then $\mathcal{U} = \phi$. Thus we only consider the case $k \leq r$.

Pick any $V_0 \in \mathcal{U}$. We'll show that there is an open neighborhood of $V_0$ in $Gr(k,m)$ that is also contained in $\mathcal{U}$. Since $\{\mathcal{U}_\tau\}_\tau$ is an atlas of $Gr(k,m)$, there exists a $k$-combination $\tau$ of $\{1,\dots,m\}$ such that $V_0 \in \mathcal{U}_\tau$. WLOG, we assume $\tau = \{1,\dots,k\}$. Then

$$W_\tau = \{(x_1,\dots,x_k,0,\dots,0) : x_i \in \mathbb{R}\} \subset \mathbb{R}^m, \text{ with basis } (u_1,\dots,u_k),$$

$$W_\tau' = \{(0,\dots,0,x_{k+1},\dots,x_m) : x_i \in \mathbb{R}\} \subset \mathbb{R}^m, \text{ with basis } (u_{k+1},\dots,u_m).$$

By the construction of smooth structure on $Gr(k,m)$ mentioned above, we have

$$V_0 \longrightarrow f_{V_0} \in \mathcal{L}(W_\tau, W_\tau') \longrightarrow \varphi_\tau(V_0) = (a_{ij}) \in M_{(m-k)\times k}(\mathbb{R}),$$

where

$$(a_{ij}) = \begin{pmatrix} | & & | \\ [f_{V_0}(u_1)]_{W_\tau'} & \cdots & [f_{V_0}(u_k)]_{W_\tau'} \\ | & & | \end{pmatrix}$$

is the representation matrix of $f_{V_0}$. We want to find an open neighborhood $O$ of $(a_{ij})$ in $\mathbb{R}^{(m-k)k}$ such that $\varphi_\tau^{-1}(O) \subset \mathcal{U}$. For each $x \in W_\tau$,

$$x = \sum_{i=1}^{k} x_i u_i, \quad f_{V_0}(x) = \sum_{l=1}^{k} x_l f(u_l) = \sum_{l=1}^{k} \sum_{s=1}^{m-k} x_l a_{sl} u_{s+k}.$$

Each $y \in V_0$ can be written in a unique way $y = x + f_{V_0}(x)$ with $x \in W_\tau$.

$$y = \sum_{i=1}^{k} x_i u_i + \sum_{l=1}^{k} \sum_{s=1}^{m-k} x_l a_{sl} u_{s+k} \qquad (1)$$

Then $T(y) = T(x) + T(f_{V_0}(x))$

$$= \sum_{i=1}^{k} x_i \underbrace{Tu_i}_{v_i} + \sum_{s=1}^{m-k} \sum_{\ell=1}^{k} x_\ell a_{s\ell} \underbrace{Tu_{s+k}}_{= \begin{cases} v_{s+k} & \text{if } s+k \le r, \\ 0 & \text{if } s+k > r. \end{cases}}$$

Thus, $$Ty = \sum_{i=1}^{k} x_i v_i + \sum_{s=1}^{r-k} \left( \sum_{\ell=1}^{k} x_\ell a_{s\ell} \right) v_{s+k} \qquad (2)$$

Therefore, $$Ty \neq 0 \Leftrightarrow \sum_{i=1}^{k} x_i^2 + \sum_{s=1}^{r-k} \left( \sum_{\ell=1}^{k} x_\ell a_{s\ell} \right)^2 > 0 \qquad (3).$$

We have $V_0 \in \mathcal{U} \Leftrightarrow V_0 \cap \ker T = 0$

$\Leftrightarrow Ty \neq 0 \quad \forall y \in V_0$

$\Leftrightarrow$ (3) is true for all $(x_1, \dots, x_k) \in \mathbb{R}^k \setminus \{0\}$

$\Leftrightarrow$ (3) is true for all $(x_1, \dots, x_k) \in S^{k-1}$,

where $S^{k-1} = \{(x_1, \dots, x_k) : x_1^2 + \dots + x_k^2 = 1\}$. This is because the epression on the left hand side of (3) is a homogeneous polynomial of $x_1, \dots, x_k$.

Define the map $\varphi : \mathbb{R}^{(m-k)k} \times S^{k-1} \longrightarrow \mathbb{R}$,

$b_{s\ell}$?

$$\varphi((b_{ij}), (x_1, \dots, x_k)) = \underbrace{\sum_{i=1}^{k} x_i^2}_{1} + \sum_{s=1}^{r-k} \left( \sum_{\ell=1}^{k} x_\ell a_{s\ell} \right)^2$$

for all $(b_{ij}) \in \mathbb{R}^{(m-k)k}$ and $(x_1, \dots, x_k) \in S^{k-1}$.

We have $\varphi((a_{ij}), (x_1, \dots, x_k)) > 0$ for all $(x_1, \dots, x_k) \in S^{k-1}$. We want to find an open neighborhood $O$ of $(a_{ij})$ in $\mathbb{R}^{(m-k)k}$ such that

12 It was enough to note that (3) is continuous in $(a_{ij})$

$$\varphi((b_{ij}),(x_1,\dots,x_k)) > 0 \quad \forall (x_1,\dots,x_k) \in S^{k-1}, \ \forall (b_{ij}) \in \mathcal{O}$$

Put $X = \mathbb{R}^{(m-k)k}$ and $Y = S^{k-1}$ and $a = (a_{ij})$. Then

- $\varphi : X \times Y \to \mathbb{R}$ is continuous,
- $Y$ is compact,
- $\varphi(a,y) > 0 \quad \forall y \in Y$.

Because $Y$ is compact, we can define a map $\Psi : X \to \mathbb{R}$, $\Psi(x) = \min_{y\in Y} \varphi(x,y)$.

We have $\Psi(a) > 0$. To show that there is an open neighborhood $\mathcal{O}$ of $a$ such that $\Psi(b) > 0$ for all $b \in \mathcal{O}$, it suffices to show the continuity of $\Psi$ at $a$. Take a closed ball $\overline{\mathbb{B}}(a,1)$ in $X$. The map

$$\varphi : \overline{\mathbb{B}}(a,1) \times Y \to \mathbb{R}$$

is uniformly continuous because the domain is compact. Thus,

$$\forall \varepsilon > 0, \exists \delta > 0 : \ |\varphi(x,y) - \varphi(a,y)| < \varepsilon, \ \forall d(x,a) < \delta, \ y \in Y.$$

Then $\Psi(a) \leq \varphi(a,y) < \varphi(x,y) + \varepsilon$ for all $d(x,a) < \delta$, $y \in Y$.

Then $\Psi(a) < \min_{y \in Y} \varphi(x,y) + \varepsilon = \Psi(x) + \varepsilon$.

Similarly, $\Psi(x) \leq \varphi(x,y) < \varphi(a,y) + \varepsilon$ for all $d(x,a) < \delta$, $y \in Y$.

Then $\Psi(x) < \min_{y \in Y} \varphi(a,y) + \varepsilon = \Psi(a) + \varepsilon$.

Therefore $|\Psi(a) - \Psi(x)| < \varepsilon$ for all $x \in X$, $d(x,a) < \delta$ $(<1)$.

Therefore $\Psi$ is continuous at $a$.

(6) We'll show that the map $\lambda : \mathcal{U} \to Gr(k,n)$, $\lambda(V) = T(V)$ is smooth. Of course, we consider the case $\mathcal{U} \neq \phi$ only. Consequently, $k \leq r$. First, we check $\lambda$ is well-defined.

$$\forall V \in \mathcal{U},\ k = \dim V = \dim T(V) + \dim(V \cap \ker T) = \dim T(V).$$

Thus $T(V) \in Gr(k,n)$. Therefore $\lambda$ is well-defined.

Because we have to consider the smooth structures on $Gr(k,m)$ and $Gr(k,n)$, we will append indices to the notations in the construction mentioned earlier.

| Name in general construction | Name in $\mathbb{R}^m$ | Name in $\mathbb{R}^n$ |
|---|---|---|
| $\tau$ | $\tau$ <br> $k$-combination of $\{1,\dots,m\}$ | $\sigma$ <br> $k$-combination of $\{1,\dots,n\}$ |
| $W_\tau$ | $W_{m,\tau}$ | $W_{n,\sigma}$ |
| $W_\tau'$ | $W_{m,\tau}'$ | $W_{n,\sigma}'$ |
| $\pi_\tau$ | $\pi_{m,\tau}$ | $\pi_{n,\sigma}$ |
| Chart <br> $\varphi_\tau : \mathcal{U}_\tau \to \mathbb{R}^{(m-k)k}$ | Chart <br> $\varphi_{m,\tau} : \mathcal{U}_{m,\tau} \to \mathbb{R}^{(m-k)k}$ | Chart <br> $\varphi_{n,\sigma} : \mathcal{U}_{n,\sigma} \to \mathbb{R}^{(n-k)k}$ |

To show that $\lambda : \mathcal{U} \to Gr(k,n)$ is smooth, we'll show that $\lambda|_{\mathcal{U} \cap \mathcal{U}_{m,\tau}}$ is smooth for every $k$-combination $\tau$ of $\{1,\dots,m\}$. We'll show this for the case $\tau = \{1, \dots, k\}$. The method for other cases is the same, but the notations get more complicated (e.g. it's not convenient to write out certain matrices).

For each $V \in \mathcal{U}_{m,\tau}$, we put $\varphi_{m,\tau}(V) = (a_{ij}) \in M_{(m-k)\times k}(\mathbb{R})$. As mention in equation (1) in part (a), each $y \in V$ is of the form

$$y = \sum_{i=1}^{k} x_i u_i + \sum_{s=1}^{m-k}\left(\sum_{l=1}^{k} x_l a_{sl}\right) u_{s+k}$$

Thus, $V = \left\{\left(x_1, \dots, x_k, \sum_{l=1}^{k} x_l a_{1l}, \dots, \sum_{l=1}^{k} x_l a_{(m-k),l}\right) : x_i \in \mathbb{R}\right\}$.

Denote the block matrix

$$\begin{pmatrix} I_k \\ (a_{ij}) \end{pmatrix} := \begin{pmatrix} \boxed{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}} \\ \boxed{\begin{matrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{m-k,1} & \cdots & a_{m-k,k} \end{matrix}} \end{pmatrix} \in M_{m\times k}(\mathbb{R}).$$

Thus each element in $V$ is of the form $\underbrace{\begin{pmatrix} I_k \\ (a_{ij}) \end{pmatrix}}_{m\times k} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}}_{k\times 1}$.

This means $V$ is the column space of the matrix $\begin{pmatrix} I_k \\ (a_{ij}) \end{pmatrix}$. More generally, if $\tau = \{i_1, \dots, i_k\}$ is any $k$-combination of $\{1, \dots, m\}$ (rather then $\{1, \dots, k\}$), then each $V \in \mathcal{U}_{m,\tau}$ is the column space of an $m\times k$ matrix whose rows $i_1, \dots, i_k$ make up $I_k$ and other rows make up $(a_{ij}) = \varphi_{m,\tau}(V)$. This rule is of course also true when $m$ is replaced by $n$.

We can think of $\begin{pmatrix} I_k \\ (a_{ij}) \end{pmatrix} \in M_{n\times k}(\mathbb{R})$ as an alternative representation of $(a_{ij}) \in M_{(n-k)\times k}(\mathbb{R})$.

Consider $V \in \mathcal{U} \cap \mathcal{U}_\tau$. As shown in equation (2) in part (a), we have

$$T(V) = \{Ty : y \in V\} = \left\{ \sum_{i=1}^{k} x_i v_i + \sum_{s=1}^{r-k} \left( \sum_{l=1}^{k} x_l a_{sl} \right) v_{s+k} \;\middle|\; x_i \in \mathbb{R} \right\}$$

Thus $T(V)$ is the column space of the matrix

$$\begin{array}{c} k\left\{\begin{array}{c} \\ \\ \end{array}\right. \\ r-k\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ n-r\left\{\begin{array}{c} \\ \\ \end{array}\right. \end{array} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{(r-k),1} & \cdots & a_{(r-k),k} \\ 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix} \in M_{n\times k}(\mathbb{R})$$

As a rule we mentioned earlier, $T(V) \in \mathcal{U}_{n,\sigma}$ with $\sigma = \{1, \dots, k\}$ and

$$\varphi_{n,\sigma}(T(V)) = (c_{ij}) = \begin{pmatrix} a_{11} & \cdots & a_{k1} \\ \vdots & & \\ a_{r-k,1} & \cdots & a_{r-k,k} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{array}{c} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} r-k \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} n-r \end{array} \in M_{(n-k)\times k}(\mathbb{R})$$

Put $$B = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & 0 & \end{pmatrix} \begin{array}{c} \left.\begin{array}{c} \\ \\ \end{array}\right\} r-k \\ \}\, n-r \end{array} \in M_{(n-k)\times k}(\mathbb{R})$$

We have $(c_{ij}) = B(a_{ij})$. Now we can consider $\lambda$ in coordinate charts.

With $\tau = \{1, \dots, k\}$ and $\sigma = \{1, \dots, k\}$, we showed that $\lambda(\mathcal{U} \cap \mathcal{U}_{m,\tau}) \subset \mathcal{U}_{n,\sigma}$.

$$\begin{tikzcd}
\mathcal{U} \cap \mathcal{U}_{m,\tau} \arrow[r, "\lambda"] \arrow[d, "\varphi_{m,\tau}"'] & \mathcal{U}_{n,\sigma} \arrow[d, "\varphi_{n,\sigma}"] \\
\varphi_{m,\tau}(\mathcal{U} \cap \mathcal{U}_{m,\tau}) \arrow[r, dashed, "\hat{\lambda}"] \arrow[d, phantom, sloped, "\subset"] & \mathbb{R}^{(n-k)k} \\
\mathbb{R}^{(m-k)k} &
\end{tikzcd}$$

We have $\hat{\lambda}(A) = BA$. Thus $\hat{\lambda}$ is simply a restriction of a linear map from $\mathbb{R}^{(m-k)k}$ to $\mathbb{R}^{(n-k)k}$ on the open subset $\varphi_{m,\tau}(\mathcal{U} \cap \mathcal{U}_{m,\tau})$. Thus $\hat{\lambda}$ is smooth. Therefore $\lambda$ is smooth.

5 Consider $\mathbb{R}^3$ with Cartesian coordinate system in which each point is represented by a triple $(x, y, z)$. Let

$$X_1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} \quad \text{and} \quad X_2 = \frac{\partial}{\partial z}$$

be two vector fields on $\mathbb{R}^3$. Consider a smooth distribution $D : \mathbb{R}^3 \to Gr(2,3)$ given by $D(p) = \text{span}(X_1(p), X_2(p))$ for all $p \in \mathbb{R}^3$.

Suppose that $f : S^1 \to \mathbb{R}^3$ is an embedding that is tangent to $D$.

This means 1) $f$ is an injective and smooth map,

2) $\forall q \in S^1$, $\exists (U, t \in \mathbb{R}) \ni q$ such that the tangent map $df_q : T_qU \to T_{f(q)}\mathbb{R}^3$ satisfies

- $\text{rank}(df_q) = 1$,
- $\text{Im}(df_q) \subset D(f(q))$.

Let $(U, t \in \mathbb{R})$ be a chart of $S^1$. Each $q \in U$ has coordinate $t \in \mathbb{R}$. Put $p = f(q)$.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d] & \mathbb{R}^3 \\ \mathbb{R}, t \arrow[ur, "{(x(t), y(t), z(t))}"'] & \end{tikzcd}$$

we know that $T_q U$ is a 1-dimensional vector space generated by $\left.\frac{d}{dt}\right|_t$.

$$df_q\left(\left.\frac{d}{dt}\right|_t\right) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)\begin{pmatrix} \partial/\partial x|_p \\ \partial/\partial y|_p \\ \partial/\partial z|_p \end{pmatrix}.$$

We have $\operatorname{rank}\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \operatorname{rank}(df_q) = 1$. Thus $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are not allowed to vanish at once.

$$\operatorname{Im}(df_q) \subset D(p) \Leftrightarrow df_q\left(\left.\frac{d}{dt}\right|_t\right) \in \operatorname{span}(X_1(p), X_2(p))$$

$$\Leftrightarrow \exists\, c_1(t), c_2(t) \in \mathbb{R} : df_q\left(\left.\frac{d}{dt}\right|_t\right) = c_1(t) X_1(p) + c_2(t) X_2(p) \quad (*)$$

we have $(*) \Leftrightarrow \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)\begin{pmatrix} \partial/\partial x|_p \\ \partial/\partial y|_p \\ \partial/\partial z|_p \end{pmatrix} = c_1(t)\begin{pmatrix} \partial/\partial x|_p \\ z\,\partial/\partial y|_p \\ 0 \end{pmatrix} + c_2(t)\begin{pmatrix} 0 \\ 0 \\ \partial/\partial z|_p \end{pmatrix}$

$$\Leftrightarrow \begin{cases} \frac{dx}{dt} = c_1(t) \\ \frac{dy}{dt} = z c_1(t) \\ \frac{dz}{dt} = c_2(t) \end{cases}$$

Since $x, y, z$ are smooth functions of $t$, $c_1$ and $c_2$ are supposed to be smooth.

The condition $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \neq 0$ becomes $(c_1(t), c_2(t)) \neq 0$.

In case $c_1(t) \neq 0$, we have $dx/dt \neq 0$. By the inverse function theorem,

$t$ is a function of $x$. Then $y = y(t) = y(t(x))$ is a function of $x$. By chain rule, $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy/dt}{dx/dt} = \frac{zG(t)}{G(t)} = z$.

Thus, in any coordinate chart $(U, t \in \mathbb{R})$ of $S^1$, whenever $G(t) \neq 0$, the projection on $xy$-plane of points in ~~Im(f)~~ $f(U)$ makes up a curve $y = y(x)$ satisfying the differential equation $\frac{dy}{dx} = z$.

As a result, if we know the picture of $Im(f)$ projected on $xy$-plane, we can calculate the slope $\frac{dy}{dx}$ of the curve, which will give us the "height" of the corresponding point of $Im(f)$ in $\mathbb{R}^3$. As we travel along the curve $y = y(x)$, ~~the pair of functions $(G(t), G(t))$~~ the points $(x, y, \frac{dy}{dx})$ draw a part of $f(S^1)$. However, we may not be able draw the whole $f(S^1)$. For example, we may reach a point when $y$ is no longer a function of $x$ (the picture below).