Compact Operators on Hilbert Space

1 Definitions

Let H is a Hilbert space, and T be a bounded linear operator from H to H. We will denote by $B(H)$ the set of all bounded linear operators from H to H, or in brief, on H.

① T is called <u>compact</u> if the image of every bounded set of H under T is pre-compact. Equivalently,

$$T \text{ is compact} \iff \overline{T(B(0,1))} \text{ is compact.}$$

Observe that the identity map is compact iff H is finite-dimensional.

② T is called <u>self-adjoint</u> if

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x,y \in H.$$

③ H is called <u>separable</u> if it has a complete orthonormal sequence, sometimes called a "basis", i.e. there exists an orthonormal sequence $\{e_i\}$ such that

$$x = \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i \qquad \forall x \in H.$$

(2)

(4) A ~~space~~ vector space is called pre-Hilbert if it is an inner ~~pro~~ product space, but the metric induced by this inner product ~~mays~~ may not be complete.

$\boxed{2}$ Clever expression for the operator's norm

Let $T \in B(H)$ be self-adjoint. Then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

Proof Put

$$s = \sup_{\|x\|=1} |\langle Tx, x\rangle| .$$

For each x with ~~norm~~ unit norm, by Cauchy-Schwarz Inequality, we have $|\langle Tx, x\rangle| \leq \|Tx\| \, \|x\| = \|Tx\| \leq \|T\| \|x\| = \|T\|$

Thus, $s \leq \|T\|$. (1)

For each $x, y \in H$ with unit norms, we have

$$2\langle Ty, x\rangle = \langle Ty, x\rangle + \langle y, Tx\rangle = \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle$$

and thus

$$2|\langle Ty, x\rangle| \leq |\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle|$$

$$= \|x+y\|^2 \left|\left\langle T\left(\frac{x+y}{|x+y|}\right), \frac{x+y}{|x+y|}\right\rangle\right| + |x-y|^2 \left|\left\langle T\left(\frac{x-y}{|x-y|}\right), \frac{x-y}{|x-y|}\right\rangle\right|$$

$$\leq |x+y|^2 s + |x-y|^2 s = \left(|x+y|^2 + |x-y|^2\right) s$$

By Parallogram laws, $|x+y|^2 + |x-y|^2 = 2(|x|^2+|y|^2) = 2$.

Then $2|\langle Ty, x\rangle| \leq 2s$, or $|\langle Ty, x\rangle| \leq s$. Thus,

$$|Ty| = \sup_{|x|=1} \langle Ty, x\rangle \leq s \text{ , and}$$

$$|T| = \sup_{|y|=1} |Ty| \leq s. \qquad (2)$$

We conclude from (1) and (2) that $|T| = s$.

$\boxed{3}$ Spectral Theorem for self-adjoint compact Operators

Let $T \in B(H)$ be a self-adjoint compact operator. For each complex λ, let $X(\lambda)$ be the λ-eigenspace of T on H

$$X(\lambda) = \{x \in X : Tx = \lambda x\}$$

Then we have the following statements

1) Either $\|T\|$ or $-\|T\|$ is an eigenvalue of T,

2) All the eigenvalues are real,

3) The eigenspaces ~~are finite-dimensional~~ corresponding to non-zero eigenvalues are finite-dimensional.

4) The only possible accummulation point of the set of eigenvalues is 0, ~~and if H is infinite-dimensional, it is an accummulation~~

~~point~~. Consequently the set of eigenvalues is at most countable.

6) $H = \overline{\bigoplus X(\lambda)}$. Consequently, there is an orthonormal basis consisting of eigenvectors.

Proof: 1) By replacing T by $T/\|T\|$, we can assume $\|T\|=1$.

By Point $\boxed{2}$, $1 = \sup_{\|x\|=1} |\langle Tx, x\rangle|$. Thus, there is a sequence $\{x_n\}$ such that $|x_n| = 1$ and $|\langle Tx_n, x_n\rangle| \to 1$. Since H is reflexive, up to a subsequence, we can assume $x_n \rightharpoonup x_0 \in H$. Since T is compact, $Tx_n \to Tx_0$. Thus $|\langle Tx_n, x_n\rangle| \to |\langle Tx_0, x_0\rangle|$, and

$$|\langle Tx_0, x_0\rangle| = 1$$

Since $\langle Tx_0, x_0\rangle \in \mathbb{R}$, we get either $\langle Tx_0, x_0\rangle = 1$ or $\langle Tx_0, x_0\rangle = -1$.

Moreover, $1 = |\langle Tx_0, x_0\rangle| \leq |Tx_0|\,|x_0|$, and

$$|x_0| \leq \liminf_{n\to\infty} |x_n| = 1. \text{ Then}$$

$$1 \leq |Tx_0|\,|x_0| \leq |Tx_0| \leq \|T\|\,|x_0| \leq 1$$

thus we must have $|Tx_0| = |x_0| = 1$.

If $\langle Tx_0, x_0\rangle = 1$ then

$$|Tx_0 - x_0|^2 = \langle Tx_0 - x_0, Tx_0 - x_0\rangle = |Tx_0|^2 - 2\langle Tx_0, x_0\rangle + |x_0|^2$$
$$= 0$$

Then $Tx_0 = x_0$, and $\|T\| = 1$ is an eigenvalue of T.

If $\langle Tx_0, x_0\rangle = -1$ then

$$|Tx_0 + x_0|^2 = |Tx_0|^2 + 2\langle Tx_0, x_0\rangle + |x_0|^2 = 0.$$

Then $Tx_0 = -x_0$, and $-\|T\| = -1$ is an eigenvalue of T.

2) Let λ be an eigenvalue of T, i.e. there exists $x_0 \neq 0$ such that $Tx_0 = \lambda x_0$. Then

$$\langle Tx_0, x_0\rangle = \lambda \langle x_0, x_0\rangle, \text{ and } \quad \lambda = \frac{\langle Tx_0, x_0\rangle}{|x_0|^2} \in \mathbb{R}.$$

3) Let λ be a non-zero eigenvalue of T. We'll show that $X(\lambda)$ is finite-dimensional. Equivalently, we'll show that the closed unit ball of $(X(\lambda), \|.\|)$, called $\overline{B_\lambda(0,1)}$, is compact. Let (x_n) be a sequence in $\overline{B_\lambda(0,1)}$. Then $Tx_n = \lambda x_n$. Since (x_n) is bounded and T is compact, there exists a subsequence (Tx_{n_k}) of (Tx_n) that is convergent. Since

$$x_{n_k} = \lambda^{-1} T x_{n_k},$$

(x_{n_k}) is also convergent. This completes to proof.

4) Suppose that $\lambda_0 \neq 0$ is an accummulation point of the set

⑥

of eigenvalues of T. Then there exists a sequence of pairwise distinct eigenvalues (λ_n) such that $\lambda_n \to \lambda_0$. ~~By the previous part~~, we can assume that $\lambda_n \neq 0$ $\forall n$. By the previous results $X(\lambda_n)$ is finite dimensional and non-zero. Take $x_n \in X(\lambda_n)$ with unit norm.

$$T x_n = \lambda_n x_n$$

Since $\{x_n\}$ is bounded and T is compact, there exists a subsequence $\{x_{n_k}\}$ such that $\{T x_{n_k}\}$ is convergent. Since $x_{n_k} = \lambda_{n_k}^{-1} T x_{n_k}$, $\{x_{n_k}\}$ is also convergent to $x_0 \in H$. Then $T x_{n_k} \to T x_0$.

$$T x_{n_k} = \lambda_{n_k} x_{n_k}$$

Let $k \to \infty$, we get $T x_0 = \lambda_0 x_0$. For each $k \in \mathbb{N}$,

$$\langle T x_{n_k}, x_0 \rangle = \cancel{x_{n_k}, Tx} \; \lambda_{n_k} \langle x_{n_k}, x_0 \rangle$$

$$\langle x_{n_k}, T x_0 \rangle = \lambda_0 \langle x_{n_k}, x_0 \rangle$$

Since T is self-adjoint, $\lambda_{n_k} \langle x_{n_k}, x_0 \rangle = \lambda_0 \langle x_{n_k}, x_0 \rangle$. Since $\lambda_{n_k} \neq \lambda_0$, we get $\langle x_{n_k}, x_0 \rangle = 0$ $\forall k \in \mathbb{N}$.

Thus $\langle x_0, x_0 \rangle = \lim_{k \to \infty} \langle x_{n_k}, x_0 \rangle = 0$, or $|x_0| = 0$, or $x_0 = 0$

This is a contradiction.

Next, we'll show that the set S of all eigenvalues of T is at most countable. Suppose by contradiction that S is uncountably infinite. Then so is $S \setminus \{0\}$.

$$S\setminus\{0\} = \bigcup_{n=1}^{\infty}\left(\left[\tfrac{1}{n}, n\right]\cap S\right) \cup \bigcup_{n=1}^{\infty}\left(\left[-n, -\tfrac{1}{n}\right]\cap S\right)$$

Thus, there exists some $n_0 \in \mathbb{N}$ such that either

$$\left[\tfrac{1}{n_0}, n_0\right]\cap S \quad \text{or} \quad \left[-n_0, -\tfrac{1}{n_0}\right]\cap S$$

is ~~contains~~ infinite~~ly many elements of~~. Then S must have a non-zero accumulation point. This is a contradiction.

6) Put

$$M = \overline{\bigoplus X(\lambda)}$$

If $M \subsetneq H$ then $M^{\perp} \neq \{0\}$ and $H = M^{\perp} \oplus M$. We'll show that $T(M^{\perp}) \subset M^{\perp}$. Let $x_0 \in M^{\perp}\setminus\{0\}$ with unit norm, ~~and~~ and $y \in M$. There exists a sequence $y_n \in \bigoplus X(\lambda)$ such that $y_n \to y$. Then $Ty_n \to Ty$. We can write y_n in the form $y_n = \alpha_{i_1} x_{i_1} + \dots + \alpha_{i_n} x_{i_n}$ where x_j's are eigenvectors. Then

$$\begin{aligned} Ty_n &= \alpha_1 T x_1 + \dots + \alpha_{i_n} T x_{i_n} \\ &= \alpha_1 \lambda_1 x_1 + \dots + \alpha_{i_n}\lambda_{i_n} x_{i_n} \in \bigoplus X(\lambda) \end{aligned}$$

Thus, $Ty \in M$.

We have

$$\langle Tx_0, y\rangle = \langle x_0, Ty\rangle = 0$$

Thus, $Tx_0 \in M^\perp$. We define

$$T_1^*: M^\perp \to M^\perp$$
$$T_1(x) = T(x)$$

then T_1 is a compact, self-adjoint operator on Hilbert space $M^\perp$.

By 1), either $\|T_1\|$ or $-\|T_1\|$ is an eigenvalue of T_1. There exists $x_0' \in M^\perp \setminus \{0\}$ that is an eigenvector of T_1 (and thus T). then $x_0' \in M$ by the definition of M. This is a contradiction because $M \cap M^\perp = \{0\}$. In conclusion

$$H = \overline{\bigoplus X(\lambda)}.$$

Let $\{e_i^\lambda\}$ be an orthonormal basis of $X(\lambda)$. We'll show that

$$\mathcal{B} = \bigcup_\lambda \{e_i^\lambda\}$$

is an orthonormal basis of H. If λ_1 and λ_2 are two distinct eigenvalue and $e_1 \in X(\lambda_1)$, $e_2 \in X(\lambda_2)$ then

$$\langle e_1, Te_2\rangle = \langle e_1, \lambda_2 e_2\rangle = \lambda_2\langle e_1, e_2\rangle$$
$$\langle Te_1, e_2\rangle = \langle \lambda_1 e_1, e_2\rangle = \lambda_1\langle e_1, e_2\rangle$$

Thus $\lambda_2\langle e_1, e_2\rangle = \lambda_1\langle e_1, e_2\rangle$, and therefore $\langle e_1, e_2\rangle = 0$.

Hence $\mathcal{B}$ is orthogonal (and then orthonormal). For each $x \in H$ and $\varepsilon > 0$. There exists $x_i \in X(\lambda_i)$ $\forall i=\overline{1,k}$ such that

$$\left| x - \sum_{i=1}^{k} x_i \right| < \varepsilon$$

For each $i = 1, \dots, k$, there exists $\alpha_i^1, \dots, \alpha_i^{N_i} \in \mathbb{C}$ such that

$$\left| x_i - \sum_{j=1}^{N_i} \alpha_i^j e_j^{\lambda_i} \right| < \frac{\varepsilon}{k}$$

thus

$$\left| x - \sum_{i=1}^{k} \sum_{j=1}^{N_i} \alpha_i^j e_j^{\lambda_i} \right| < \varepsilon + \left(\frac{\varepsilon}{k} + \dots + \frac{\varepsilon}{k} \right) = 2\varepsilon$$

That means $\mathcal{B}$ is a basis of H, which consists of eigenvectors.

$\boxed{4}$ The space of compact operators

Let $K(H)$ be the space of all compact operators on H. Then we have the following statements

1) $K(H)$ is a closed subvector-space of $B(H)$,

2) The set of all finite-rank operators on H is dense in $K(H)$,

3) $K(H)$ is an ideal of the ring $(B(H), +, \circ)$.

Proof 1) It is easy to see that the sum of two compact operators, and the scalar multiplication between a ~~scal~~ complex and a compact operator are also compact operators. Hence $K(H)$

is a subspace of $B(H)$. Let (T_n) be a sequence in $K(H)$ that converges to $T \in B(H)$ in norm. We'll show that $T \in K(H)$.

Let (x_n) be a bounded sequence in H. Suppose by contradiction that (Tx_n) has no convergent subsequence. Then it has no Cauchy subsequence. There exists $\varepsilon > 0$ such that $B(Tx_m, \varepsilon)$ contains only finitely many elements of (Tx_n), $\forall m \in \mathbb{N}$. ~~Since~~ We have

$$\|T_n - T\| = \sup_{|x|=1} |T_n x - Tx| \longrightarrow 0 \quad \text{as } n \to \infty$$

There exists $n_0 \in \mathbb{N}$ such that $\|T_{n_0} - T\|_{B(H)} < \frac{\varepsilon}{3M}$ where M is an upper bound of (x_n). Then

$$|\langle T_{n_0} - T, x_m \rangle| = |x_m| \left| \left\langle T_{n_0} - T, \frac{x_m}{|x_m|} \right\rangle \right| \leq M \frac{\varepsilon}{3M} = \frac{\varepsilon}{3}$$

or $$|T_{n_0}(x_m) - T(x_m)| < \frac{\varepsilon}{3}$$

thus, $B\left(T_{n_0}(x_m), \frac{\varepsilon}{3}\right) \subset B(T(x_m), \varepsilon)$, and $B\left(T_{n_0}(x_m), \frac{\varepsilon}{3}\right)$ also contains finitely many elements of (Tx_m). Thus, $B\left(T_{n_0}(x_m), \frac{\varepsilon}{3}\right)$ contains only finitely many elements of $(T_{n_0} x_m)$. Then the sequence $(T_{n_0} x_m)$ has no convergence subsequente. This is a contradiction because T_{n_0} is compact.

2) Let T ~~be a~~ $\in K(H)$. We'll find a sequence of finite-rank operators converging to T in $B(H)$. Since $\overline{T(B(0,1))}$ is compact, it is pre-compact, i.e. for each $\varepsilon > 0$, there exists $y_1^\varepsilon, \dots, y_n^\varepsilon \in H$ such that $\overline{T(B(0,1))} \subset B(y_1, \varepsilon) \cup \dots \cup B(y_n, \varepsilon)$. Let K_ε be the space spaned by $y_1 \dots y_n$. Then

$$\overline{T(B(0,1))} \subset K_\varepsilon + B(0,\varepsilon)$$

~~We define~~ For each $x \in H$, we define $T_\varepsilon x$ as the projection of Tx on K_ε

$$T_\varepsilon x = \sum_{i=1}^{n} \langle Tx, y_i \rangle y_i$$

Then $T_\varepsilon \in B(H)$ and finite-rank. For each $x \in H$, $|x| = 1$, we have

$$Tx \in \overline{T(B(0,1))} \subset K_\varepsilon + B(0,\varepsilon)$$

there exists $y \in K_\varepsilon$ such that $|Tx - y| < \varepsilon$. Moreover,

$$|Tx - T_\varepsilon x| = \min_{z \in K_\varepsilon} |Tx - z| \leq |Tx - y| < \varepsilon$$

thus

$$\|T - T_\varepsilon\|_{B(H)} = \sup_{|x|=1} |Tx - T_\varepsilon x| \leq \varepsilon$$

therefore $T_\varepsilon \longrightarrow T$ in $B(H)$.

3) Let $T \in K(H)$ and $f \in B(H)$. First we'll show that $T \circ f \in K(H)$.

Let (x_n) be a bounded sequence in H. Since f is bounded, fx_n is also bounded. Since T is compact, $\{Tf(x_n)\}$ has a convergent subsequence. Therefore $T \circ f$ is compact.

Second, we'll show that $f \circ T$ is compact. Let (x_n) be a bounded sequence in H. Then (Tx_n) has a convergent subsequence

$$Tx_{n_k} \to y \in H$$

Since f is continuous, $fTx_{n_k} \to fy \in H$. Therefore $f \circ T$ is compact.

$\boxed{5}$ Representation of compact operators on separable Hilbert space

Let H be a separable Hilbert space and $T \in K(H)$. Then we have the following statements

1) H has an orthonormal basis $\{f_m\}_{m \in \mathbb{N}}$ consisting of eigenvectors of $L = T^*T$.

2) $$T = \sum_{m \in J \subset \mathbb{N}} \rho_m \langle \cdot, f_m \rangle g_m$$, where $\{g_m\}_{m \in J}$ is orthonormal and $\rho_m \geq 0$ $\forall m$

Proof 1) Since $T^* \in B(H)$, $L \in K(H)$. Moreover, $L^* = T^*T = L$, i.e. L is self-adjoint. By Statement 6, Point $\boxed{3}$, H has an orthonormal basis consisting of eigenvectors of L, called $\{e_i\}_{i \in I}$. We'll show that Φ_i is at most countable. Since H is separable, it has a countable orthonormal basis $\{u_n\}_{n\in\mathbb{N}}$. For each $n\in\mathbb{N}$, there exists complex numbers $\alpha_{1n}, \alpha_{2n}, \ldots$ such that

$$u_n = \sum_{j=1}^{\infty} \alpha_{jn} \cdot e_{i_{jn}}$$

Put $\mathcal{B} = \{e_{i_{jn}} : j\in\mathbb{N}, n\in\mathbb{N}\}$ Then $\mathcal{B}$ is countable and $\mathcal{B}\subset\{e_i\}_{i\in I}$ and $\overline{\langle\mathcal{B}\rangle} \ni u_n$ $\forall n\in\mathbb{N}$. Thus

$$\langle\{u_1,\ldots,u_n\}\rangle \subset \overline{\langle\mathcal{B}\rangle}$$

and hence $H = \overline{\langle\{u_1, u_2, \ldots\}\rangle} \subset \overline{\langle\mathcal{B}\rangle}$, and therefore $H = \overline{\langle\mathcal{B}\rangle}$. Thus $\{e_i\}_{i\in I} = \mathcal{B}$, which is a countable set. We ~~rename~~ denote $\mathcal{B} = \{f_m\}_{m\in\mathbb{N}}$.

2) Put $\rho_m = |Tf_m| \geq 0$. ~~We can write~~ ~~For each $f_m \geq 0$, we~~

$$\cancel{Tf_m = \rho_m g_m \quad \text{where } |g_m| = 1.}$$

~~put $g_m = \dfrac{Tf_m}{\rho_m}$. Then $|g_m| = 1$~~

Put $J = \{m\in\mathbb{N} : \rho_m > 0\}$. For each $m\in J$, we put

$$g_m = \frac{Tf_m}{\rho_m}. \quad \text{Then } |g_m| = 1.$$

For $m, n\in J$, $m\neq n$, $\langle Tf_n, Tf_m\rangle = \langle f_n, T^*Tf_m\rangle = \langle f_n, \lambda_m f_m\rangle = 0$

Thus $0 = \langle Tf_n, Tf_m \rangle = \langle s_n g_n, s_m g_m \rangle = s_n s_m \langle g_n, g_m \rangle$

Hence $\langle g_m, g_n \rangle = 0$, and $\{g_m\}_{m \in J}$ is an orthonormal set.

For each $x \in H$, since $\{f_m\}_{m \in \mathbb{N}}$ is a basis of H,

$$\cancel{x = \sum_{m \in \mathbb{N}} \langle f_n, f_m \rangle f_m} \quad x = \sum_{m=1}^{\infty} \langle x, f_m \rangle f_m.$$

Since T is continuous,

$$Tx = T\left(\sum_{m=1}^{\infty} \langle x, f_m \rangle f_m\right) = \sum_{m=1}^{\infty} \langle x, f_m \rangle Tf_m$$

$$= \sum_{m \in J} \langle x, f_m \rangle Tf_m = \sum_{m \in J} \langle x, f_m \rangle s_m g_m$$

Thus, $T = \sum_{m \in J} s_m \langle \cdot, f_m \rangle g_m$.

$\boxed{6}$ Some useful facts

1) Let $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ where X is Banach, reflexive be a ~~compact operator~~ linear continuous. Then T is compact iff T is continuous if viewed as a mapping from (X, τ_{X,X^*}) to $(Y, \|\cdot\|_Y)$.

2) Let $T : H \to H$ be a ~~self-adjoint~~ linear operator over the complex field. Then T is self-adjoint iff $\langle Tx, x \rangle \in \mathbb{R}$ $\forall x \in H$.

3) Let $T_n \in B(H)$, $T \in B(H)$ ~~an~~ such that $T_n \to T$ pointwise. Let $f \in K(H)$. then $T_n f \to Tf$ in $B(H)$.

4) Let $\{u_n\}$ be an orthonormal sequence ~~Subset~~ ~~basis~~ of H an $\{\alpha_n\}$ a bounded sequence of complex numbers. The operator A on H is defined as

$$Ax = \sum_{n=1}^{\infty} \alpha_n \langle x, u_n \rangle u_n$$

Then A is compact iff $\lim \alpha_n = 0$.

5) Let $\{u_n\}$ be an orthonormal sequence in H, and $T \in K(H)$.
Then $\lim_{n\to\infty} Tu_n = 0$.

Proof 1) The backward part: suppose that

$$T : (X, \tau_{X,X^*}) \to (Y, \|.\|_Y)$$

is continuous. Let (x_n) be a bounded sequence in X. Since X is reflexive, there is a convergent subsequence $\{x_{n_k}\}$ in weak topology

$$x_{n_k} \rightharpoonup x_0 \in X$$

Since T is continuous, $Tx_{n_k} \to Tx_0$.

The forward part: suppose that T is compact. Let (x_n) be a sequence in X such that $x_n \rightharpoonup x_0$. Then (x_n) is bounded.

Then there exists a ~~subs~~ convergent subsequence of (Tx_n), called $\{Tx_{n_k}\}$.

$$Tx_{n_k} \to y \in Y \qquad (3)$$

Since T is continuous from $(X, \|.\|_X)$ to $(Y, \|.\|_Y)$, it is also

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continuous from (X, τ_{X,X^*}) to (Y, τ_{Y,Y^*}). Indeed, put $z_k = x_{n_k}$, then $z_k \to x_0$ in (X, τ_{X,X^*}). The adjoint of T is $T^*: Y^* \to X^*$ which is also linear continuous. Thus, for each $y^* \in Y^*$, $T^* y^* \in X^*$ and hence

$$\langle T^* y^*, z_k \rangle \to \langle T^* y^*, x_0 \rangle$$

or

$$\langle y^*, T z_k \rangle \to \langle y^*, T x_0 \rangle$$

Thus, $T z_k \to T x_0$ in (Y, τ_{Y,Y^*}). By (3), $y = T x_0$. We get

$$T x_{n_k} \to T x_0 \quad \text{as} \quad k \to \infty$$

2) The forward: Suppose that T is self-adjoint. For each $x \in H$, we'll show that $\langle Tx, x \rangle \in \mathbb{R}$. We have

$$\langle Tx - x, Tx - x \rangle = |Tx|^2 + |x|^2 - \langle Tx, x \rangle - \langle x, Tx \rangle$$
$$= |Tx|^2 + |x|^2 - 2 \langle Tx, x \rangle$$

or

$$\langle Tx, x \rangle = \frac{1}{2} \left(|Tx|^2 + |x|^2 - |Tx - x|^2 \right) \in \mathbb{R}$$

The backward: Suppose that $\langle Tz, z \rangle \in \mathbb{R}, \forall z \in H$.

Let $x, y \in H$, we'll show that $\langle Tx, y \rangle = \langle x, Ty \rangle$. We have

$$\langle Tx - Ty, x - y \rangle = \langle Tx, x \rangle + \langle Ty, y \rangle - \langle Tx, y \rangle - \langle Ty, x \rangle$$

or

$$\langle Tx, y \rangle + \langle Ty, x \rangle = \langle Tx, x \rangle + \langle Ty, y \rangle - \langle T(x-y), x-y \rangle \in \mathbb{R}$$

or $\langle Tx, y\rangle + \overline{\langle x, Ty\rangle} \in \mathbb{R}$

Thus, $\operatorname{Im}\langle Tx, y\rangle = \operatorname{Im}\langle x, Ty\rangle$. By replacing x by ix, we get

$$\operatorname{Im} i\langle Tx, y\rangle = \operatorname{Im} i\langle x, Ty\rangle,$$

or $$\operatorname{Re}\langle Tx, y\rangle = \operatorname{Re}\langle x, Ty\rangle.$$ Therefore,

$$\langle Tx, y\rangle = \langle x, Ty\rangle.$$

3) It is sufficient to show that there is a subsequence $(T_{n_k} f)$ such that $T_{n_k} f \longrightarrow Tf$ in $B(H)$. Let $S = f(\partial B(0,1))$, then S is compact. For each $n \in \mathbb{N}$,

$$\|T_n f - Tf\|_{B(H)} = \sup_{|x|=1} |T_n f(x) - Tf(x)|$$

There exists a sequence $\{x_{mn}\}_m$ in $\partial B(0,1)$ such that

$$|T_n f(x_{mn}) - Tf(x_{mn})| \longrightarrow \|T_n f - Tf\|_{B(H)} \quad \text{as } m \to \infty$$

The sequence $\{f(x_{mn})\}_m$ is in the compact set S. Hence, it has a convergent subsequence to $y_n \in \cancel{H}\, S$. Then

$$\cancel{T_{n_k} y_{n_k}}\; |T_n y_n - Ty_n| \Longrightarrow \|T_n f - Tf\|_{B(H)} \qquad (4)$$

Since $\{y_n\}$ is in the compact set S, it has a convergent subsequence

$y_{n_k} \longrightarrow y_0 \in S.$

Then $$|T_{n_k} y_{n_k} - T y_{n_k}| \le |T_{n_k}(y_{n_k} - y_0)| + |T_{n_k} y_0 - T y_0| + |T(y_{n_k} - y_0)|$$

Since $T_{n_k} \to T$ pointwise, by Banach-Steinhaus Theorem, the sequence $\{T_{n_k}\}$ is bounded in $B(H)$. There exists ~~$M>0$ s.t.~~ $M<\infty$ such that $\|T_{n_k}\|, \|T\| \le M$ $\forall k \in \mathbb{N}$. Then

$$|T_{n_k} y_{n_k} - T y_{n_k}| \le M \underbrace{|y_{n_k} - y_0|}_{\downarrow\, 0} + \underbrace{|T_{n_k} y_0 - T y_0|}_{\downarrow\, 0} + M \underbrace{|y_{n_k} - y_0|}_{\downarrow\, 0}$$

Thus $|T_{n_k} y_{n_k} - T y_{n_k}| \to 0$ as $k \to \infty$.

By (4), $\|T_{n_k} f - Tf\|_{B(H)} \to 0$, or $T_{n_k} f \to Tf$ in $B(H)$.

4) First, we'll prove that A is well-defined, i.e. the series $\sum_{n=1}^{\infty} \alpha_n \langle x, u_n\rangle u_n$ is convergent. Since $\{\alpha_n\}$ is bounded, there exists $M<\infty$ such that $|\alpha_n| \le M$. $\forall n \in \mathbb{N}$. Then

$$|\cancel{\alpha_n \langle x}| \sum_{k=n}^{m} \alpha_k \langle x, u_k\rangle u_k \Big|^2 = \sum_{k=n}^{m} |\alpha_k|^2 |\langle x, u_k\rangle|^2 \le M^2 \sum_{k=n}^{m} |\langle x, u_k\rangle|^2$$

By Bessel's Inequality, the series
$$\sum_{k=1}^{\infty} |\langle x, u_k\rangle|^2 \le |x|^2 \quad \text{(and thus convergent)}$$

~~Thus~~ Given $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k=n}^{m} |\langle x, u_k\rangle|^2 < \frac{\varepsilon^2}{M^2} \quad \forall m, n > N(\varepsilon)$$

Then
$$\left|\sum_{k=n}^{m} \alpha_k \langle x, u_k\rangle u_k\right|^2 \leq M^2 \frac{\varepsilon^2}{M^2} = \varepsilon^2, \quad \text{or}$$

$$\left|\sum_{k=n}^{m} \alpha_k \langle x, u_k\rangle u_k\right| \leq \varepsilon \quad \forall m, n > N(\varepsilon)$$

Thus, the series $\sum_{k=1}^{\infty} \alpha_k \langle x, u_k\rangle u_k$ is convergent.

Second, we'll show that A is linear and continuous. Since A is well-defined, it's obvious that A is linear. By the first part, we know that

$$\left|\sum_{n=1}^{m} \alpha_n \langle x, u_n\rangle u_n\right|^2 \leq M^2 \sum_{n=1}^{m} |\langle x, u_n\rangle|^2 \leq M^2 |x|^2$$

or
$$\left|\sum_{n=1}^{m} \alpha_n \langle x, u_n\rangle u_n\right| \leq M|x|$$

Let $m \to \infty$, we get $|Ax| \leq M|x| \quad \forall x \in H$. Thus, A is continuous. Now comes the forward part. If A is compact, we have
$$Au_m = \sum_{n=1}^{\infty} \alpha_n \langle u_m, u_n\rangle u_n = \alpha_m.$$
Since $\{u_m\}_m$ is orthonormal, by ~~prior~~ statement 5 (which will be proven)

$$\lim_{m\to\infty} Au_m = 0.$$

The backward part: For each $m \in \mathbb{N}$, we define

$$A_m x = \sum_{n=1}^{m} \alpha_n \langle x, u_n \rangle u_n$$

Then A_m is linear, continuous and finite-rank. We'll show that

$A_m \to A$ in $B(H)$.

$$(A - A_m)x = \sum_{n=m+1}^{\infty} \alpha_n \langle x, u_n \rangle u_n$$

By the previous part, we have shown that

$$\| A - A_m \|_{B(H)} \leq \left(\sup_{n>m} |\alpha_n| \right) \longrightarrow 0 \text{ as } m \to \infty$$

Thus $A_m \to A$. Since A is a norm-limit of a sequence of finite-rank operators, A is compact.

5) It is sufficient to show that (Tu_n) has a subsequence convergent to 0. Since $\{u_n\}$ is bounded, it has a subsequence weakly convergent to $u_0 \in H$

$$u_{n_k} \rightharpoonup u_0$$

By Statement 1, T is continuous from (H, τ_{H,H^*}) to $(H, \|.\|)$. Then

$$Tu_{n_k} \to Tu_0 \qquad (5)$$

For each $y \in H$, $\langle Tu_{n_k}, y \rangle = \langle u_{n_k}, T^* y \rangle$. Thus

$$\sum_{k=1}^{\infty} \langle Tu_{n_k}, y \rangle u_{n_k} = \sum_{k=1}^{\infty} \langle u_{n_k}, T^* y \rangle u_{n_k} \text{ is convergent}$$

Hence $|\langle Tu_{m_k}, y\rangle u_{m_k}| \to 0$, or $\langle Tu_{m_k}, y\rangle \to 0$,

or $Tu_{m_k} \to 0$. By (5), $Tu_0 = 0$ and $Tu_{m_k} \to 0$.

7 Trace-class operators

Let H be a separable Hilbert space and T be a compact operator on H. By Point 5, H has a countable basis $\{f_m\}_{\mathbb{N}}$, and there exists an orthonormal set $\{g_m\}_{m\in J\subset\mathbb{N}}$ such that

$$Tx = \sum_{m\in J} \rho_m \langle x, f_m\rangle g_m$$

where $\rho_m = |Tf_m|$, $J = \{m\in\mathbb{N}: \rho_m > 0\}$. Then T is said to be a trace class if $\sum_{m=1}^{\infty} \rho_m < \infty$.

We have the following statements:

1) $\rho_m = \sqrt{\lambda_m}$ where λ_m is the eigenvalue of T^*T corresponding to eigenvector f_m.

2) If T is a trace class operator then the sum

$$\sum_{n=1}^{\infty} \langle \varphi_n, T\varphi_n\rangle$$

is independent of the choice of orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}}$ of H.

3) If T is non-negative self-adjoint, T is in trace class if there exists a basis $\{e_n\}_{n\in\mathbb{N}}$ of H such that $\sum_{n=1}^{\infty}\langle Te_n, e_n\rangle < \infty$.

(22)

Proof 1) We have $T^*T(f_m) = \lambda_m f_m$. Then

$$\langle T^*T(f_m), f_m\rangle = \langle \lambda_m f_m, f_m\rangle = \lambda_m$$

or $\langle Tf_m, Tf_m\rangle = \lambda_m$, or $|Tf_m|^2 = \lambda_m$, or $\rho_m = \sqrt{\lambda_m}$.

2) Since
$$T = \sum_{m\in J} \rho_m \langle \cdot, f_m\rangle g_m,$$

$$T\varphi_n = \sum_{m\in J} \rho_m \langle \varphi_n, f_m\rangle g_m \quad \text{and}$$

$$\langle \varphi_n, T\varphi_n\rangle = \sum_{m\in J} \rho_m \langle \varphi_n, f_m\rangle\langle \varphi_n, g_m\rangle$$

Thus
$$\sum_{n=1}^{\infty} \langle \varphi_n, T\varphi_n\rangle = \sum_{n=1}^{\infty}\sum_{m\in J} \rho_m \langle \varphi_n, f_m\rangle\langle \varphi_n, g_m\rangle \qquad (6)$$

We have

$$|\langle \varphi_n, f_m\rangle\langle \varphi_n, g_m\rangle| \leq \left|\frac{\langle \varphi_n, f_m\rangle + \langle \varphi_n, g_m\rangle}{2}\right|^2 + \left|\frac{\langle \varphi_n, f_m\rangle - \langle \varphi_n, g_m\rangle}{2}\right|^2$$

$$= \cancel{\tfrac{1}{4}}\, \left|\left\langle \varphi_n, \frac{f_m+g_m}{2}\right\rangle\right|^2 + \left|\left\langle \varphi_n, \frac{f_m-g_m}{2}\right\rangle\right|^2$$

$$\cancel{\leq \left|\frac{f_m+g_m}{2}\right|^2 + \left|\frac{f_m-g_m}{2}\right|^2}$$

$$\cancel{= (1/2)|f_m|^2 + (1/2)|g_m|^2 = 1}$$

Thus,
$$\sum_{n=1}^{\infty} |\rho_m \langle \varphi_n, f_m\rangle\langle \varphi_n, g_m\rangle| \leq$$

$$\leq s_m \left\{ \sum_{n=1}^{\infty} \left| \left\langle \varphi_n, \frac{f_m + g_m}{2} \right\rangle \right|^2 + \sum_{n=1}^{\infty} \left| \left\langle \varphi_n, \frac{f_m - g_m}{2} \right\rangle \right|^2 \right\}$$

$$\leq s_m \left(\left| \frac{f_m + g_m}{2} \right|^2 + \left| \frac{f_m - g_m}{2} \right|^2 \right) \leq 2 s_m$$

Hence

$$\sum_{m \in J} \sum_{n=1}^{\infty} \left| s_m \langle \varphi_n, f_m \rangle \langle \varphi_n, g_m \rangle \right| \leq 2 \sum_{m \in J} s_m < \infty$$

By Fubini's Theorem, we can convert the order of integration in (6)

$$\sum_{n=1}^{\infty} \langle \varphi_n, T\varphi_n \rangle = \sum_{m \in J} \sum_{n=1}^{\infty} s_m \langle \varphi_n, f_m \rangle \langle \varphi_n, g_m \rangle$$

$$= \sum_{m \in J} \sum_{n=1}^{\infty} \langle \varphi_n, f_m \rangle \langle \varphi_n, Tf_m \rangle$$

$$= \sum_{m=1}^{\infty} \left\langle \sum_{n=1}^{\infty} \langle \varphi_n, f_m \rangle \varphi_n , Tf_m \right\rangle$$

$$= \sum_{m=1}^{\infty} \langle f_m, Tf_m \rangle$$

Thus, the series $\sum_{n=1}^{\infty} \langle \varphi_n, T\varphi_n \rangle$ is independent of the choice of orthonormal basis (φ_n).

3) ~~First we'll show that~~ $Tf_n = s_n f_n$ ~~$\forall n \in \mathbb{N}$.~~

~~If $n \in \mathbb{N} \setminus J$ then~~ $s_n = |Tf_n| = 0$. ~~We only have to consider~~

~~$n \in I$, i.e. $f_n \geq 0$~~. Since T is self-adjoint, by Statement 6, Point 3, H has an orthonormal basis consisting of eigenvectors of T, called $\{u_n\}_{n\in\mathbb{N}}$. Let α_n be the corresponding eigenvalue

$$Tu_n = \alpha_n u_n$$

Then $$T^*Tu_n = T(Tu_n) = T(\alpha_n u_n) = \alpha_n Tu_n = \alpha_n^2 u_n$$

thus, α_n^2 is an eigenvalue of T^*T and u_n is the corresponding eigenvector of T^*T. Note that

$$\alpha_n = \langle Tu_n, u_n\rangle \geq 0$$

To show that T is a trace-class, we have to show

$$\sum_{n=1}^{\infty} \alpha_n < \infty$$

Since $\{e_m\}_{m\in\mathbb{N}}$ is also an orthonormal basis of H,

$$u_n = \sum_{m=1}^{\infty} \langle u_n, e_m\rangle e_m$$

Then $$Tu_n = \sum_{m=1}^{\infty} \langle u_n, e_m\rangle Te_m$$

and $$\alpha_n = \langle u_n, Tu_n\rangle = \sum_{m=1}^{\infty} \langle u_n, e_m\rangle \langle Te_m, u_n\rangle$$

$$= \sum_{m=1}^{\infty} \langle u_n, e_m\rangle \langle e_m, Tu_n\rangle$$

Put $a_{mn} = \langle u_n, e_m\rangle\langle e_m, Tu_n\rangle$. We see that

$$a_{mn} = \langle u_n, e_m\rangle \langle e_m, \alpha_n u_n\rangle = \alpha_n \langle u_n, e_m\rangle \overline{\langle e_n, e_m\rangle}$$

$$= \alpha_n |\langle u_n, e_m\rangle|^2 \geq 0$$

Then

$$\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn} \overset{\text{Fubini}}{=\!=\!=} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \langle u_n, e_m\rangle \langle e_m, T u_n\rangle$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \cancel{\langle e_m, Tu_n} \langle u_n, e_m\rangle \langle Te_m, u_n\rangle$$

$$= \sum_{m=1}^{\infty} \left\langle \sum_{n=1}^{\infty} \langle Te_m, u_n\rangle u_n, e_m \right\rangle$$

$$= \sum_{m=1}^{\infty} \langle Te_m, e_m\rangle < \infty$$

$\boxed{8}$ Norm on the space of trace class operators

Hereafter, the term "positive semidefinite" will refer to a property of an operator $T \in B(H)$ such that $\langle Tu, u\rangle \geq 0 \ \forall u \in H$.

Then we can rephrase the definition of trace class operator mentioned in Point $\boxed{7}$ as follow:

Let $T \in K(H)$. $|T|$ is by definition a bounded positive semidefinite operator such that $|T|^2 = T^*T$. Then T is called a trace class operator if $\sum_{k=1}^{\infty} \langle |T| e_k, e_k \rangle < \infty$

for some (thus any, by Point 7, Number 2) orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of H.

Now we put
$$\|T\|_1 = \sum_{k=1}^{\infty} \langle |T| e_k, e_k \rangle$$
$$B_1(H) = \{T \in K(H) : \|T\|_1 < \infty\}$$

Then we have the following statements:

1) Let $T \in B(H)$. Then T^*T is compact iff T is compact.

2) $$\|T\|_1 = \sup\left\{ \sum_{n=1}^{\infty} |\langle Tx_n, y_n \rangle| \,/\, (x_n), (y_n) \text{ are orthonormal bases of } H \right\}$$

3) $(B_1(H), \|.\|_1)$ is a norm space.

4) For each $T \in B_1(H)$, we define
$$Tr(T) = \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle$$
for any othonormal basis (e_k) of H. Then Tr is well-defined and linear from $B_1(H)$ to $\mathbb{C}$, and continuous on $(B_1(H), \|.\|_1)$.

5) $\|T\|_1 = \|T^*\|_1$ and $Tr(T) = \overline{Tr(T^*)}$. Consequently, $T \in B_1(H)$ iff $T^* \in B_1(H)$.

6) $B_1(H)$ is an ideal of $B(H)$. In particular, if $A \in B_1(H)$ and $B \in B(H)$ then $\|BA\|_1 \leq \|B\| \, \|A\|_1$ and $\|AB\|_1 \leq \|B\| \, \|A\|_1$

7) Let $A \in B_1(H)$, $B \in B(H)$ such that $A = A^*$. Then $tr(AB) = tr(BA)$.

8) Let $A \in B_1(H)$, $B \in B(H)$. Then $Tr(AB) = Tr(BA)$.

Proof

1) To show that T is compact, for each sequence (x_n) in $B(0,1)$, we show that (Tx_n) has a convergent subsequence. Since T^*T is compact, the sequnce $\{T^*Tx_n\}$ has a convergent subsequence. Up to a subsequence, we can assume that $\{T^*Tx_n\}$ is convergent, i.e.

$$T^*Tx_n \longrightarrow u \in H$$

For each $v \in H$, we have

$$\langle T^*Tx_n, v\rangle \longrightarrow \langle u, v\rangle$$

or
$$\langle x_n, T^*Tv\rangle \longrightarrow \langle u, v\rangle$$

Since $\{x_n\}$ is bounded and H is reflexive, $\{x_n\}$ is (up to a subsequence) weakly convergent to $x_0 \in H$. Thus

$$\langle x_n, T^*Tv\rangle \longrightarrow \langle x_0, T^*Tv\rangle$$

Thus, $\langle x_0, T^*Tv\rangle = \langle u, v\rangle$, or $\langle T^*Tx_0, v\rangle = \langle u, v\rangle$.

Since v is arbitrary, $T^*Tx_0 = u$. Then

$$T^*Tx_n \to T^*Tx_0, \quad \text{or} \quad T^*(Tx_n - Tx_0) \longrightarrow 0$$

We have

$$|Tx_n - Tx_0|^2 = \langle T^*(Tx_n - Tx_0), x_n - x_0 \rangle \leq \underbrace{|T^*(Tx_n - Tx_0)|}_{\to 0} \underbrace{|x_n - x_0|}_{\text{bounded}}$$

Thus $Tx_n \to Tx_0$.

2) Put

$$S = \left\{ \sum_{n=1}^{\infty} |\langle Tx_n, y_n \rangle| \,/\, (x_n), (y_n) \text{ are orthonormal bases of } H \right\}$$

Because T is a compact operator, there exists an orthonormal basis (f_n) of H and an orthonormal set $\{g_n\}_{n \in J}$ in H such that

$$T = \sum_{m \in J} \rho_m \langle \cdot, f_m \rangle g_m$$

where $\rho_m \geq 0$ (by Point 5, Number 2). Let $\{g_m\}_{m \in \mathbb{N}}$ be an orthonormal basis of H. We can put $\rho_m = 0$ $\forall m \in \mathbb{N} \setminus J$. Then

$$Tf_n = \rho_n g_n$$

Thus, $\langle Tf_n, g_n \rangle = \langle \rho_n g_n, g_n \rangle = \rho_n$. Hence

$$\|T\|_1 = \sum_{n=1}^{\infty} \rho_n = \sum_{n=1}^{\infty} |\langle Tf_n, g_n \rangle| \leq \sup S \qquad (7)$$

For each orthonormal basis (x_n) and (y_n) of H, we have

$$Tx_n = \sum_{n=1}^{\infty} \rho_m \langle x_n, f_m \rangle g_m$$

$$\langle Tx_n, y_n \rangle = \sum_{m=1}^{\infty} \rho_m \langle x_n, f_m \rangle \langle g_m, y_n \rangle$$

Thus,

$$|\langle Tx_n, y_n\rangle| \le \sum_{m=1}^{\infty} \rho_m |\langle x_n, f_m\rangle| |\langle g_m, y_n\rangle|$$

$$\le \frac{1}{2}\sum_{m=1}^{\infty} \rho_m \left(|\langle x_n, f_m\rangle|^2 + |\langle g_m, y_n\rangle|^2 \right)$$

Thus, $$\sum_{n=1}^{\infty} |\langle Tx_n, y_n\rangle| \le \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \rho_m \left(|\langle x_n, f_m\rangle|^2 + |\langle g_m, y_n\rangle|^2 \right)$$

$$= \frac{1}{2}\sum_{m=1}^{\infty} \rho_m \left(\sum_n |\langle x_n, f_m\rangle|^2 + \sum_n |\langle g_m, y_n\rangle|^2 \right)$$

$$= \frac{1}{2}\sum_{m=1}^{\infty} \rho_m \left(\|f_m\|^2 + \|g_m\|^2 \right)$$

$$= \sum_{m=1}^{\infty} \rho_m = \|T\|_1$$

Hence $\sup S \le \|T\|_1$ (8)

By (7) and (8) we get $\|T\|_1 = \sup S$.

3) First, we show that $B_1(H)$ is a vector space.

Because $B_1(H)$ is a subset of $K(H)$, which is a vector space, we only have to show that $B_1(H)$ is closed under scalar multiplication and addition. For each $c \in \mathbb{C}$ and $T \in B_1(H)$. By Number 2, we have

$$\|cT\|_1 = \sup\left\{ \sum_{n=1}^{\infty} |\langle cTx_n, y_n\rangle| \,/\, (x_n), (y_n) \text{- orthonormal bases} \right\}$$

$$= |c| \sup\left\{ \sum_{n=1}^{\infty} |\langle Tx_n, y_n\rangle| \,/\, (x_n), (y_n) \text{- orthonormal bases} \right\}$$

$$= |c|\, \|T\|_1 < \infty$$

Thus, $cT \in B_1(H)$. For each $T_1, T_2 \in B_1(H)$, by Number 2, we have

$$\|T_1+T_2\|_1 = \sup\left\{ \sum_{n=1}^{\infty} |\langle (T_1+T_2)x_n, y_n\rangle| \;/\; (x_n),(y_n)\text{-orthonormal bases} \right\}$$

$$\leq \sup\left\{ \sum_{n=1}^{\infty} |\langle T_1 x_n, y_n\rangle| + \sum_{n=1}^{\infty} |\langle T_2 x_n, y_n\rangle| \;/\; (x_n),(y_n)\text{- orthonormal bases} \right\}$$

$$\leq \|T_1\|_1 + \|T_2\|_1 < \infty$$

Thus, $T_1+T_2 \in B_1(H)$. To show that $\|\ \|_1$ is a norm on $B_1(H)$, we only need to show that if $\|T\|_1 = 0$ then $T \equiv 0$. We have

$$0 = \sup\left\{ \sum_{n=1}^{\infty} |\langle Tx_n, y_n\rangle| \;/\; (x_n),(y_n) \text{ are orthonormal bases} \right\}$$

$$\geq |\langle Tx_n, y_n\rangle| \quad \text{for every } x_n, y_n \in H \text{ with unit norms.}$$

Thus, $\langle Tx, y\rangle = 0 \;\; \forall x, y \in H,\ \|x\| = \|y\| = 1$. If $Tx \neq 0$, then we can choose $y = Tx/|Tx|$.

$$0 = \langle Tx, y\rangle = \left\langle Tx, \frac{Tx}{|Tx|} \right\rangle = |Tx|$$

which is a contradiction.

4) By Point 7, Number 2, to each trace class operator $T \in B_1(H)$, there corresponds a sum named $Tr(T) = \sum_{k=1}^{\infty} \langle Te_k, e_k\rangle$

that is independent of the choice of orthonormal basis (e_n) of H. Thus Tr is a well-defined map from $B_1(H)$ to $\mathbb{C}$. Moreover, Tr is linear because

$$Tr(cT_1+T_2) = \sum_k \langle (cT_1+T_2)e_k, e_k\rangle = \sum_k \left(c\langle T_1e_k, e_k\rangle + \langle T_2e_k, e_k\rangle\right)$$

$$= c\sum_k \langle T_1e_k, e_k\rangle + \sum_k \langle T_2e_k, e_k\rangle$$

$$= c\,Tr(T_1) + Tr(T_2)$$

By Number 2, $|Tr\,T| \leq \|T\|_1$. Thus Tr is continuous on $(B_1(H), \|.\|_1)$.

5) By Number 2, we have

$$\|T\|_1 = \sup\left\{ \sum_n |\langle Tx_n, y_n\rangle| \;/\; (x_n), (y_n) - \text{orthonormal bases}\right\}$$

$$= \sup\left\{ \sum_n |\langle x_n, T^*y_n\rangle| \;/\; (x_n), (y_n) - \text{orthonormal bases}\right\}$$

$$= \sup\left\{ \sum_n |\langle T^*y_n, x_n\rangle| \;/\; (x_n), (y_n) - \text{orthonormal bases}\right\}$$

$$= \|T^*\|_1$$

we have

$$Tr(T) = \sum \langle Te_k, e_k\rangle = \sum \langle e_k, T^*e_k\rangle = \sum \overline{\langle T^*e_k, e_k\rangle} = \overline{Tr(T^*)}.$$

6) Since $A \in B_1(H)$, there exists orthonormal bases $\{f_n\}, \{g_n\}$ and a nonnegative sequence $\{\rho_n\}$ such that

$$A = \sum_{m=1}^{\infty} \rho_m \langle \cdot, f_m\rangle g_m$$

and $\|A\|_1 = \sum_{m=1}^{\infty} \rho_m < \infty$

Then $BAx = B\left(\sum_m \rho_m \langle x, f_m\rangle g_m\right) = \sum_m \rho_m \langle x, f_m\rangle Bg_m$

For each orthonormal bases $(x_n), (y_n)$, we have

$$\langle BAx_n, y_n\rangle = \sum_m \rho_m \langle x_n, f_m\rangle \langle Bg_m, y_n\rangle$$

$$\cancel{= \sum_m \rho_m \langle Bg_m, \langle x_n, f_m\rangle y_n\rangle}$$

Put $z_m = \begin{cases} \|Bg_m\|^{-1}(Bg_m) & \text{if } Bg_m \neq 0 \\ 0 & \text{if } Bg_m = 0 \end{cases}$

We get $|z_m| \leq 1$ and

$$\langle BAx_n, y_n\rangle = \sum_m \rho_m \langle x_n, f_m\rangle \langle z_m, y_n\rangle |Bg_m|$$

Thus, $$|\langle BAx_n, y_n\rangle| \leq \sum_m \rho_m |\langle x_n, f_m\rangle| |\langle z_m, y_n\rangle| \, |Bg_m|$$

$$\leq \|B\| \sum_m \rho_m |\langle x_n, f_m\rangle| |\langle z_m, y_n\rangle|$$

$$\leq \frac{1}{2}\|B\| \sum_m \rho_m \left(|\langle x_n, f_m\rangle|^2 + |\langle z_m, y_n\rangle|^2\right)$$

Thus, $$\sum_n |\langle BAx_n, y_n\rangle| \leq \frac{1}{2}\|B\| \sum_n \sum_m \rho_m \left(|\langle x_n, f_m\rangle|^2 + |\langle z_m, y_n\rangle|^2\right)$$

$$= \frac{1}{2}\|B\| \sum_m \rho_m \sum_n \left(|\langle x_n, f_m\rangle|^2 + |\langle z_m, y_n\rangle|^2\right)$$

$$= \frac{1}{2} \|B\| \sum_m 2\rho_m = \|B\| \, \|A\|_1 .$$

Therefore, $\|BA\|_1 \leq \|B\| \, \|A\|_1$. By Number 5, we have

$$\|AB\|_1 = \|(AB)^*\|_1 = \|B^*A^*\|_1 \leq \|B^*\| \, \|A^*\|_1 = \|B\| \, \|A\|_1 .$$

bounded   trace class

7) By Number 6, Point 3, H has an orthonormal basis consisting of eigenvectors of A, called $(u_i)_{i\in\mathbb{N}}$. We have

$$Au_i = \lambda_i u_i , \text{ where } \lambda_i \in \mathbb{R},$$

and thus

$$Tr(BA) = \sum_i \langle BAu_i, u_i \rangle = \sum_i \langle \lambda_i Bu_i, u_i \rangle = \sum_i \lambda_i \langle Bu_i, u_i \rangle$$

$$Tr(AB) = \sum_i \langle ABu_i, u_i \rangle = \sum_i \langle Bu_i, A^*u_i \rangle = \sum_i \langle Bu_i, Au_i \rangle$$

$$= \sum_i \langle Bu_i, \lambda_i u_i \rangle = \sum_i \lambda_i \langle Bu_i, u_i \rangle$$

Therefore, $Tr(BA) = Tr(AB)$.

8) Now the condition $A = A^*$ is obmitted. But we still have to show that $Tr(AB) = Tr(BA)$. Put

$$A_s = \frac{A+A^*}{2}, \qquad A_a = \frac{A-A^*}{2},$$

$$B_s = \frac{B+B^*}{2}, \qquad B_a = \frac{B-B^*}{2}.$$

Then A_s, B_s are Hermitian, i.e. $A_s^* = A_s$, $B_s^* = B_s$, and while A_a, B_a are anti-Hermitian, i.e. $A_a^* = -A_a$, $B_a^* = -B_a$. We have

$$A = A_s + A_a, \quad B = B_s + B_a$$

and

$$AB = A_s B_s + A_s B_a + A_a B_s + A_a B_a$$

$$BA = B_s A_s + B_s A_a + B_a A_s + B_a A_a$$

By Number 5, $A_s, A_a \in B_1(H)$. By Number 4,

$$Tr(AB) = Tr(A_s B_s) + Tr(A_s B_a) + Tr(A_a B_s) + Tr(A_a B_a)$$

$$Tr(BA) = Tr(B_s A_s) + Tr(B_s A_a) + Tr(B_a A_s) + Tr(B_a A_a)$$

By Number 7, we have

$$Tr(A_s B_s) = Tr(B_s A_s)$$

$$Tr(A_s B_a) = Tr(B_a A_s)$$

We only have to show that $Tr(A_a B_s) = Tr(B_s A_a)$ and $Tr(A_a B_a) = Tr(B_a A_a)$. Put $\tilde{A}_a = i A_a$ where i is the immagmary unit, we have

$$\tilde{A}_a^* = \bar{i} A_a^* = (-i)(-A_a) = i A_a = \tilde{A}_a$$

Thus, $\tilde{A}_a$ is Hermitian. Then By Number 7, we have

$$Tr(\tilde{A}_a B_s) = Tr(B_s \tilde{A}_a)$$

or $$Tr(iA_a B_s) = Tr(B_s(iA_a))$$

or $$iTr(A_a B_s) = iTr(B_s A_a)\,,\quad \text{or}\quad Tr(A_a B_s) = Tr(B_s A_a).$$

We have, similarly,

$$Tr(\tilde{A}_a B_a) = Tr(B_a \tilde{A}_a)$$

or $$Tr(iA_a B_a) = Tr(B_a(iA_a))$$

or $$iTr(A_a B_a) = iTr(B_a A_a)$$

or $$Tr(A_a B_a) = Tr(B_a A_a).$$