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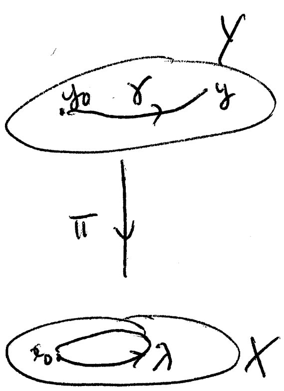
Math 8302: Topology & Manifolds
Take-home Final

(1) Let (X, x_0) be a connected smooth ~~m~~-manifold, and $\pi : (Y, y_0) \to (X, x_0)$ be its universal covering map. Assume there is a smooth structure on Y such that Y is a smooth m-manifold and π is a smooth covering map. The latter means: for each $p \in X$, there are an open connected neighborhood U of p, a discrete space F and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\pi"] \arrow[d, "\varphi"'] & U \\ U \times F \arrow[ur, "proj"'] & \end{tikzcd}$$

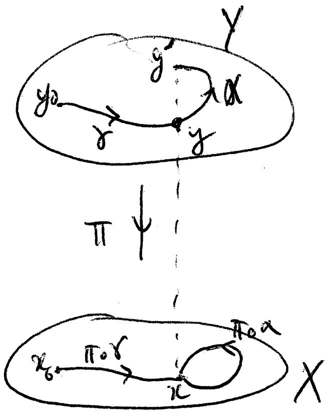
Because Y is a topological covering space of X, we knew that there is a right action of $\pi_1(X, x_0)$ on Y as follows.



For each $y \in Y$ and $[\lambda] \in \pi_1(X, x_0)$, let γ be a path from y_0 to y in Y. Then $\pi \circ \gamma$ is a path from x_0 to $\pi(y)$. Then $\lambda . (\pi \circ \gamma)$ is also a path from x_0 to $\pi(y)$. For each path α in X starting from x_0, we denote by $\tilde{\alpha}$ the lift of α at y_0. Then

$$y . [\lambda] := \widetilde{\lambda . (\pi \circ \gamma)}(1).$$

By this definition, the action is transitive on each fiber. Indeed, suppose we have $y, y' \in \pi^{-1}(x)$. Let α be a path in Y from y to y'.



Since $\pi(y) = \pi(y') = x$, ~~$\alpha = \pi\circ\alpha$ is~~ $\pi\circ\alpha$ is a loop based at x in X. Let γ be a path from y_0 to y. Then

$$\lambda = (\pi\circ\gamma)\cdot(\pi\circ\alpha)\cdot(\pi\circ\gamma)^{-1}$$

is a loop based at x_0 in X. Then by definition,

$$\begin{aligned} y\cdot[\lambda] &= \widetilde{\lambda\cdot(\pi\circ\gamma)}(1) = \widetilde{(\pi\circ\gamma)\cdot(\pi\circ\alpha)}(1) \\ &= \widetilde{\pi\circ(\gamma\cdot\alpha)}(1) \\ &= (\gamma\cdot\alpha)(1) \\ &= y'. \end{aligned}$$

This concludes that $\pi(X, x_0)$ acts transitively on Y. For each $[\lambda] \in \pi_1(X, x_0)$, the map $f_{[\lambda]}: Y \to Y$, $y \mapsto y\cdot[\lambda]$ is continuous. Moreover, the inverse of $f_{[\lambda]}$ is $f_{[\lambda]^{-1}}$, which is also continuous. Thus $f_{[\lambda]}$ is a homeomorphism. Assume that $f_{[\lambda]}$ is smooth for every $[\lambda] \in \pi_1(X, x_0)$. Then $(f_{[\lambda]})^{-1} = f_{[\lambda]^{-1}}$ is also smooth. Thus $f_{[\lambda]}$ is a diffeomorphism on Y. Moreover,

$$\pi\circ f_{[\lambda]}(y) = \pi\left(\widetilde{\lambda\cdot(\pi\circ\gamma)}(1)\right) = \lambda\cdot(\pi\circ\gamma)(1) = (\pi\circ\gamma)(1) = \pi(y).$$

Thus $\pi\circ f_{[\lambda]} = \pi$. $\qquad (*)$

So far, we know that $\pi_1(X, x_0)$ acts transitively on Y by diffeomorphisms and $\pi \circ f_{[\lambda]} = \pi$ for all $[\lambda] \in \pi_1(X, x_0)$. From now on, we will not refer to any other properties of $\pi_1(X, x_0)$. Thus, we should regard $\pi_1(X, x_0)$ as a group G acting transitively on Y by diffeomorphisms. Also, by switching the order of path composition (write $[\gamma] * [\eta]$ instead of $[\eta].[\gamma]$ for path composition), the right action becomes a left action. Each element $g \in G$ can be viewed as a diffeomorphism $g: Y \to Y$, $g(y) = g \cdot y$. Then $(*)$ implies $\pi \circ g = \pi$.

For each $n \geqslant 0$, let $\Omega^n(X)$ and $\Omega^n(Y)$ be the real vector spaces of differential forms on X and Y respectively. Since each $g: Y \to Y$ is a diffeomorphism, it induces a linear isomorphism $g^*: \Omega^n(Y) \to \Omega^n(X)$. Then a form $\eta \in \Omega^n(Y)$ is said to be G-invariant if $g^*(\eta) = \eta$ for all $g \in G$. Let $[\Omega^n(Y)]^G$ be the subset of $\Omega^n(Y)$ containing all G-invariant forms.

First, we'll show that $[\Omega^n(Y)]^G$ is a vector subspace of $\Omega^n(Y)$. Since $g^*(0) = 0$ for all $g \in G$, $0 \in [\Omega^n(Y)]^G$. Suppose we have $c \in \mathbb{R}$, $\eta_1, \eta_2 \in [\Omega^n(Y)]^G$. Then

$$g^*(\eta_1) = \eta_1 \, , \quad g^*(\eta_2) = \eta_2 \quad \forall g \in G.$$

Since g^* is a linear map, $g^*(\eta_1+\eta_2) = g^*(\eta_1)+g^*(\eta_2) = \eta_1+\eta_2$,
$$g^*(c\eta_1) = c\,g^*(\eta_1) = c\eta_1.$$
Thus, $\eta_1+\eta_2,\ c\eta_1 \in [\Omega^n(Y)]^G$. This means $[\Omega^n(Y)]^G$ is a vector subspace of $\Omega^n(Y)$.

Since $\pi: Y \to X$ is a smooth map, it induces a linear map $\pi^*: \Omega^n(\Omega) \to \Omega^n(Y)$. We want to show that π^* is injective and $\mathrm{Im}(\pi^*) = [\Omega^n(Y)]^G$.

<u>Show that π^* is injective</u>

Suppose that $\omega \in \Omega^n(X)$ satisfies $\pi^*(\omega) = 0$. Take $p \in X$ arbitrarily. We want to show that $\omega_p = 0$. Recall that $\omega_p \in \Lambda^n(T_pX)$ is an alternating ~~linear map~~ multilinear map from $(T_pX)^n$ to $\mathbb{R}$. Let $u_1, \dots, u_n \in T_pX$. We want to show $\omega_p(u_1, \dots, u_n) = 0$.

Take $q \in \pi^{-1}(p)$. For any $v_1, \dots, v_n \in T_qY$, we have
$$\pi^*(\omega)_q(v_1, \dots, v_n) = 0$$
Thus, $\omega_p(d\pi_q(v_1), \dots, d\pi_q(v_n)) = 0$, where $d\pi_q$ is the linear map between the tangent spaces $d\pi_q: T_q\cancel{X} \to T_pX$. It suffices to show that for each $i = 1, 2, \dots, n$, there is $v_i \in T_qY$ such that $u_i = d\pi_q(v_i)$. Since π is a smooth covering map, it is a local diffeomorphism by definition. Thus $d\pi_q$ is a linear isomorphism. Thus the existence

of v_i is always guaranteed.

Show that $Im(\pi^*) \subset [\Omega^n(Y)]^G$

For each $\omega \in \Omega^n(X)$, we put $\eta = \pi^*(\omega)$. We want to show that $\eta \in [\Omega^n(Y)]^G$. Take any $g \in G$, we want to show $g^*(\eta) = \eta$. We have

$$
\begin{aligned}
g^*(\eta) = g^*(\pi^*(\omega)) = (g^* \circ \pi^*)(\omega) = \\
&= (\pi \circ g)^*(\omega) \text{ (by the functorial property of pullbacks)} \\
&= \pi^*(\omega) \\
&= \eta .
\end{aligned}
$$

Show that $[\Omega^n(Y)]^G \subset Im(\pi^*)$

Take any $\eta \in [\Omega^n(Y)]^G$. We want to find $\omega \in \Omega^n(X)$ such that $\eta = \pi^*(\omega)$. We have

$$
\begin{aligned}
\eta = \pi^*(\omega) &\Leftrightarrow \forall q \in Y,\ \eta_q = \pi^*(\omega)_q \\
&\Leftrightarrow \forall q \in Y,\ \forall v_1, \dots, v_n \in T_qY,\ \eta_q(v_1, \dots, v_n) = \pi^*(\omega)_q(v_1, \dots, v_n) \\
&\Leftrightarrow \forall q \in Y,\ \forall v_1, \dots, v_n \in T_qY,\ \eta_q(v_1, \dots, v_n) = \omega_p(d\pi_q(v_1), \dots, d\pi_q(v_n)) \qquad (**)
\end{aligned}
$$

For each $p \in X$, we want to define $\omega_p \in \Lambda^n(T_pX)$ so that $(**)$ is satisfied. Pick any $q \in \pi^{-1}(p)$. For $u_1, \dots, u_n \in T_pX$, we define

$$\omega_p(u_1, \dots, u_n) := \eta_q((d\pi_q)^{-1}(u_1), \dots, (d\pi_q)^{-1}(u_n)) \qquad (1)$$

If the definition of ω_p does not depend on the choice of q in $\pi^{-1}(p)$

then $(**)$ is certainly satisfied.

Suppose $q, r \in \pi^{-1}(p)$. We'll show that for any $u_1, \dots, u_n \in T_pX$,

$$\eta_q\left((d\pi_q)^{-1}(u_1), \dots, (d\pi_q)^{-1}(u_n)\right) = \eta_r\left((d\pi_r)^{-1}(u_1), \dots, (d\pi_r)^{-1}(u_n)\right) \qquad (2)$$

Since q and r belong to the fiber of p and G acts on Y transitively in each fiber, there exists $g \in G$ such that $r = g(q)$. Since $\eta \in [\Omega^n(Y)]^G$, we have $\eta = g^*\eta$. Now evaluating both sides at q, we get

$$\eta_q(v_1, \dots, v_n) = \eta_{g(q)}\left(dg_q(v_1), \dots, dg_q(v_n)\right) \quad \forall v_1, \dots, v_n \in T_qY \qquad (3)$$

Choose $v_i = (d\pi_q)^{-1}(u_i)$. Then (3) becomes

$$\eta_q\left((d\pi_q)^{-1}(u_1), \dots, (d\pi_q)^{-1}(u_n)\right) = \eta_r\left(dg_q \circ (d\pi_q)^{-1}(u_1), \dots, dg_q \circ (d\pi_q)^{-1}(u_n)\right) \quad (4)$$

Since $\pi \circ g = \pi$, we have $d\pi_r \circ dg_q = d\pi_q$. Thus $dg_q \circ (d\pi_q)^{-1} = (d\pi_r)^{-1}$.

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} T_qY \arrow[rr, "dg_q"] \arrow[dr, "d\pi_q"'] & & T_rY \arrow[dl, "d\pi_r"] \\ & T_pX & \end{tikzcd}$$

Thus, (4) becomes

$$\eta_q\left((d\pi_q)^{-1}(u_1), \dots, (d\pi_q)^{-1}(u_n)\right) = \eta_r\left((d\pi_r)^{-1}(u_1), \dots, (d\pi_r)^{-1}(u_n)\right)$$

Therefore, (2) is proved. By (1), ω_p is alternating and multilinear because $d\eta_q$ is so. Thus $\omega_p \in \Lambda^n(T_pX)$.

we get a section $\omega : X \longrightarrow \coprod_{p \in X} \Lambda^n(T_pX)$. To show that $\omega \in \Omega^n(X)$, it remains to show that ω is smooth.

Take $p_0 \in X$ arbitrarily. We'll show that ω is smooth at p_0. Pick $q_0 \in \pi^{-1}(p_0)$. Since π is a local diffeomorphism, there are open neighborhoods V of q_0 in Y, and U of p_0 in X such that $\pi|_V : V \to U$ is a diffeomorphism. By shrinking U and V if necessary, we can assume that there are smooth charts (U, ϕ) and (V, ψ).

$$\begin{tikzcd}
V \subset Y \ni q_0 \arrow[r, "\pi"] \arrow[d, "\psi"'] & U \subset X \ni p_0 \arrow[d, "\phi"] \\
V' \subset \mathbb{R}^m \arrow[r, dashed, "\hat{\pi}"] & U' \subset \mathbb{R}^m \\
y^1, \dots, y^m & x^1, \dots, x^m
\end{tikzcd}$$

Let $(y^1, \dots, y^m)$ be the coordinates on V, and $(x^1, \dots, x^n)$ be the coordinates on U. Then the coordinate representation of π, namely $\hat{\pi} : V' \longrightarrow U'$, is a diffeomorphism. Thus, for each $i = 1, 2, \dots, m$, y^i is a smooth map of $x^1, \dots, x^m$.

$$y^i = h_i(x^1, \dots, x^m).$$

Then $\hat{\pi}^{-1}(x^1, \dots, x^m) = (h_1(x^1, \dots, x^m), \dots, h_m(x^1, \dots, x^m))$.

For each $p \in U$, we put $q = (\pi|_V)^{-1}(p)$. Then

$$(d\pi_q)_p^{-1} = \left(d_q(\pi|_V)\right)_q^{-1} = d(\pi|_V)^{-1}_p = \left(\frac{\partial h_j}{\partial x^i}\right)_{1\le i,j\le n}.$$

Thus,
$$(d\pi_q)^{-1}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial h_j}{\partial x^i}\frac{\partial}{\partial y^j} \qquad (5)$$

Since η is smooth on V, for each tuple $1\le j_1<j_2<\dots<j_n\le m$, there is a smooth function $\eta_{j_1\dots j_n}: V\to\mathbb{R}$ such that

$$\eta_q = \eta_{j_1\dots j_n}(q)\, dy^{j_1}\wedge\dots\wedge dy^{j_n}.$$

We have
$$\omega_p = \underbrace{\omega_p\left(\frac{\partial}{\partial x^{i_1}}\Big|_p, \dots, \frac{\partial}{\partial x^{i_n}}\Big|_p\right)}_{\omega_{i_1\dots i_n}(p)} dx^{i_1}\wedge\dots\wedge dx^{i_n}$$

We want to show that each $\omega_{i_1\dots i_n}: U\to\mathbb{R}$ is smooth.

In (1), we choose $u_1 = \frac{\partial}{\partial x^{i_1}}, \dots, \cancel{\frac{\partial}{\partial x^{i_n}}}\ u_n = \frac{\partial}{\partial x^{i_n}}$. Then (1) gives

$$\begin{aligned}
\omega_{i_1\dots i_n}(p) &= \eta_q\left((d\pi_q)^{-1}\left(\frac{\partial}{\partial x^{i_1}}\right)\Big|_q, \dots, (d\pi_q)^{-1}\left(\frac{\partial}{\partial x^{i_n}}\right)\Big|_q\right), \\
&\overset{(5)}{=} \eta_q\left(\frac{\partial h_{j_1}}{\partial x^{i_1}}\Big|_p \frac{\partial}{\partial y^{j_1}}\Big|_q, \dots, \frac{\partial h_{j_n}}{\partial x^{i_n}}\Big|_p \frac{\partial}{\partial y^{j_n}}\Big|_q\right) \\
&= \frac{\partial h_{j_1}}{\partial x^{i_1}}\Big|_p \cdots \frac{\partial h_{j_n}}{\partial x^{i_n}}\Big|_p \underbrace{\eta_q\left(\frac{\partial}{\partial y^{j_1}}\Big|_q, \dots, \frac{\partial}{\partial y^{j_n}}\Big|_q\right)}_{\eta_{j_1\dots j_n}(q)} \\
&= \frac{\partial h_{j_1}}{\partial x^{i_1}}\Big|_p \cdots \frac{\partial h_{j_n}}{\partial x^{i_n}}\Big|_p \eta_{j_1\dots j_n}\circ(\pi|_V)^{-1}(p).
\end{aligned}$$

Thus $\omega_{i_1\dots i_n}$ is smooth on U. Therefore ω is smooth and $\omega\in\Omega^n(X)$.

(2) Let M be a smooth n-manifold, and $\Omega^k(M)$ be the space of all differential k-forms on M. Put $\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$.

We know that the wedge product $\wedge$ is an operator on $\Omega(M)$ which satisfies the following rules.

(i) If $\omega_1 \in \Omega^p(M)$, $\omega_2 \in \Omega^q(M)$ then $\omega_1 \wedge \omega_2 \in \Omega^{p+q}(M)$.

(ii) Bilinearity:

$$(a\omega_1 + b\omega_2) \wedge \eta = a(\omega_1 \wedge \eta) + b(\omega_2 \wedge \eta),$$

$$\eta \wedge (a\omega_1 + b\omega_2) = a(\eta \wedge \omega_1) + b(\eta \wedge \omega_2),$$

$$\forall a, b \in \mathbb{R},\ \forall \omega_1, \omega_2 \in \Omega^p(M),\ \eta \in \Omega^q(M).$$

(iii) Associativity:

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi \quad \forall \omega \in \Omega^p(M), \eta \in \Omega^q(M), \xi \in \Omega^r(M).$$

For each $k \geq 0$, there is a linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$, which gives rise to a chain complex of real vector spaces.

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots$$

Also, d satisfies $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta) \quad \forall \omega \in \Omega^k(M), \eta \in \Omega^\ell(M)$

(*)

The de Rham cohomology groups were defined as

$$H^k_{dR}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

Put $H_{dR}(M) = \bigoplus_{k=0}^{\infty} H^k_{dR}(M)$.

To make $H_{dR}(M)$ a ring, we have to define a product law on it. First, we'll show that $\wedge : \Omega^p(M) \times \Omega^q(M) \longrightarrow \Omega^{p+q}(M)$ induces a map $\bar{\wedge} : H^p_{dR}(M) \times H^q_{dR}(M) \longrightarrow H^{p+q}_{dR}(M)$.

Let ω_1, ω_1' be closed p-forms, and ω_2, ω_2' be closed q-forms on M.

Suppose that $\omega_1' - \omega_1 = d\eta_1$ for some $\eta_1 \in \Omega^{p-1}(M)$, and

$\omega_2' - \omega_2 = d\eta_2$ for some $\eta_2 \in \Omega^{q-1}(M)$.

we'll show that $\omega_1' \wedge \omega_2' - \omega_1 \wedge \omega_2 = d\eta$ for some $\eta \in \Omega^{p+q-1}(M)$.

We have $\omega_1' = \omega_1 + d\eta_1$ and $\omega_2' = \omega_2 + d\eta_2$. Then

$$\omega_1' \wedge \omega_2' = (\omega_1 + d\eta_1) \wedge (\omega_2 + d\eta_2)$$

$$= (\omega_1 \wedge \omega_2) + (d\eta_1) \wedge \omega_2 + \omega_1 \wedge (d\eta_2) + (d\eta_1) \wedge (d\eta_2) \quad (**)$$

(bilinearity of wedge product)

We have

$$(d\eta_1) \wedge \omega_2 = d(\eta_1 \wedge \omega_2) - (-1)^p \eta_1 \wedge (d\omega_2) \quad (\text{by } (*))$$

$$= d(\eta_1 \wedge \omega_2), \quad (\text{because } d\omega_2 = 0)$$

$$\omega_1 \wedge (d\eta_2) = (-1)^p [\, d(\omega_1 \wedge \eta_2) - (d\omega_1) \wedge \eta_2 \,] \quad (\text{by } (*))$$

$$= (-1)^p d(\omega_1 \wedge \eta_2) \quad (\text{because } d\omega_1 = 0)$$

$$= d[(-1)^p \omega_1 \wedge \eta_2], \quad (d \text{ is linear})$$

$$(d\eta_1) \wedge (d\eta_2) = d(\eta_1 \wedge d\eta_2) - (-1)^{p-1}(\eta_1 \wedge d^2\eta_2) \quad (\text{by } (*))$$

$$= d(\eta_1 \wedge d\eta_2). \quad (\text{because } d^2\eta_2 = 0).$$

Therefore, (**) becomes

$$\omega_1' \wedge \omega_2' = \omega_1 \wedge \omega_2 + d(\eta_1 \wedge \omega_2) + d[(-1)^p \omega_1 \wedge \eta_2] + d(\eta_1 \wedge d\eta_2)$$
$$= \omega_1 \wedge \omega_2 + d\eta,$$

where $\eta = \eta_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \eta_2 + \eta_1 \wedge d\eta_2 \in \Omega^{p+q-1}(\Omega)$.

Note that $\omega_1 \wedge \omega_2$ and $\omega_1' \wedge \omega_2'$ are closed $(p+q)$-forms because

$$d(\omega_1 \wedge \omega_2) = \underbrace{(d\omega_1)}_{0} \wedge \omega_2 + (-1)^p \omega_1 \wedge \underbrace{(d\omega_2)}_{0} = 0,$$

$$d(\omega_1' \wedge \omega_2') = \underbrace{(d\omega_1')}_{0} \wedge \omega_2' + (-1)^p \omega_1' \wedge \underbrace{(d\omega_2')}_{0} = 0.$$

By what we have showed, $\wedge$ decends to a map

$$\bar{\wedge} : H^p_{dR}(M) \times H^q_{dR}(M) \longrightarrow H^{p+q}_{dR}(M)$$
$$[\omega_1] \,\bar{\wedge}\, [\omega_2] := [\omega_1 \wedge \omega_2].$$

Thus $\bar{\wedge}$ is an operator on $H_{dR}(M)$. To show that $(H_{dR}(M), +, \bar{\wedge})$ is a ring, we have to show that $\bar{\wedge}$ is assocative and distributive over addition.

<u>Associativity</u>

Let ω_1 be a p-closed form, ω_2 be a q-closed form, and ω_3 be an r-closed form. Then

$$([\omega_1] \,\bar{\wedge}\, [\omega_2]) \,\bar{\wedge}\, [\omega_3] = [\omega_1 \wedge \omega_2] \,\bar{\wedge}\, [\omega_3] = [(\omega_1 \wedge \omega_2) \wedge \omega_3]$$
$$= [\omega_1 \wedge (\omega_2 \wedge \omega_3)] \quad (\text{associativity in } \Omega(M))$$
$$= [\omega_1] \,\bar{\wedge}\, [\omega_2 \wedge \omega_3]$$
$$= [\omega_1] \,\bar{\wedge}\, ([\omega_2] \,\bar{\wedge}\, [\omega_3]).$$

Distributivity

Let ω_1, ω_2 be p-closed forms, and ω_3 be a ~~q-form~~ closed q-form.

Then
$$([\omega_1]+[\omega_2])\,\bar{\wedge}\,[\omega_3] = [\omega_1+\omega_2]\,\bar{\wedge}\,[\omega_3]$$
$$= [(\omega_1+\omega_2)\wedge\omega_3]$$
$$= [(\omega_1\wedge\omega_3)+(\omega_2\wedge\omega_3)] \quad (\text{distributivity in } \Omega(M))$$
$$= [\omega_1\wedge\omega_3] + [\omega_2\wedge\omega_3]$$
$$= [\omega_1]\,\bar{\wedge}\,[\omega_3] + [\omega_2]\,\bar{\wedge}\,[\omega_3].$$

$$[\omega_3]\,\bar{\wedge}\,([\omega_1]+[\omega_2]) = [\omega_3]\,\bar{\wedge}\,[\omega_1+\omega_2]$$
$$= [\omega_3\wedge(\omega_1+\omega_2)]$$
$$= [(\omega_3\wedge\omega_1)+(\omega_3\wedge\omega_2)] \quad (\text{distributivity in } \Omega(M))$$
$$= [\omega_3\wedge\omega_1] + [\omega_3\wedge\omega_2]$$
$$= [\omega_3]\,\bar{\wedge}\,[\omega_1] + [\omega_3]\,\bar{\wedge}\,[\omega_2].$$

Therefore, $(H_{dR}(M), +, \bar{\wedge})$ is a ring.

About the unitality

If $M=\emptyset$ then $H_{dR}(M)=\{0\}$. The unit element is 0.

If $M\neq\emptyset$ then we define $\mathbb{1}: M\to\mathbb{R}$ to be the constant map $\mathbb{1}(p)=1$ for all $p\in M$. Then $\mathbb{1}\in\Omega^0(M)$. Since $d(\mathbb{1})=0$, $[\mathbb{1}]\in H^0_{dR}(M)$ and thus $[\mathbb{1}]\in H_{dR}(M)$. For any $[\omega]\in H_{dR}(M)$, we have

$$[\mathbb{1}]\,\bar{\wedge}\,[\omega] = [\mathbb{1}\wedge\omega] = [\omega], \qquad [\omega]\,\bar{\wedge}\,[\mathbb{1}] = [\omega\wedge\mathbb{1}] = [\omega].$$

Therefore, $[\mathbb{1}]$ is the unit element of $H_{dR}(M)$.