The first additional problem of Math 8302, Manifolds and Topology II posted on April 29, 2013 by Tyler Lawson.

Solution

According to the assignment, M may not be path-connected. Thus we decompose M into path-connected components. Since M is locally Euclidean, the path-connected components are the same as the connected components and are open subspaces of M. Thus each component itself is a smooth manifold. Then we will solve the problem on each of them separately but with the same method. This means we can assume from the beginning that M is connected.

Our claim is that ω *satisfies the conditions stated in the problem if and only if* $\omega(t)$ *is independent of* t *and* ω *is an exact 1-form on* M. In other words, there is a smooth map $\eta : M \to \mathbb{R}$ such that $\omega = d\eta$.

We will break our proof into 3 steps, namely $M = \mathbb{R}$, $M = \mathbb{R}^n$ and M arbitrary.

Step 1. $M = \mathbb{R}$

Then $\omega(t) = f(x,t)dx$, where $f : \mathbb{R} \times (a,b) \to \mathbb{R}$ is a smooth function. The property of ω can be restated as follow:

For any $[c,d] \subset (a,b)$, for any smooth maps $x_1, x_2 : [c,d] \to \mathbb{R}$ such that $x_1(c) = x_2(c)$ and $x_1(d) = x_2(d)$, we have

$$\int_c^d f(x_1(t),t){x_1}'(t)dt = \int_c^d f(x_2(t),t){x_2}'(t)dt$$

By taking x_2 to be a constant function, the right hand side vanishes. Then the above property can be restated as follow:

For any $[c,d] \subset (a,b)$, for any smooth maps $x : [c,d] \to \mathbb{R}$ such that $x(c) = x(d)$, we have

$$\int_c^d f(x(t),t)x'(t)dt = 0 \tag{1}$$

Now fix an interval $[c,d] \subset (a,b)$. We will show that $f(x,t)$ is independent of t. Take $x_0 \in \mathbb{R}$ arbitrarily. We will show that the map $t \in [c,d] \to f(x_0,t) \in \mathbb{R}$ is constant. Put

$$\tau = \pi\frac{t-c}{d-c} \quad \forall t \in [c,d],$$

$$g : [0,\pi] \to \mathbb{R}, \quad g(\tau) = f(x_0,t) = f\left(x_0, \frac{d-c}{\pi}\tau + c\right).$$

For each $\epsilon > 0$, $n \in \mathbb{N}$, we define the maps

$$x_{\varepsilon,n} : [c,d] \to \mathbb{R}, \quad x_{\varepsilon,n}(t) = x_0 + \varepsilon \sin\left(n\pi\frac{t-c}{d-c}\right),$$

$$y_{\varepsilon,n} : [0,\pi] \to \mathbb{R}, \quad y_{\varepsilon,n}(\tau) = x_{\varepsilon,n}(t) = x_0 + \varepsilon \sin(n\tau).$$

Then $x_{\varepsilon,n}$ is smooth and $x_{\varepsilon,n}(c) = x_{\varepsilon,n}(d) = x_0$. Equivalently, $y_{\varepsilon,n}$ is smooth and $y_{\varepsilon,n}(0) = y_{\varepsilon,n}(\pi) = x_0$. With $x = x_{\varepsilon,n}$, Eq. (1) becomes

$$0 = \int_c^d f(x_{\varepsilon,n}(t), t){x_{\varepsilon,n}}'(t)dt = \int_0^\pi f(y_{\varepsilon,n}(\tau), t){y_{\varepsilon,n}}'(\tau)\frac{\pi}{d-c}d\tau$$

Thus,

$$\int_0^\pi f\left(x_0 + \varepsilon \sin(n\tau), \frac{d-c}{\pi}\tau + c\right)\cos(n\tau)d\tau = 0 \tag{2}$$

Because this is true for all $\epsilon > 0$, we can take the limit as $\epsilon \to 0$ on both sides of Eq. (2). Note that f is smooth, so we can bring the limit into the integral sign. Then Eq. (2) gives

$$\int_0^\pi f\left(x_0, \frac{d-c}{\pi}\tau + c\right)\cos(n\tau)d\tau = 0,$$

which means

$$\int_0^\pi g(\tau)\cos(n\tau)d\tau = 0 \quad \forall n \in \mathbb{N}$$

We know that g has a Fourier Cosine series on $[0, \pi]$ and

$$g(\tau) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\tau)$$

where $a_n = \frac{2}{\pi}\int_0^\pi g(\tau)\cos(n\tau)d\tau$. By what we have just proved, $a_n = 0$ for all $n \in N$. Thus $g(\tau)$ is a constant function. Therefore, $f(x_0, t)$ is constant, i.e. $f(x, t)$ does not depend on t. We can write $f(x, t) = f(x)$. Then $\omega(t) = \omega = f(x)dt$. This is always an exact 1-form because $f(x)$ has an antiderivative on $\mathbb{R}$. Conversely, with $\omega = f(x)dt$, we have

$$\int_c^d f(x(t))x'(t)dx = \int_{x(c)}^{x(d)} f(u)du,$$

which depends only on the values of x at the starting and ending points. Thus, such an ω satisfies the property stated in the problem.

Step 2. $M = \mathbb{R}^n$
Then

$$\omega(t) = \sum_{k=1}^{n} f_k(x^1, \ldots, x^n, t)dx^k$$

where each $f_k : \mathbb{R}^n \times (a,b) \to \mathbb{R}$ is smooth. The property of ω can be restated as follow:

For any $[c,d] \subset (a,b)$, for any smooth maps $\gamma_1, \gamma_2 : [c,d] \to \mathbb{R}^n$ such that $\gamma_1(c) = \gamma_2(c)$ and $\gamma_1(d) = \gamma_2(d)$, written as $\gamma_1(t) = (x^1(t), \dots, x^n(t))$ and $\gamma_2(t) = (y^1(t), \dots, y^n(t))$ we have

$$\sum_{k=1}^{n} \int_c^d f_k(x^1(t), \dots, x^n(t), t) \frac{dx^k}{dt} dt = \sum_{k=1}^{n} \int_c^d f_k(y^1(t), \dots, y^n(t), t) \frac{dy^k}{dt} dt$$

By taking γ_2 to be a constant path, the right hand side vanishes. Then we can restate the above property as follow:

For any $[c,d] \subset (a,b)$, for any smooth maps $\gamma : [c,d] \to \mathbb{R}^n$ such that $\gamma(c) = \gamma(d)$, written as $\gamma(t) = (x^1(t), \dots, x^n(t))$, we have

$$\sum_{k=1}^{n} \int_c^d f_k(x^1(t), \dots, x^n(t), t) \frac{dx^k}{dt} dt = 0$$

We can restate the above property as follow:

For any $[c,d] \subset (a,b)$, for any smooth maps $x^1, \dots, x^n : [c,d] \to \mathbb{R}$ such that $x^k(c) = x^k(d)$ for every $k = 1, 2, \dots, n$, we have

$$\sum_{k=1}^{n} \int_c^d f_k(x^1(t), \dots, x^n(t), t) \frac{dx^k}{dt} dt = 0 \tag{3}$$

Fix an interval $[c,d] \subset [a,b]$. We will show that each $f_k(x^1, \dots, x^n, t)$ does not depend on t. By symmetry, it suffices to prove this is true for $f_1(x^1, \dots, x^n, t)$.

Take $p_0 = (x_0^1, \dots, x_0^n) \in \mathbb{R}^n$ arbitrarily. We will show that the map $t \in [c,d] \to f(p_0, t) \in \mathbb{R}$ is constant. Choose $x^k(t) \equiv x_0^k$ for all $k = 2, \dots, n$. Then Eq. (3) becomes

$$\int_c^d f_1(x^1(t), x_0^2, \dots, x_0^n, t) \frac{dx^1}{dt} dt = 0$$

for all smooth map $x^1 : [c,d] \to \mathbb{R}$ such that $x^1(c) = x^1(d) = x_0^1$. Put $g : \mathbb{R} \times (a,b) \to \mathbb{R}$, $g(x^1, t) = f_1(x^1, x_0^2, \dots, x_0^n, t)$. Then

$$\int_c^d g(x^1(t), t) \frac{dx^1}{dt} dt = 0$$

for all smooth map $x^1 : [c,d] \to \mathbb{R}$ such that $x^1(c) = x^1(d) = x_0^1$. Now we return to the case $M = \mathbb{R}$, where we proved that g is independent of t via Fourier Cosine series. Therefore, f_1 is also independent of t. We can write $f_k(x^1, \dots, x^n, t) = f_k(x^1, \dots, x^n)$ for all $k = 1, \dots, n$. Then

$$\omega(t) = \omega = \sum_{k=1}^{n} f_k(x^1, \dots, x^n) dx^k$$

We know that $\int_\gamma \omega$ depends only on the starting and ending points of the path. Fix $p_0 = (0, 0, \dots, 0) \in \mathbb{R}^n$. For each point $p = (a^1, \dots, a^n) \in \mathbb{R}^n$, we define a map $F : \mathbb{R}^n \to \mathbb{R}$,

$$F(p) := \int_\gamma \omega = \sum_{k=1}^{n} \int_c^d f_k(x^1(t), .., x^n(t)) \frac{dx^k}{dt} dt,$$

where $\gamma : [c, d] \to \mathbb{R}^n$ is any smooth path from p_0 to p. We will show that $\frac{\partial F}{\partial x^k}(p) = f_k(p)$ for every $k = 1, \dots, n$. It suffices to show that this is true for $k = 1$. Choose γ to be a path from p_0 to p such that $x^k(t) = a^k$ for all $(c + d)/2 \le t \le d$, for all $k = 2, \dots, n$ and $x^1((c + d)/2) = 0$. Then

$$F(p) - F\left(p\left(\frac{c+d}{2}\right)\right) = \int_{(c+d)/2}^{d} f_1(x^1(t), a^2, .., a^n) \frac{dx^1}{dt} dt$$

Thus,

$$\begin{aligned} F(p) &= F\left(0, a^2, \dots, a^n\right) + \int_{(c+d)/2}^{d} f_1(x^1(t), a^2, \dots, a^n) \frac{dx^1}{dt} dt \\ &= F\left(0, a^2, \dots, a^n\right) + \int_{x^1((c+d)/2)}^{x^1(d)} f_1(u, a^2, \dots, a^n) du \\ &= F\left(0, a^2, \dots, a^n\right) + \int_0^{a^1} f_1(u, a^2, .., a^n) du \end{aligned}$$

Thus, $\frac{\partial F}{\partial x^1}(p) = f_1(a^1, a^2, \dots, a^n) = f_1(p)$. Therefore,

$$\omega = \sum_{k=1}^{n} \frac{\partial F}{\partial x^k}(x^1, \dots, x^n) dx^k.$$

which is an exact 1-form on $\mathbb{R}^n$. Conversely, if $\omega = dF$ then

$$\int_\gamma \omega = \int_\gamma dF = \int_c^d \frac{dF}{dt} dt = F(\gamma(d)) - F(\gamma(c)),$$

which depends only on the starting and ending points of γ. Thus, such an ω satisfies the desired property.

Step 3. M is an arbitrary smooth connected n-manifold.

Pick any smooth chart $(U, \phi : U \to \mathbb{R}^n)$ on M. On U, ω has the form

$$\omega(t) = \sum_{k=1}^{n} f_k(x^1, \ldots, x^n, t) dx^k$$

Since ω satisfies the property stated in the problem for $(x^1, \ldots, x^n, t) \in M \times (a, b)$, it must satisfy the same property for $(x^1, \ldots, x^n, t) \in U \times (a, b)$. Now we return to the case $M = \mathbb{R}^n$. We conclude that, on U, ω is independent of t and there exists a smooth map $F_U : U \to \mathbb{R}$ such that $\omega|_U = dF_U$. Since $\omega(t)$ is independent of t on every chart U, it is so on M. Thus, $\omega(t) = \omega$ is simply a 1-form on M.

Fix $a \in M$. For each $p \in M$, there is a path γ from a to p. Since M is a smooth manifold, there is at least such a piecewise smooth path γ. Since $\int_\gamma \omega$ depends only on the ending point of γ, we can define a map

$$\eta : M \to \mathbb{R}, \quad \eta(p) = \int_\gamma \omega$$

We will show that η is smooth and $d\eta = \omega$. On each chart $(U, (x^i))$, we fix a point p_U. Then for any point $p \in U$, p can be connected from p_U by a smooth path λ in U. Then

$$\eta(p) - \eta(p_U) = \int_\lambda \omega$$

We know that $\omega|_U = dF_U$. Thus

$$\eta(p) - \eta(p_U) = \int_\lambda \omega = F_U(p) - F_U(p_U)$$

Then we get

$$\eta(p) = F_U(p) + \eta(p_U) - F_U(p_U) \tag{4}$$

Since F_U is smooth, η is also smooth on U. Hence η is smooth on M. By the local property of the exterior derivative, Eq. (4) gives us $d\eta = dF_U = \omega$. Therefore ω is an exact 1-form.

Conversely, suppose that $\omega(t) = \omega = d\eta$. Let $\gamma : [c, d] \to M$ be a smooth path. Since $\gamma([c, d])$ is compact, it can be covered by finitely many coordinate charts $U_1, \ldots, U_r$ on M, each of which is diffeomorphic to $\mathbb{R}^n$. Moreover, we can even assume that there are $c = t_0 < t_1 < \ldots < t_r = d$ such that $\gamma([t_{i-1}, t_i]) \subset U_i$ for $i = 1, \ldots, r$. Then

$$\begin{aligned} \int_\gamma \omega = \sum_{i=1}^{r} \int_{\gamma([t_{i-1}, t_i])} d\eta &= \sum_{i=1}^{r} (\eta(\gamma(t_i)) - \eta(\gamma(t_{i-1}))) \\ &= \eta(\gamma(t_r)) - \eta(\gamma(t_0)) = \eta(\gamma(d)) - \eta(\gamma(c)) \end{aligned}$$

This means the integral depends only on the starting and ending point of γ.

The second additional problem of Math 8302, Manifolds and Topology II posted on April 29, 2013 by Tyler Lawson.

Solution

First, we define an equivalence relation on $\mathbb{C}^{n+1}$ as follows

$$(z_0, z_1, \ldots, z_n) \sim (z_0', z_1', \ldots, z_n') \Leftrightarrow \exists \lambda \in \mathbb{C}^* : (z_0, z_1, \ldots, z_n) = \lambda(z_0', z_1', \ldots, z_n').$$

Denote by $[z_0 : z_1 : \ldots : z_n]$ the equivalence class of $(z_0, z_1, \ldots, z_n)$. Then $\mathbb{CP}^n$ is defined to be the set of the equivalence classes. For each $0 \leq k \leq n$, we put

$$U_k = \{[z_0 : z_1 : \ldots : z_n] | z_k \neq 0\}$$

There is a bijective map $\phi_k : U_k \to \mathbb{C}^n$ given by

$$\phi_k([z_0 : z_1 : \ldots : z_n]) = \left(\frac{z_0}{z_k}, \ldots, \widehat{\frac{z_k}{z_k}}, \ldots, \frac{z_n}{z_k}\right)$$

On the other hand, we can identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the bijection $(x_1+iy_1, \ldots, x_n+iy_n) \mapsto (x_1, y_1, \ldots, x_n, y_n)$. Thus we can think of ϕ_k as a bijection from U_k to $\mathbb{R}^{2n}$. The cover $\{(U_k, \phi_k) | k = 0, 1, .., n\}$ defines a smooth structure on $\mathbb{CP}^n$ and makes it a $2n$-manifold.

$\mathbb{CP}^1$ can be viewed as a submanifold of $\mathbb{CP}^n$ via the inclusion $i : [z_0 : z_1] \mapsto [z_0 : z_1 : 0 : \ldots : 0]$. Given that $H^2_{dR}(\mathbb{CP}^n) \simeq \mathbb{R}$, we will find a generator for this group. This is equivalent to finding a nonzero element in $H^2_{dR}(\mathbb{CP}^n)$. We know that $\mathbb{CP}^1 \simeq S^2$, which has no boundary. This means $\mathbb{CP}^1$ is a 2-cycle in $\mathbb{CP}^n$. By de Rham theorem, it suffices to find a closed 2-form on $\mathbb{CP}^n$ whose integral over $\mathbb{CP}^1$ is nonzero. We break the solution into two steps, namely $n = 1$ and $n \geq 2$.

Step 1. $n = 1$

First we will construct a diffeomorphism from $\mathbb{CP}^1$ to S^2. This gives a group isomorphism from $H^2_{dR}(S^2)$ to $H^2_{dR}(\mathbb{CP}^1)$ given by the pullbacks. Then we will find a generator of $H^2_{dR}(S^2)$, which gives a generator of $H^2_{dR}(\mathbb{CP}^1)$. Then we express that 2-form in coordinates.

By definition,

$$\mathbb{CP}^1 = \left\{[z_0 : z_1] | |z_0|^2 + |z_1|^2 \neq 0\right\}$$

has an atlas consisting of two charts (U_+, ϕ_+) and (U_-, ϕ_-) defined as follows.

$$U_+ = \{[z_0 : z_1] | z_1 \neq 0\},$$

$$U_- = \{[z_0 : z_1] | z_0 \neq 0\},$$

$$\phi_+ : U_+ \to \mathbb{C}, \quad \phi_+([z_0 : z_1]) = \frac{z_0}{z_1},$$

$$\phi_- : U_- \to \mathbb{C}, \quad \phi_-([z_0 : z_1]) = \frac{z_1}{z_0}.$$

The inverse maps are $\phi_+^{-1}(z) = [z : 1]$ and $\phi_-^{-1}(z) = [1 : z]$. By definition, the sphere

$$S^2 = \left\{(a, b, c)|a^2 + b^2 + c^2 = 1\right\}$$

has an atlas consisting of two charts (V_+, ψ_+) and (V_-, ψ_-) defined as follows.

$$V_+ = \left\{(a, b, c) \in S^2|c \neq 1\right\},$$

$$V_- = \left\{(a, b, c) \in S^2|c \neq -1\right\},$$

$$\psi_+ : V_+ \to \mathbb{C}, \quad \psi_+(a, b, c) = \frac{a + ib}{1 - c},$$

$$\psi_- : V_- \to \mathbb{C}, \quad \psi_-(a, b, c) = \frac{a - ib}{1 + c}.$$

The inverse maps are obtained by computation as

$$\psi_+^{-1}(x, y) = \frac{(2x, 2y, x^2 + y^2 - 1)}{x^2 + y^2 + 1},$$

$$\psi_-^{-1}(x, y) = \frac{(2x, -2y, 1 - x^2 - y^2)}{x^2 + y^2 + 1}$$

Define a function $f : \mathbb{CP}^1 \to S^2$

$$f([z_0 : z_1]) = \frac{(2\mathrm{Re}(z_0\bar{z}_1), 2\mathrm{Im}(z_0\bar{z}_1), |z_0|^2 - |z_1|^2)}{|z_0|^2 + |z_1|^2}.$$

Then f is bijective with $f(U_+) = V_+$ and $f(U_-) = V_-$. We can check that the transtion maps $f_\pm = \psi_\pm \circ f \circ \phi_\pm^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ are the identity. This means the coordinate representations of any differential form of S^2 on $V_\pm$ are the same as the representations of its pullback on $U_\pm$.

Consider a 2-form on S^2 given by $\rho = a\ db \wedge dc + b\ dc \wedge da + c\ da \wedge db$. Since S^2 is a 2-manifold, ρ is closed. Moreover, we can compute the integral of ρ over S^2 via spherical parametrization (the computation was done as an example in class on April 10, 2013). Here we only give the result $\int_{S^2} \rho = -4\pi \neq 0$. Thus $[\rho]$ is a nonzero element in $H^2_{dR}(S^2)$. Its pullback $[\eta] = f^*[\rho]$ is also a nonzero element in $H^2_{dR}(\mathbb{CP}^1)$. This is the 2-form we are looking for. We will express it in coordinate charts. In the chart V_+,

$$(a, b, c) = \psi_+^{-1}(x, y) = \frac{(2x, 2y, x^2 + y^2 - 1)}{x^2 + y^2 + 1}$$

Thus,

$$\begin{aligned}
\rho &= \left[a\left(\frac{\partial b}{\partial x}\frac{\partial c}{\partial y} - \frac{\partial b}{\partial y}\frac{\partial c}{\partial x}\right) + b\left(\frac{\partial c}{\partial x}\frac{\partial a}{\partial y} - \frac{\partial c}{\partial y}\frac{\partial a}{\partial x}\right) + c\left(\frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}\right)\right] dx \wedge dy \\
&= -\frac{4}{(x^2 + y^2 + 1)^2} dx \wedge dy
\end{aligned}$$

Similarly, we also obtain the representation of ρ in V_-, which turns out to be of the same formula as in V_+. Therefore, the representations of η in the $U_\pm$ are

$$\eta = -\frac{4}{(x^2+y^2+1)^2} dx \wedge dy. \tag{1}$$

Step 2. $n \geq 2$

The idea is as follows. We will define a 2-form ω in $\mathbb{R}^{2n}$ and then pull it back to a 2-form in each U_k via the coordinate chart $\phi : U_k \to \mathbb{R}^{2n}$. Then we get $(n+1)$ different 2-forms in $U_0, U_1, \ldots, U_n$. To say that these are just the restrictions of a single 2-form in $\mathbb{CP}^n$, we have to make sure that they are consistent on the overlaps. Thus ω is supposed to be invariant under the transition maps $\phi_k \circ \phi_j^{-1}$. With this constraint in mind, we will define ω in $\mathbb{R}^{2n} = \{(x_1, y_1, \ldots, x_n, y_n) | x_j, y_j \in \mathbb{R}\}$ which is symmetric in sense that (x_j, y_j) and (x_k, y_k) can commute without changing ω. Then we say it suffices to check that ω is invariant under $\phi = \phi_1 \circ \phi_0^{-1}$. Put

$$S = 1 + x_1^2 + y_1^2 + .. + x_n^2 + y_n^2$$

We claim that

$$\begin{aligned} \omega \;\; = \;\; & \frac{1}{S^2} \left\{ \sum_{j=1}^{n} (S - x_j^2 - y_j^2) dx_j \wedge dy_j + \sum_{1 \leq j < l \leq n} [(x_l y_j - x_j y_l) dx_l \wedge dx_j - \right. \\ - \;\; & \left. (x_j x_l + y_j y_l) dx_j \wedge dy_l - (x_j x_l + y_j y_l) dx_l \wedge dy_j + (x_l y_j - x_j y_l) dy_l \wedge dy_j \;] \; \right\} \end{aligned} \tag{2}$$

is such a 2-form. Put $U = \{(w_1, .., w_n) \in \mathbb{C}^n : w_1 \neq 0\}$. Then $\phi_0(U_0 \cap U_1) = \phi_1(U_0 \cap U_1) = U$. Then $\phi : U \to U$ and

$$\phi(w_1, w_2, \ldots, w_n) = \left(\frac{1}{w_1}, \frac{w_2}{w_1}, .. \frac{w_n}{w_1} \right),$$

where $w_k = x_k + i y_k$. If we write $\phi(w_1, \ldots, w_n) = (w_1', \ldots, w_n')$ then

$$x_1' = \frac{x_1}{{x_1}^2 + y_1^2}, \quad y_1' = \frac{-y_1}{{x_1}^2 + y_1^2},$$

$$x_j' = \frac{x_j x_1 + y_j y_1}{{x_1}^2 + y_1^2}, \quad y_j' = \frac{y_j x_1 - x_j y_1}{{x_1}^2 + y_1^2} \quad \forall 2 \leq j \leq n$$

The pullback of ω in U by ϕ is thus

$$\begin{aligned} \phi^*\omega \;\; = \;\; & \frac{1}{S'^2} \left\{ \sum_{j=1}^{n} (S' - {x_j'}^2 - {y_j'}^2) dx_j' \wedge dy_j' + \sum_{1 \leq j < l \leq n} \left[(x_l' y_j' - x_j' y_l') dx_l' \wedge dx_j' - \right. \right. \\ - \;\; & \left. (x_j' x_l' + y_j' y_l') dx_j' \wedge dy_l' - (x_j' x_l' + y_j' y_l') dx_l' \wedge dy_j' + (x_l' y_j' - x_j' y_l') dy_l' \wedge dy_j' \;] \; \right\} \end{aligned} \tag{3}$$

where $S' = 1 + {x_1'}^2 + {y_1'}^2 + \ldots + {x_n'}^2 + {y_n'}^2$. We are going to show that $\phi^*\omega = \omega$.

Technically, we can use the chain rule to express $dx'_j,\ dy'_l$ in terms of $dx_k,\ dy_s$ and then use the antisymmetry property of exterior product to simplify the big expression. However, actual work shows that the calculation is too much and that the corresponding components of $\phi^*\omega$ and ω are equal is not easy to prove (since their formulae are too long!) Instead, we will use a casual manipulation as follows. With the identification of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ mentioned above, we define the notations

$$dw_k = dx_k + idy_k, \quad d\bar{w}_k = dx_k - idy_k.$$

We then assume that all laws of exterior derivatives (bilinearity, antisymmetry and the chain rule) work for this notation. For example, we allow ourselves to write

$$\begin{aligned} dw_k \wedge d\bar{w}_l &= (dx_k + idy_k) \wedge (dx_l - idy_l) \\ &= (dx_k \wedge dx_l) + i(dy_k \wedge dx_l) - i(dx_k \wedge dy_l) + (dy_k \wedge dy_l) \end{aligned}$$

This looks like we are pulling back a "differential form" in $\mathbb{C}^n$ to the one in $\mathbb{R}^{2n}$. Then Eq. (2) can be written as

$$-2i\omega = \sum_{j=1}^{n} \frac{S - |w_j|^2}{S^2} dw_j \wedge d\bar{w}_j - \sum_{1 \le l \ne j \le n} \frac{w_j \bar{w}_l}{S^2} dw_l \wedge d\bar{w}_j \tag{4}$$

The pullback version of Eq. (4) is

$$\begin{aligned} -2i\phi^*\omega &= \sum_{j=1}^{n} \frac{S' - |w'_j|^2}{S'^2} dw'_j \wedge d\bar{w}'_j - \sum_{1 \le l \ne j \le n} \frac{w'_j \bar{w}'_l}{S'^2} dw'_l \wedge d\bar{w}'_j && (5) \\ &= A + B - C - D - E, && (6) \end{aligned}$$

where

$$A = \frac{S' - |w'_1|^2}{S'^2} dw'_1 \wedge d\bar{w}'_1, \quad B = \sum_{j=2}^{n} \frac{S' - |w'_j|^2}{S'^2} dw'_j \wedge d\bar{w}'_j$$

$$C = \sum_{l=2}^{n} \frac{w'_1 \bar{w}'_l}{S'^2} dw'_l \wedge d\bar{w}'_1, \quad D = \sum_{j=2}^{n} \frac{w'_j \bar{w}'_1}{S'^2} dw'_1 \wedge d\bar{w}'_j, \quad E = \sum_{2 \le l \ne j \le n} \frac{w'_j \bar{w}'_l}{S'^2} dw'_l \wedge d\bar{w}'_j$$

We have

$$S' = 1 + |w'_1|^2 + .. + |w'_n|^2 = \frac{S}{|w_1|^2}, \quad dw'_1 = -\frac{1}{w_1^2} dw_1, \quad d\bar{w}'_1 = -\frac{1}{\bar{w}_1^2} d\bar{w}_1,$$

$$dw'_r = d\left(\frac{w_r}{w_1}\right) = \frac{1}{w_1} dw_r - \frac{w_r}{w_1^2} dw_1, \quad \forall 2 \le r \le n$$

$$d\bar{w}'_r = d\left(\frac{\bar{w}_r}{\bar{w}_1}\right) = \frac{1}{\bar{w}_1} d\bar{w}_r - \frac{\bar{w}_r}{\bar{w}_1^2} d\bar{w}_1, \quad \forall 2 \le r \le n$$

Using these identities, we obtain A, B, C, D, E in terms of $w_j,\ \bar{w}_l$ as follows.

$$A = \frac{1}{|w_1|^2} \frac{S - 1}{S^2} dw_1 \wedge d\bar{w}_1,$$

$$B=\sum_{j=2}^{n}\frac{S-|w_j|^2}{S^2}\left(dw_j\wedge d\bar{w}_j-\frac{w_j}{w_1}dw_1\wedge d\bar{w}_j-\frac{\bar{w}_j}{\bar{w}_1}dw_j\wedge d\bar{w}_1+\frac{|w_j|^2}{|w_1|^2}dw_1\wedge d\bar{w}_1\right),$$

$$C=\sum_{l=2}^{n}\frac{1}{S^2}\left(-\frac{\bar{w}_l}{\bar{w}_1}dw_l\wedge d\bar{w}_1+\frac{|w_l|^2}{|w_1|^2}dw_1\wedge d\bar{w}_1\right),$$

$$D=\sum_{j=2}^{n}\frac{1}{S^2}\left(-\frac{w_j}{w_1}dw_1\wedge d\bar{w}_j+\frac{|w_j|^2}{|w_1|^2}dw_1\wedge d\bar{w}_1\right),$$

$$E=\sum_{2\le l\ne j\le n}\frac{1}{S^2}\left(w_j\bar{w}_l dw_l\wedge d\bar{w}_j-\frac{|w_l|^2w_j}{|w_1|^2}dw_1\wedge d\bar{w}_j-\frac{|w_j|^2\bar{w}_l}{\bar{w}_1}dw_l\wedge d\bar{w}_1+\frac{|w_j|^2|w_l|^2}{|w_1|^2}dw_1\wedge d\bar{w}_1\right)$$

Now we check that $A+B-C-D-E$ equals $RHS(4)$ by comparing the coefficients of $dw_k\wedge d\bar{w}_s$.

Coefficients of $dw_1\wedge d\bar{w}_1$.
That of $RHS(4)$ is

$$\frac{S-|w_1|^2}{S^2}$$

That of $A+B-C-D-E$ is

$$\underbrace{\frac{1}{|w_1|^2}\frac{S-1}{S^2}}_{from\ A}+\underbrace{\sum_{j=2}^{n}\frac{S-|w_j|^2}{S^2}\frac{|w_j|^2}{|w_1|^2}}_{from\ B}-\underbrace{\sum_{l=2}^{n}\frac{1}{S^2}\frac{|w_l|^2}{|w_1|^2}}_{from\ C}-\underbrace{\sum_{j=2}^{n}\frac{1}{S^2}\frac{|w_j|^2}{|w_1|^2}}_{from\ D}-\underbrace{\sum_{2\le l\ne j\le n}\frac{|w_l|^2|w_j|^2}{S^2|w_1|^2}}_{from\ E}$$

$$=\frac{1}{|w_1|^2S^2}\left(S-1+\sum_{j=2}^{n}\left(S-|w_j|^2\right)|w_j|^2-2\sum_{j=2}^{n}|w_j|^2-\sum_{2\le l\ne j\le n}|w_l|^2|w_j|^2\right)$$

$$=\frac{1}{|w_1|^2S^2}\left(S+S(S-|w_1|^2-1)-\left(1+\sum_{j=2}^{n}|w_j|^2\right)^2\right)$$

$$=\frac{S-|w_1|^2}{S^2}$$

Thus the coefficients are equal.

Coefficients of $dw_1\wedge d\bar{w}_j$ where $j\ge 2$.
That of $RHS(4)$ is

$$-\frac{w_j\bar{w}_1}{S^2}$$

That of $A+B-C-D-E$ is

$$\underbrace{\frac{S-|w_j|^2}{S^2}\left(-\frac{w_j}{w_1}\right)}_{from\ B}-\underbrace{\left(-\frac{1}{S^2}\frac{w_j}{w_1}\right)}_{from\ D}-\underbrace{\left(-\sum_{\substack{l=2\\ l\ne j}}^{n}\frac{|w_l|^2w_j}{S^2w_1}\right)}_{from\ E}$$

$$= \frac{w_j}{S^2 w_1}\left(|w_j|^2 - S + 1 + \sum_{\substack{l=2\\ l\neq j}}^{n} |w_l|^2\right) = \frac{w_j}{S^2 w_1}\left(-S + 1 + \sum_{l=2}^{n} |w_l|^2\right)$$

$$= \frac{-w_j|w_1|^2}{S^2 w_1} = -\frac{w_j \bar{w}_1}{S^2}$$

Thus the coefficients are equal.

Coefficients of $dw_l \wedge d\bar{w}_1$ where $l \geq 2$.

That of $RHS(4)$ is

$$-\frac{w_1\bar{w}_l}{S^2}$$

That of $A + B - C - D - E$ is

$$\underbrace{\frac{S - |w_l|^2}{S^2}\left(-\frac{\bar{w}_l}{\bar{w}_1}\right)}_{from\ B} - \underbrace{\left(-\frac{1}{S^2}\frac{\bar{w}_l}{\bar{w}_1}\right)}_{from\ C} - \underbrace{\left(-\sum_{\substack{l=2\\ l\neq j}}^{n} \frac{|w_l|^2 w_j}{S^2 \bar{w}_1}\right)}_{from\ E}$$

$$= \frac{\bar{w}_l}{S^2 \bar{w}_1}\left(|w_l|^2 - S + 1 + \sum_{\substack{j=2\\ j\neq l}}^{n} |w_j|^2\right) = \frac{\bar{w}_l}{S^2 \bar{w}_1}\left(-S + 1 + \sum_{j=2}^{n} |w_j|^2\right)$$

$$= \frac{-\bar{w}_l|w_1|^2}{S^2 \bar{w}_1} = -\frac{\bar{w}_l w_1}{S^2}$$

Thus the coefficients are equal.

Coefficients of $dw_j \wedge d\bar{w}_j$ where $j \geq 2$.

That of $RHS(4)$ is $\frac{S-|w_j|^2}{S^2}$. That of $A + B - C - D - E$ is $\frac{S-|w_j|^2}{S^2}$ (from B). Thus the coefficients are equal.

Coefficients of $dw_l \wedge d\bar{w}_j$ where $j, l \geq 2$ and $j \neq l$.

That of $RHS(4)$ is $-\frac{w_j\bar{w}_l}{S^2}$. That of $A + B - C - D - E$ is $-\frac{w_j\bar{w}_l}{S^2}$ (from E). Thus the coefficients are equal.

Therefore we have proved that $A + B - C - D - E = RHS(4)$, i.e. $\phi^*\omega = \omega$. This means ω is invariant under the transition map $\phi_1 \circ \phi_0^{-1}$. Since the formula of ω at Eq. (2) is symmetric, ω is also invariant under any other transition map $\phi_k \circ \phi_j^{-1}$. Therefore, the pullbacks of ω to $U_0, U_1, \ldots, U_n$ give rise to a differential 2-form on $\mathbb{CP}^n$. We will still denote it by ω.

Next we will show that ω is closed. Again, we could compute $d\omega$ by Eq. (2), but the expression would be very complicated. Instead, we will compute $d\omega$ by Eq. (4). First we can separate the difference in the first sum of RHS(4) to get

$$-2i\omega = \frac{1}{S}\sum_{j=1}^{n} dw_j \wedge d\bar{w}_j - \sum_{j,l} \frac{w_j\bar{w}_l}{S^2} dw_l \wedge d\bar{w}_j \tag{7}$$

Now take the d both sides.

$$-2id\omega = -\frac{1}{S^2}\sum_{j=1}^{n} dS \wedge dw_j \wedge d\bar{w}_j + \frac{2}{S^3}\sum_{j,l} w_j\bar{w}_l dw_l \wedge d\bar{w}_j - \sum_{j,l} \frac{w_j d\bar{w}_l + \bar{w}_l dw_j}{S^2} dw_l \wedge d\bar{w}_j \tag{8}$$

We know that

$$dS = d\left(1 + \sum_{k=1}^{n} w_k\bar{w}_k\right) = \sum_{k=1}^{n} (w_k d\bar{w}_k + \bar{w}_k dw_k)$$

Then Eq. (8) becomes

$$-2id\omega = -\frac{1}{S^2}\underbrace{\sum_{j,k}(w_k d\bar{w}_k + \bar{w}_k dw_k) \wedge dw_j \wedge d\bar{w}_j}_{A}$$

$$+\frac{2}{S^3}\underbrace{\sum_{j,l.k}(w_j\bar{w}_l w_k d\bar{w}_k \wedge dw_l \wedge d\bar{w}_j + w_j\bar{w}_l\bar{w}_k dw_k \wedge dw_l \wedge d\bar{w}_j)}_{B}$$

$$-\frac{1}{S^2}\underbrace{\sum_{j,l}(w_j d\bar{w}_l \wedge dw_l \wedge d\bar{w}_j + \bar{w}_l dw_j \wedge dw_l \wedge d\bar{w}_j)}_{C}$$

We have

$$A = \sum_{j,k} w_k d\bar{w}_k \wedge dw_j \wedge d\bar{w}_j + \sum_{j,k} \bar{w}_k dw_k \wedge dw_j \wedge d\bar{w}_j$$

Renaming the indices (j,k) in the first sum by (l,j), and renaming the indices (j,k) in the second sum by (j,l), we get $A = -C$. Also,

$$B = \sum_{j,l,k} w_j\bar{w}_l w_k d\bar{w}_k \wedge dw_l \wedge d\bar{w}_j + \sum_{j,l,k} w_j\bar{w}_l\bar{w}_k dw_k \wedge dw_l \wedge d\bar{w}_j$$

The first sum is zero because when we interchange the indices (k,j), the sign of the sum switches due to antisymmetry of wedge product. The second sum is also zero because when we interchange the indices (k,l), the sign of the sum switches. Thus $B = 0$. Therefore,

$$-2id\omega = -\frac{1}{S^2}A + \frac{2}{S^3}B - \frac{1}{S^2}C = 0$$

Thus $d\omega = 0$, and ω is a closed form.

Next we will show that ω is not exact. To do so, we need to show that the integral of ω over some 2-cycle is nonzero. Recall that $\mathbb{CP}^1$ can be viewed as a submanifold of $\mathbb{CP}^n$ by the inclusion map

$$i : \mathbb{CP}^1 \to \mathbb{CP}^n, \quad [z_0 : z_1] \mapsto [z_0 : z_1 : 0 : \ldots : 0]$$

Thus $i(\mathbb{CP}^1)$ is a 2-cycle in $\mathbb{CP}^n$ and is contained in $U_0 \cup U_1$. We will show that the integral of ω over $i(\mathbb{CP}^1)$ is nonzero. On $U_0 \cup U_1$, we can define a projection map

$$proj : U_0 \cup U_1 \to \mathbb{CP}^1, \quad [z_0 : z_1 : \ldots : z_n] \mapsto [z_0 : z_1]$$

We have $proj \circ i = id_{\mathbb{CP}^1}$. Recall that we have a 2-form in $\mathbb{CP}^1$ given by Eq. (1). It can be written in the following form

$$-\frac{\eta}{4} = \frac{1}{(x^2 + y^2 + 1)^2} dx \wedge dy.$$

We will show that $\omega|_{i(\mathbb{CP}^1)}$ is the pullback of $-\eta/4$ by $proj$. The coordinates in U_0 are $(w_1, .., w_n) = (z_1/z_0, z_2/z_0, \ldots, z_n/z_0)$. In $U_0 \cap i(\mathbb{CP}^1)$, $(w_1, .., w_n) = (z_1/z_0, 0, \ldots, 0)$. Therefore, only the coordinates x_1 and y_1 in the formula of ω given by Eq. (2) can be nonzero. Thus,

$$\omega|_{U_0 \cap i(\mathbb{CP}^1)} = \frac{1}{(x_1^2 + y_1^2 + 1)^2} dx_1 \wedge dy_1$$

Similarly, the coordinates in U_1 are $(w_1, .., w_n) = (z_0/z_1, z_2/z_1, \ldots, z_n/z_1)$. In $U_1 \cap i(\mathbb{CP}^1)$, $(w_1, .., w_n) = (z_1/z_0, 0, \ldots, 0)$. Therefore, only the coordinates x_1 and y_1 in the formula of ω given by Eq. (2) can be nonzero. Thus,

$$\omega|_{U_1 \cap i(\mathbb{CP}^1)} = \frac{1}{(x_1^2 + y_1^2 + 1)^2} dx_1 \wedge dy_1$$

This means $\omega|_{i(\mathbb{CP}^1)}$ is the pullback of $-\eta/4$ by $proj$. Then

$$\int_{i(\mathbb{CP}^1)} \omega = \int_{\mathbb{CP}^1} -\frac{\eta}{4} = -\frac{1}{4} \int_{\mathbb{CP}^1} \eta = \pi \neq 0$$

Therefore, ω is a 2-form in $\mathbb{CP}^n$ that we were looking for.

Case for the Grassmannian. We remind the definition of the complex Grassmannian

$$Gr_{\mathbb{C}}(2,4) := \{< u, v >: u, v \in \mathbb{C}^4, \text{linearly independent}\}$$

Also we define

$$\wedge^2(\mathbb{C}^4) := \{\text{set of alternating 2-tensors on} \mathbb{C}^4\}$$

Now $\wedge^2(\mathbb{C}^4)$ is a vector space over $\mathbb{C}$ of dimension 6. If we choose a canonical base $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{C}$, then $\wedge^2(\mathbb{C}^4) =< e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 >$ and this can be identified with $\mathbb{C}^6$ where this ordered base maps to the canonical $\{(1,0,0,0,0,0), \cdots, (0,0,0,0,0,1)\}$

We will exhibit a smooth embedding

$$
\begin{array}{llll}
Gr_{\mathbb{C}}(2,4) \longrightarrow \wedge^2(\mathbb{C}^4) = \mathbb{C}^6 & & & \longrightarrow \mathbb{CP}^5 \\
< v_1, v_2 > \longrightarrow < v_1 \wedge v_2 > & = \displaystyle\sum_{1 \le i < j \le 4} (v_1^i v_2^j - v_1^j v_2^i) e_i \wedge e_j & & \longrightarrow [v_1^1 v_2^2 - v_1^2 v_2^1 : \cdots : v_1^3 v_2^4 - v_1^4 v_2^3]
\end{array}
$$

The function is well defined: If $< \lambda_1 v_1 + \lambda_2 v_2, \mu_1 v_1 + \mu_2 v_2 > = < v_1, v_2 >$ then $< (\lambda_1 v_1 + \lambda_2 v_2) \wedge (\mu_1 v_1 + \mu_2 v_2) > = < (\lambda_2 \mu_2 - \lambda_2 \mu_1) v_1 \wedge v_2 >$ which is the same (complex) subspace of $\mathbb{C}^6$ since $(\lambda_1 \mu_2 - \lambda_2 \mu_1) \neq 0$ and thus defines the same point in $\mathbb{CP}^5$. It is obviously smooth and it is easy to see that it is an embedding since the subspace $< v_1, v_2 >$ is exactly the set of vectors w such that $w \wedge (v_1 \wedge v_2) = 0$

It is well known that this embedding of $Gr_{\mathbb{C}}(2,4)$ can be given as the zero set of the following polynomial in $\mathbb{CP}^5$.

$$Gr_{\mathbb{C}}(2,4) \equiv \{[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] \in \mathbb{CP}^5 \mid z_1 z_4 - z_0 z_5 - z_2 z_3 = 0\}$$

where z_0 corresponds to $e_1 \wedge e_2$ etc. We notice here that although the polynomial is defined in $\mathbb{C}^6$ its zero set makes sense (is well defined) in $\mathbb{CP}^5$ because it is homogeneous.

Now $\mathbb{CP}^1$ is also embedded in $\mathbb{CP}^5$ in an obvious way, that is, projection to the first two homogeneous coordinates. If we express this $\mathbb{CP}^1$ as the zero set of the system of the following (again homogeneous) equations.

$$\mathbb{CP}^1 = \{[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] \in \mathbb{CP}^5 \mid z_2 = z_3 = z_4 = z_5 = 0\}$$

we observe that $z_2 = z_3 = z_4 = z_5 = 0$ implies $z_1 z_4 - z_0 z_5 - z_2 z_3 = 0$ which means that this way $\mathbb{CP}^1$ is embedded in $\mathbb{CP}^5$ as a subset (actually submanifold) of $Gr_{\mathbb{C}}(2,4)$.

We consider the inclusions $\mathbb{CP}^1 \overset{j}{\hookrightarrow} Gr_{\mathbb{C}}(2,4)$ and the inclusion $Gr_{\mathbb{C}}(2,4) \overset{i}{\hookrightarrow} \mathbb{CP}^5$ and their composition $i \circ j = \lambda$. It is important that λ is actually the projection on the first two variables which is exactly the inclusion that we used in the first part of the problem.

The next part of the proof proceeds as follows. We have a $2-$form ω on $\mathbb{CP}^5$. The pullback $i^*\omega$ is a $2-$form on $Gr_{\mathbb{C}}(2,4)$. We claim $i^*\omega$ is a generator of $H^2_{dR}(Gr_{\mathbb{C}}(2,4))$ (given that $H^2_{dR}(Gr_{\mathbb{C}}(2,4)) = \mathbb{R}$).

Indeed, since ω is closed $d\omega = 0 \Rightarrow i^*(d\omega) = 0 \Rightarrow d(i^*\omega) = 0 \Rightarrow i^*\omega$ is a closed $2-$form on $Gr_{\mathbb{C}}(2,4)$. It is only left to show that $i^*\omega$ is not exact.

Since $\mathbb{CP}^1$ is a $2-$cycle in $Gr_{\mathbb{C}}(2,4)$ it is enough to show that $\int_{\mathbb{CP}^1} i^*\omega \neq 0$ where the inclusion of $\mathbb{CP}^1$ is given by j. By the very definition of integrating a $k-$form defined on a greater-dimension manifold M over an immersed $k-$dimensional submanifold we have

$$\int_{\mathbb{CP}^1} \omega = \int_{\mathbb{CP}^1} \lambda^*\omega \tag{9}$$

and

$$\int_{\mathbb{CP}^1} i^*\omega = \int_{\mathbb{CP}^1} (j^* \circ i^*)\omega \tag{10}$$

We know though that $j^* \circ i^* = (i \circ j)^* = \lambda^*$, so from Eq. (9) and Eq. (10) we get

$$\int_{\mathbb{CP}^1} i^*\omega = \int_{\mathbb{CP}^1} \omega \neq 0 \text{ by the first part of the problem.}$$

Finally, we will describe how one can give an explicit expression for $i^*\omega$. Strictly speaking we need only describe i in local coordinates but we will do a bit more. The fact that we have expressed $Gr_{\mathbb{C}}(2,4)$ as a submanifold of $\mathbb{CP}^5$ is very useful here because we can consider local coordinates induced by the ambient space as opposed to the classical ones.

$\mathbb{CP}^5$ has 6 coordinate charts $\{(U_j, \phi_j) : j = 0, \cdots, 5\}$. Then $\{U_j \cap Gr_{\mathbb{C}}(2,4)\}_{j=0,\cdots,5}$ is an open cover of $Gr_{\mathbb{C}}(2,4)$. These will give us coordinate charts on $Gr_{\mathbb{C}}(2,4)$.

In U_0 the local coordinates of $\mathbb{CP}^5$ are

$$w_1 = \frac{z_1}{z_0}, w_2 = \frac{z_2}{z_0}, \cdots, w_5 = \frac{z_5}{z_0}$$

In $U_0 \cap Gr_{\mathbb{C}}(2,4)$ the extra relation $z_1 z_4 - z_0 z_5 - z_2 z_3 = 0$ holds. This in local coordinates means (divide by z_0^2).

$$w_1 w_4 - w_5 - w_2 w_3 = 0 \iff w_5 = w_1 w_4 - w_2 w_3$$

Then we have the diffeomorphism

$$\psi : \mathbb{C}^4 \to U_0 \cap Gr_C$$
$$(u_1, \cdots, u_4) \to (w_1, \cdots, w_5)$$

where $(w_1, w_2, w_3, w_4) = (u_1, u_2, u_3, u_4)$ and $w_5 = u_1 u_4 - u_2 u_3$. This diffeomorphism actually describes the inclusion j of the grassmannian in the complex projective space in local coordinates for $Gr_{\mathbb{C}}(2,4)$ induced by the ambient local coordinates of $\mathbb{CP}^5$. The inclusion map is similarly described in the other 5 charts. We will moreover describe the pullback $i^*\omega$ on $U_0 \cap Gr_{\mathbb{C}}(2,4)$.

By the first part of the problem, we have a 2-form on $\mathbb{CP}^5$ given in Eq. (7).

$$\omega = \frac{-1}{2i}\left[\frac{1}{S}\sum_{j=1}^{5} dw_j \wedge d\bar{w}_j - \sum_{1 \leq j,l \leq 5} \frac{w_j \bar{w}_l}{S^2} dw_l \wedge d\bar{w}_j\right]$$

Now $i^*\omega$ in $U_0 \cap Gr_{\mathbb{C}}(2,4)$ will be given by

$$\begin{aligned}
i^*\omega = \frac{-1}{2i}\Big[&\frac{1}{S}\sum_{j=1}^{4} du_j \wedge d\bar{u}_j + \frac{(u_4 du_1 + u_1 du_4 - u_3 du_2 - u_2 du_3) \wedge (\bar{u}_4 d\bar{u}_1 + \bar{u}_1 d\bar{u}_4 - \bar{u}_3 d\bar{u}_2 - \bar{u}_2 d\bar{u}_3)}{S} \\
&- \sum_{1 \leq j,l \leq 4} \frac{u_j \bar{u}_l}{S^2} du_l \wedge d\bar{u}_j - \sum_{1 \leq j \leq 4} \frac{u_j(\bar{u}_1 \bar{u}_4 - \bar{u}_2 \bar{u}_3)}{S^2}(u_4 du_1 + u_1 du_4 - u_3 du_2 - u_2 du_3) \wedge d\bar{u}_j \\
&- \sum_{1 \leq l \leq 4} \frac{(u_i u_4 - u_2 u_3)\bar{u}_l}{S^2} du_l \wedge (\bar{u}_4 d\bar{u}_1 + \bar{u}_1 d\bar{u}_4 - \bar{u}_3 d\bar{u}_2 - \bar{u}_2 d\bar{u}_3) \\
&- \frac{|u_1 u_4 - u_2 u_3|^2}{S^2}(u_4 du_1 + u_1 du_4 - u_3 du_2 - u_2 du_3) \wedge (\bar{u}_4 d\bar{u}_1 + \bar{u}_1 d\bar{u}_4 - \bar{u}_3 d\bar{u}_2 - \bar{u}_2 d\bar{u}_3)\Big]
\end{aligned}$$

Note that $S = 1 + |w_1|^2 + \ldots + |w_5|^2 = 1 + |u_1|^2 + |u_2|^2 + |u_3|^2 + |u_4|^2 + |u_1 u_4 - u_2 u_3|^2$.