Wedge Product

I Definition, universal property and basis

1 Definition of n-fold wedge product

Let V be a vector space and $(V\otimes\cdots\otimes V, \phi)$ be an n-fold tensor product. Here ϕ is an n-linear mapping from $V\times\cdots\times V$ to $V\otimes\cdots\otimes V$. On $V\otimes\cdots\otimes V$, we define a vector space S spanned by all elements of the form $v_1\otimes\cdots\otimes v_n$ where each $v_i\in V$ and $v_i = v_j$ for some $i\neq j$

$$S = \langle \{ v_1\otimes\cdots\otimes v_n \,/\, v_1,\ldots,v_n\in V,\ v_i = v_j \text{ for some } i\neq j \} \rangle$$

Then we have the quotient vector space $V\otimes\cdots\otimes V/S$, which is denoted by $V\wedge V\wedge\cdots\wedge V$. There is a natural projection

$$\begin{aligned} \pi:\ V\otimes\cdots\otimes V &\longrightarrow V\wedge\cdots\wedge V \\ f &\longmapsto f+S \end{aligned}$$

Put $\psi = \pi\phi$.

$$\begin{tikzcd} V\times\cdots\times V \arrow[r, "\phi"] \arrow[dr, dashed, "\psi"'] & V\otimes\cdots\otimes V \arrow[d, "\pi"] \\ & V\wedge\cdots\wedge V \end{tikzcd}$$

Then the pair $(V\wedge\cdots\wedge V, \psi)$ is called n-fold wedge product of V. Hereafter, we will the notation

$$v_1 \wedge \cdots \wedge v_n := v_1 \otimes \cdots \otimes v_n + S$$

2 Universal property

(a) Definition of an alternating map

A map $f: V\times\cdots\times V \longrightarrow \mathbb{C}$ is called alternating if

$$f(v_1,\dots,v_n) = (-1)^{\sigma} f(R^{\sigma}(v_1,\dots,v_n))$$

for all $(v_1,\dots,v_n) \in V\times\cdots\times V$, for all permutation $\sigma \in S_n$, where $R^{\sigma}(v_1,\dots,v_n) = (v_{\sigma(1)},\dots,v_{\sigma(n)})$. R^{σ} is called a rotation on $V\times\cdots\times V$.

(b) ψ is alternating multilinear

Proof We know that ϕ is multilinear and π is linear. Thus, $\psi = \pi\phi$ is multilinear. To show that ψ is alternating, it is sufficient to show that $\psi(\dots v_i \dots v_j \dots) = -\psi(\dots v_j \dots v_i \dots)$. Here the dots mean all entries (arguments) at positions except i and j are fixed; only the entries at i and j are exchanged. Because, ψ is multilinear, we have

$$\psi(\dots v_i \dots v_j \dots) + \psi(\dots v_j \dots v_i \dots)$$

$$= \psi(\dots v_i+v_j \dots v_i+v_j \dots) - \psi(\dots v_i \dots v_i \dots) - \psi(\dots v_j \dots v_j \dots)$$

$$= \pi(\underbrace{\dots \otimes(v_i+v_j)\otimes \dots \otimes (v_i+v_j)\otimes \dots}_{\in S}) - \pi(\underbrace{\dots \otimes v_i \otimes \dots \otimes v_i \otimes \dots}_{\in S})$$

$$- \pi(\underbrace{\dots \otimes v_j \otimes \dots \otimes v_j \otimes \dots}_{\in S})$$

$$= 0$$

Thus, $\psi(\dots v_i \dots v_j \dots) = -\psi(\dots v_j \dots v_i \dots)$

(c) For each alternating multilinear map $h: V\times\dots\times V \to W$, there exists uniquely a linear map $\tilde{h}: V\wedge\dots\wedge V \to W$ such that the following diagram is commutative

$$\begin{tikzcd} V\times\dots\times V \arrow[r, "\psi"] \arrow[d, "h"'] & (V\otimes\dots\otimes V)/S \arrow[dl, dashed, "\tilde{h}"] \\ W & \end{tikzcd}$$

Proof Since h is multilinear, by the universal property of tensor product, there exists a unique linear map $\tilde{g}: V\otimes\dots\otimes V \to W$ such that $h = \tilde{g}\phi$. We'll show that if $u-v \in S$ for some $u, v \in V\otimes\dots\otimes V$ then $\tilde{g}(u) = \tilde{g}(v)$. Indeed, we can write

$$u - v = \sum_i \alpha_i(\dots \otimes v_i \otimes \dots \otimes v_i \otimes \dots)$$

④

$$= \sum \alpha_i \phi(\cdots v_i \cdots v_i \cdots)$$

Then $\tilde{g}(u-v) = \tilde{g}\left(\sum_i \alpha_i \phi(\cdots v_i \cdots v_i \cdots)\right) = \sum_i \alpha_i \tilde{g}\phi(\cdots v_i \cdots v_i \cdots)$

$$= \sum \alpha_i h(\cdots v_i \cdots v_i \cdots)$$

$$= 0 \quad \text{because } h \text{ is alternating.}$$

Therefore, we can define the map $\cancel{g}\tilde{h}$ $V\otimes\cdots\otimes V/S \longrightarrow W$

$$u+S \longmapsto \tilde{g}(u)$$

Then $\cancel{\tilde{g}}\tilde{h}$ is linear and $h = \cancel{g}\tilde{h}\pi\phi = \tilde{h}\psi$.

$$\begin{tikzcd}
V\times\cdots\times V \arrow[r, "\phi"] \arrow[d, "h"'] & V\otimes\cdots\otimes V \arrow[r, "\pi"] \arrow[dl, dashed, "\tilde{g}"] & V\otimes\cdots\otimes V/S \arrow[dll, dashed, "\tilde{h}"] \\
W & &
\end{tikzcd}$$

Suppose that $h_1 : V\otimes\cdots\otimes V/S \to W$ is another linear map such that $h = h_1\psi$. Then $\cancel{h_1}$ $g_1 = h_1\pi$ satisfies $h = g_1\phi$. Thus, $g_1 = \tilde{g}$, i.e. $h_1\pi = \tilde{g}$. For each $u \in V\otimes\cdots\otimes V$, we have

$$\tilde{h}(u+S) = \tilde{g}(u) = h_1\pi(u) = h_1(u+S)$$

Hence $\tilde{h} = h_1$.

③ A basis of $V\wedge\cdots\wedge V$ is $\{e_{i_1}\wedge\cdots\wedge e_{i_n} \mid i_1 < i_2 < \cdots < i_n\}$ where $\{e_1, e_2, \dots\}$ is a basis of V.

Proof Put $\mathcal{B} = \{e_{i_1} \wedge \cdots \wedge e_{i_n} \,/\, i_1 < i_2 < \cdots < i_n\}$. First we show that $\mathcal{B}$ can generate $V \wedge \cdots \wedge V$. An element of $V \wedge \cdots \wedge V$ can be written as the form

$$\tilde{v} = \sum_j \alpha_j e_{j_1} \otimes \cdots \otimes e_{j_n} + S \qquad (1)$$

We have $\cdots \otimes e_i \otimes \cdots \otimes e_j \otimes \cdots + \cdots \otimes e_j \otimes \cdots \otimes e_i \otimes \cdots$

$$= \cdots \otimes (e_i + e_j) \otimes \cdots \otimes (e_i + e_j) \otimes \cdots - \cdots \otimes e_i \otimes \cdots \otimes e_i \otimes \cdots$$
$$- \cdots \otimes e_j \otimes \cdots \otimes e_j \otimes \cdots \in S$$

~~thus,~~ $\cancel{\cdots \otimes e_i \otimes \cdots \otimes e_j \otimes \cdots = - \cdots \otimes e_j \otimes \cdots \otimes e_i \otimes \cdots + S}$

More generally, $e_{j_1} \otimes \cdots \otimes e_{j_n} - e_{\sigma(j_1)} \otimes \cdots \otimes e_{\sigma(j_n)} (-1)^\sigma \in S$ for every permutation σ. Thus, we can assume at (1) that $j_1 < j_2 < \cdots < j_n$. Then

$$\tilde{v} = \sum_j \alpha_j \big(\underbrace{e_{j_1} \otimes \cdots \otimes e_{j_n} + S}_{\in \mathcal{B}} \big)$$

Hence, $\mathcal{B}$ can generate $V \wedge \cdots \wedge V$. Next, we show that $\mathcal{B}$ is linearly independent. Suppose that

$$\sum_j \alpha_j e_{j_1} \otimes \cdots \otimes e_{j_n} + S = 0$$

Then $$0 = \sum_j \alpha_j (e_{j_1} \otimes \cdots \otimes e_{j_n} + S) = \sum_j \alpha_j \pi (e_{j_1} \otimes \cdots \otimes e_{j_n})$$

$$= \sum \alpha_j \pi \phi (e_{j_1}, \ldots, e_{j_n}) = \sum_j \alpha_j \psi (e_{j_1}, \ldots, e_{j_n})$$

We know that to each alternating multilinear map $h : V \times \cdots \times V \to W$ there corresponds a linear map $\tilde{h} : V \otimes \cdots \otimes V / S \to W$ such that $h = \tilde{h} \psi$. Then

$$\sum_j \alpha_j h(e_{j_1}, \ldots, e_{j_n}) = \sum_j \alpha_j \tilde{h} \psi (e_{j_1}, \ldots, e_{j_n})$$

$$= \tilde{h} \left[\sum_j \alpha_j \psi (e_{j_1}, \ldots, e_{j_n}) \right] = \tilde{h}(0) = 0 \qquad (2)$$

For each j, we'll show that $\alpha_j = 0$. ~~Put~~ For each $(v_1, \ldots, v_n) \in V \times \cdots \times V$, put

$$M_j(v_1, \ldots, v_n) = \begin{bmatrix} \beta_{1j_1} & \beta_{1j_2} & \cdots & \beta_{1j_n} \\ \vdots & \vdots & & \vdots \\ \beta_{nj_1} & \beta_{nj_2} & \cdots & \beta_{nj_n} \end{bmatrix}$$

where $v_i = \sum_k \beta_{ik} e_k$. We define
$$h_j : V \times \cdots \times V \longrightarrow \mathbb{C}$$
$$(v_1, \ldots, v_n) \longmapsto \det M_j(v_1, \ldots, v_n)$$

Then h_j is linear with respect to each v_i because the determinant depends linearly on each row. Moreover, h_j is alternating because the exchange of two arguments of h_j is equivalent to the exchange of two rows in the matrix $M_j(v_1, \ldots, v_n)$, which causes a minus sign of the determinant.

Thus, we can apply (2) for $h = h_j$:

$$\sum_i \alpha_i h_j(e_{i_1}, \dots, e_{i_n}) = 0$$

If $(e_{i_1}, \dots, e_{i_n}) \neq (e_{j_1}, \dots, e_{j_n})$ then we can assume without loss of generality that $e_{i_1} \neq e_{j_1}, \dots, e_{j_n}$. Then the matrix $M_j(e_{i_1}, \dots, e_{i_n})$ has the first row 0. Hence $h_j(e_{i_1}, \dots, e_{i_n}) = \det M_j(e_{i_1}, \dots, e_{i_n}) = 0$.

Moreover, $h_j(e_{j_1}, \dots, e_{j_n}) = 1$ because $M_j(e_{j_1}, \dots, e_{j_n})$ is the identity matrix. Thus $\alpha_j = 0$.

$\boxed{\text{II}}$ Hilbert-space 2-fold wedge product

Let V be a Hilbert space and $\{e_n\}_{n \in \mathbb{N}}$ an orthonormal basis.

Put $V_0 = \langle \{e_1, e_2, \dots\} \rangle$. In Part $\boxed{\text{I}}$, we know what $V_0 \wedge V_0$ is. In this part, we will introduce an inner product on $V_0 \wedge V_0$ and complete this space to get a Hilbert space.

① $V_0 \wedge V_0$ has the following inner product

$$\left\langle \sum_{i<j} \alpha_{ij} e_i \wedge e_j , \sum_{k<l} \beta_{kl} e_k \wedge e_l \right\rangle := \sum_{i<j} \alpha_{ij} \overline{\beta_{ij}} \qquad (3)$$

Proof Since $V_0 \wedge V_0$ is a vector space with basis $\{e_i \wedge e_j \mid i<j\}$, the map $\langle \cdot, \cdot \rangle : (V_0 \wedge V_0) \times (V_0 \wedge V_0) \longrightarrow \mathbb{C}$ at (3) is well-defined.

It follows immediately from (3) that $\langle\cdot,\cdot\rangle$ is linear in the first argument. Moreover,

$$\left\langle \sum_{k<l} \beta_{kl}\, e_k\wedge e_l, \sum_{i<j} \alpha_{ij}\, e_i\wedge e_j \right\rangle = \sum_{i<j} \beta_{ij}\,\overline{\alpha_{ij}} = \overline{\sum_{i<j} \alpha_{ij}\,\overline{\beta_{ij}}}$$

$$= \overline{\left\langle \sum_{i<j} \alpha_{ij}\, e_i\wedge e_j, \sum_{k<l} \beta_{kl}\, e_k\wedge e_l \right\rangle}$$

Thus, $\langle\cdot,\cdot\rangle$ is conjugate symmetric.

$$\left\langle \sum_{i<j} \alpha_{ij}\, e_i\wedge e_j, \sum_{k<l} \alpha_{kl}\, e_k\wedge e_l \right\rangle = \sum_{i<j} \alpha_{ij}\cdot\overline{\alpha_{ij}} = \sum_{i<j} |\alpha_{ij}|^2 \geq 0$$

The equality occurs if and only if $\alpha_{ij} = 0 \ \forall i<j$. Therefore, $\langle\cdot,\cdot\rangle$ is indeed an inner product on $V_0\wedge V_0$.

② Definition: A completion of $V_0\wedge V_0$ with respect to the norm induced by $\langle\cdot,\cdot\rangle$ is called a Hilbert-space 2-fold wedge product of V, and denoted $V\hat{\wedge}V$.

Consequently, as tensor product, Hilbert-space 2-fold wedge product of V is unique up to a linear isometric isomorphism.

③ An explicit construction of $V\hat{\wedge}V$

Put $\quad S_0 = \langle \{ v\otimes v \,/\, v\in V_0 \} \rangle$

$\quad\quad S = \langle \{ v\hat{\otimes} v \,/\, v\in V_0 \} \rangle$

$$G_0 = \left\{ \sum_{i<j} \gamma_{ij} e_i \otimes e_j + S_0 \,/\, (\gamma_{ij}) \in \ell_{fin} \right\}$$

Here ℓ_{fin} is the space of sequences that are zero at all but finitely many positions. We know that $G_0 = V_0 \wedge V_0$. As in Point 4, Part $\boxed{\text{III}}$, report on "HS-operator and tensor product of two vector spaces", we put

$$\cancel{\tilde{F}_0 = \langle \{ e_i \otimes e_j \,/\, i,j \in \mathbb{N} \} \rangle}$$

$$\cancel{\tilde{F} = \langle \{ e_i \hat{\otimes} e_j \,/\, i,j \in \mathbb{N} \} \rangle}$$

$$\tilde{F}_0 = \left\{ \sum \gamma_{ij} e_i \otimes e_j \,/\, (\gamma_{ij}) \in \ell_{fin} \right\}$$

$$\tilde{F} = \left\{ \sum \gamma_{ij} e_i \hat{\otimes} e_j \,/\, (\gamma_{ij}) \in \tilde{\ell} \right\}$$

Then $\tilde{F}_0 = V_0 \otimes V_0$ and $\tilde{F} = V \hat{\otimes} V$. Define the map

$$\phi_0 : \tilde{F}_0 \longrightarrow \tilde{F}$$

$$\sum \gamma_{ij} e_i \otimes e_j \mapsto \sum \gamma_{ij} e_i \hat{\otimes} e_j \qquad \forall (\gamma_{ij}) \in \ell_{fin}$$

Then also by that Point, ϕ_0 is well-defined, linear and injective. Thus $\phi_0(\tilde{F}_0) = V_0 \otimes V_0$. We will show that $S = \phi_0(S_0)$. To show that $S \subset \phi_0(S_0)$, it is sufficient to show that $v \hat{\otimes} v \in \phi_0(S_0)$ for each $v \in V_0$. We have

$$v \hat{\otimes} v = \left(\sum \alpha_i e_i\right) \hat{\otimes} \left(\sum \alpha_j e_j\right) \qquad \text{(finite sums)}$$

$$= \sum \alpha_i \alpha_j \, e_i \hat{\otimes} e_j$$

$$= \sum \alpha_i \alpha_j \, \phi_0(e_i \otimes e_j) = \phi_0\Big(\underbrace{\sum \alpha_i \alpha_j \, e_i \otimes e_j}_{\in S_0} \Big)$$

To show that $\phi_0(S_0) \subset S$, it is sufficient to show that $\phi_0(v \otimes v) \in S$ for each $v \in V_0$. We have

$$\phi_0(v \otimes v) = \phi_0\left[\left(\sum \alpha_i e_i\right) \otimes \left(\sum \alpha_j e_j\right)\right] = \phi_0\left(\sum \alpha_i \alpha_j \, e_i \otimes e_j\right)$$

$$= \sum \alpha_i \alpha_j \, \phi_0(e_i \otimes e_j) = \sum \alpha_i \alpha_j \, e_i \hat{\otimes} e_j$$

$$= \left(\sum \alpha_i e_i\right) \hat{\otimes} \left(\sum \alpha_j e_j\right) = v \hat{\otimes} v \in S$$

Therefore, $\phi_0(\tilde{F}_0)/\phi_0(S_0) = \phi_0(\tilde{F}_0)/S$ is 2-fold wedge product of V_0 because $\tilde{F}_0/S_0$ is too. We have

$$\phi_0(\tilde{F}_0)/S = \left\{ \sum \gamma_{ij} \, e_i \hat{\otimes} e_j + S \;/\; (\gamma_{ij}) \in \ell_{fin} \right\}$$

Because $\{ e_i \otimes e_j + S_0 \;/\; i<j \}$ is a basis of $\tilde{F}_0/S$, the set $\{ e_i \hat{\otimes} e_j + S \;/\; i<j \}$ is in turn a basis of $\phi_0(\tilde{F}_0)/S$. Hence we can write

$$\phi_0(\tilde{F}_0)/S = G := \left\{ \sum_{i<j} \gamma_{ij} \, e_i \hat{\otimes} e_j + S \;/\; (\gamma_{ij}) \in \ell_{fin} \right\}$$

The inner producton G is induced by ϕ_0 as follow

$$\left\langle \sum_{i<j} \gamma_{ij}\, e_i \hat{\otimes} e_j + S,\ \sum_{k<l} \gamma'_{kl}\, e_k \hat{\otimes} e_l + S \right\rangle_G$$

$$:= \left\langle \sum_{i<j} \gamma_{ij}\, e_i \otimes e_j + S_0,\ \sum_{k<l} \gamma'_{kl}\, e_k \otimes e_l + S_0 \right\rangle_{\tilde{F}_0/S_0}$$

$$= \sum_{i<j} \gamma_{ij} \overline{\gamma'_{ij}}$$

This inner product induced a norm on G. A completion of $(G, \|.\|)$ is a ~~2-fold~~ Hilbert-space 2-fold wedge product of V. Put

$$\tilde{G} = \tilde{F}/S = \left\{ \sum_{i<j} \gamma_{ij}\, e_i \hat{\otimes} e_j + S \ / \ (\gamma_{ij}) \in l^2 \right\}$$

We show that each element in $\tilde{G}$ corresponds uniquely to a sequence $(\gamma_{ij}) \in l^2$. It is equivalent to show that if

$$\sum_{i<j} \gamma_{ij}\, e_i \hat{\otimes} e_j \in S \quad \text{for } (\gamma_{ij}) \in l^2$$

then $\gamma_{ij} \equiv 0$. We have

$$\sum_{i<j} \gamma_{ij}\, e_i \hat{\otimes} e_j = \underbrace{\sum_k \alpha_k v_k \hat{\otimes} v_k}_{\text{finite sum}}$$

$$= \sum_k \alpha_k \underbrace{\left(\sum_l \beta_{kl} e_l\right)}_{\text{finite sum}} \hat{\otimes} \underbrace{\left(\sum_s \beta_{ks} e_s\right)}_{\text{finite sum}}$$

$$= \sum_{k,l,s} \alpha_k \beta_{kl} \beta_{ks}\, e_l \hat{\otimes} e_s$$

$$= \sum_{l,s} \gamma'_{ls}\, e_l \hat{\otimes} e_s \quad , \text{where } \gamma'_{ls} = \sum_k \alpha_k \beta_{kl} \beta_{ks}$$

As in Point 4, Part III, report on "HS operator and ....", we put

$$R = \left\langle \left\{ 1\!\!1_{(\sum \alpha_i e_i, \sum \beta_j e_j)} - \sum_{ij} \alpha_i \beta_j 1\!\!1_{(e_i, e_j)} \,/\, (\alpha_i), (\beta_j) \in \ell^2(\mathbb{N}) \right\} \right\rangle$$

Then

$$\sum_{i<j} \gamma_{ij} 1\!\!1_{(e_i, e_j)} - \underbrace{\sum_{l,s} \gamma'_{ls} 1\!\!1_{(e_l, e_s)}}_{\text{finite sum}} \in R$$

Thus,

$$\sum_{i<j} \gamma_{ij} 1\!\!1_{(e_i, e_j)} - \sum_{l,s} \gamma'_{ls} 1\!\!1_{(e_l, e_s)} = \sum_k \varepsilon_k \left[1\!\!1_{(\sum \alpha_i^k e_i, \sum \beta_j^k e_j)} - \sum_{ij} \alpha_i^k \beta_j^k 1\!\!1_{(e_i, e_j)} \right]$$

Hence each pair $(\sum \alpha_i^k e_i, \sum \beta_j^k e_j)$ should have the form (e_i, e_j).

Thus, the sum $\sum_{i<j} \gamma_{ij} 1\!\!1_{(e_i, e_j)}$ must be a finite sum. Then

$$\sum_{i<j} \gamma_{ij}\, e_i \hat{\otimes} e_j \in S \quad \text{with } (\gamma_{ij}) \in \ell_{fin}$$

Therefore $\gamma_{ij} \equiv 0$.

On $\tilde{G}$, we can define an inner product

$$\left\langle \sum_{i<j} \gamma_{ij}\, e_i \hat{\otimes} e_j + S,\ \sum_{k<l} \gamma'_{kl}\, e_k \hat{\otimes} e_l + S \right\rangle_{\tilde{G}} := \sum_{i<j} \gamma_{ij} \overline{\gamma'_{ij}}$$

This inner product induced a norm on $\tilde{G}$. Moreover, the inner product on G is just a restriction of the inner product ~~of~~ on $\tilde{G}$. To show that $\tilde{G} = V \hat{\wedge} V$, we have two tasks to do. First, show that

$(\tilde{G}, \|.\|)$ is Banach, and second is to show that $(G, \|.\|)$ is dense in $(\tilde{G}, \|.\|)$. Because

$$\Big\| \sum_{i<j} \gamma_{ij}\, e_i \hat{\otimes} e_j + S \Big\|_{\tilde{G}} = \Big(\sum_{i<j} |\gamma_{ij}|^2 \Big)^{1/2} = \|(\gamma_{ij})\|_{\ell^2}$$

and ℓ^2 is a Banach space, $(\tilde{G}, \|.\|)$ is also Banach. We only have to show that $(G, \|.\|)$ is dense in $(\tilde{G}, \|.\|)$. Take

$$f = \sum_{i<j} \gamma_{ij}\, e_i \hat{\otimes} e_j + S \in \tilde{G}$$

For each $n \in \mathbb{N}$, we define

$$f_n = \sum_{i<j<n} \gamma_{ij}\, e_i \hat{\otimes} e_j + S \in G$$

Then

$$\|f - f_n\|^2_{\tilde{G}} = \Big\| \sum_{\substack{j\geq n \\ i<j}} \gamma_{ij}\, e_i \hat{\otimes} e_j + S \Big\|^2 = \sum_{\substack{i<j \\ j\geq n}} |\gamma_{ij}|^2 \xrightarrow{n\to\infty} 0$$

Thus $f_n \to f$. and $(G, \|.\|)$ is dense in $(\tilde{G}, \|.\|)$.

4 With the notation $u \hat{\wedge} v := u \hat{\otimes} v + S$, we have

$$\langle v_1 \hat{\wedge} v_2 , w_1 \hat{\wedge} w_2 \rangle = \det \begin{pmatrix} \langle v_1, w_1 \rangle & \langle v_1, w_2 \rangle \\ \langle v_2, w_1 \rangle & \langle v_2, w_2 \rangle \end{pmatrix}$$

for every $v_1, v_2, w_1, w_2 \in V$.

Proof Suppose that

$v_1 = \sum \alpha_i e_i \qquad w_1 = \sum \alpha'_k e_k$ where $(\alpha_i), (\beta_j), (\alpha'_k), (\beta'_\ell) \in \ell^2(\mathbb{N})$

$v_2 = \sum \beta_j e_j \qquad w_2 = \sum \beta'_\ell e_\ell$

(14)

Then

$$v_1 \hat{\wedge} v_2 = v_1 \hat{\otimes} v_2 + S = \left(\sum \alpha_i e_i\right) \hat{\otimes} \left(\sum \beta_j e_j\right) + S$$

$$= \sum \alpha_i \beta_j \, e_i \hat{\otimes} e_j + S$$

$$= \sum_{i<j} \alpha_i \beta_j e_i \hat{\otimes} e_j + \sum_i \alpha_i \beta_i \, e_i \hat{\otimes} e_i + \sum_{i>j} \alpha_i \beta_j \, e_i \hat{\otimes} e_j + S \qquad (4)$$

Because our construction in Point 3 does not depend on the choice of orthonormal basis (e_i) in sense that if we choose another orthonormal basis (e_i') instead, all the corresponding results are isomorphic, we can rechoose (e_i) such that $(\alpha_i), (\beta_i), (\alpha_k'), (\beta_\ell') \in \ell_{fin}$. Then the sums at (4) are all finite sums. Then

$$v_1 \hat{\wedge} v_2 = \sum_{i<j} \alpha_i \beta_j e_i \hat{\otimes} e_j + \sum_{i>j} \alpha_i \beta_j \, e_i \hat{\otimes} e_j + S$$

$$= \sum_{i<j} \alpha_i \beta_j \, e_i \hat{\otimes} e_j + \sum_{i<j} \alpha_j \beta_i \, e_j \hat{\otimes} e_i + S$$

$$= \sum_{i<j} \alpha_i \beta_j \, e_i \hat{\otimes} e_j - \sum_{i<j} \alpha_j \beta_i \, e_i \hat{\otimes} e_j + S$$

$$= \sum_{i<j} (\alpha_i \beta_j - \alpha_j \beta_i) \, e_i \hat{\otimes} e_j + S$$

Likewise, $\quad w_1 \hat{\wedge} w_2 = \sum_{k<\ell} (\alpha_k' \beta_\ell' - \alpha_\ell' \beta_k') \, e_k \hat{\otimes} e_\ell + S$

By the inner product on $\widetilde{G}$ introduced in Point 3, we have

$$\langle v_1 \hat{\wedge} v_2, w_1 \hat{\wedge} w_2 \rangle = \sum_{i<j} (\alpha_i \beta_j - \alpha_j \beta_i)(\overline{\alpha_i'} \overline{\beta_j'} - \overline{\alpha_j'} \overline{\beta_i'})$$

$$= \sum_{i<j} \alpha_i \overline{\alpha_i'} \beta_j \overline{\beta_j'} + \sum_{i<j} \alpha_j \overline{\alpha_j'} \beta_i \overline{\beta_i'}$$

$$- \sum_{i<j} \alpha_i \overline{\beta_i'} \beta_j \overline{\alpha_j'} - \sum_{i<j} \alpha_j \overline{\beta_j'} \beta_i \overline{\alpha_i'}$$

$$= \sum \alpha_i \overline{\alpha_i'} \beta_j \overline{\beta_j'} - \sum \alpha_i \overline{\beta_i'} \beta_j \overline{\alpha_j'}$$

$$= \left(\sum \alpha_i \overline{\alpha_i'}\right)\left(\sum \beta_j \overline{\beta_j'}\right) - \left(\sum \alpha_i \overline{\beta_i'}\right)\left(\sum \beta_j \overline{\alpha_j'}\right)$$

$$= \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle - \langle v_1, w_2 \rangle \langle v_2, w_1 \rangle$$

$$= \det \begin{pmatrix} \langle v_1, w_1 \rangle & \langle v_1, w_2 \rangle \\ \langle v_2, w_1 \rangle & \langle v_2, w_2 \rangle \end{pmatrix}$$

$\boxed{\text{III}}$ An example of Hilbert-space 2-fold wedge product

Denote $X = \mathbb{R}^3$, $V = L^2(X)$, $\{f_n\}_{n \in \mathbb{N}}$ an orthonormal basis of V,

$$V_0 = \langle \{f_1, f_2, \dots\} \rangle$$

$$S_0 = \langle \{ v \otimes v \,/\, v \in V_0 \} \rangle$$

We know that

$$V_0 \otimes V_0 = \left\{ \sum \alpha_{ij} f_i \otimes f_j \,/\, (\alpha_{ij}) \in \ell_{fin} \right\}$$

$$V_0 \wedge V_0 = V_0 \otimes V_0 / S$$

By Point 2, Part IV, report on "HS operator and ...", we have

$$V \hat{\otimes} V = \overline{V_0 \otimes V_0} = L^2(X^2)$$

In this section, we will find $V \hat{\wedge} V$ as a subset of $L^2(X^2)$.

1 For each $i,j \in \mathbb{N}$, we denote $h_{ij} \in L^2(X^2)$ such that

$$h_{ij}(x,y) = f_i(x) f_j(y)$$

Put $Y = \{ \sum \alpha_{ij} h_{ij} \,/\, \alpha_{ij} \in \ell_{fin} \} \subset L^2(X^2)$

$$T = \langle \{ h \in L^2(X^2) \,/\, h(x,y) = f(x) f(y) \text{ for some } f \in V_0 \} \rangle$$

Then $Y/T = V_0 \wedge V_0$ and Y/T has a basis $\{ h_{ij} + T \,/\, i<j \}$.

Proof Define $\phi_1 : V_0 \otimes V_0 \longrightarrow Y$

$$\sum \alpha_{ij} f_i \otimes f_j \longmapsto \sum \alpha_{ij} h_{ij}$$

Then ϕ_1 is well-defined and a linear isomorphism. Thus, $Y = V_0 \otimes V_0$.

We show that $T = \phi_1(S_0)$. Take $h \in L^2(X^2)$ of the form

$$h(x,y) = f(x) f(y) \quad \text{for some } f \in V_0.$$

We can write $\quad f(x) = \sum \beta_i f_i(x) \quad$ (finite sum)

then

$$h(x,y) = \Big(\sum \beta_i f_i(x) \Big) \Big(\sum \beta_j f_j(y) \Big) = \sum_{ij} \beta_i \beta_j f_i(x) f_j(y)$$

$$= \sum_{ij} \beta_i \beta_j h_{ij}(x,y)$$

Thus $h = \sum_{ij} \beta_i \beta_j h_{ij} = \sum_{ij} \beta_i \beta_j \phi_1(f_i \otimes f_j)$

$$= \phi_1 \Big(\sum_{ij} \beta_i \beta_j f_i \otimes f_j \Big) = \phi_1 \Big[\Big(\sum \beta_i f_i \Big) \otimes \Big(\sum \beta_j f_j \Big) \Big]$$

$$= \phi_1(f \otimes f) \in \phi_1(S_0)$$

Thus, $T \subset \phi_1(S_0)$. Take $f = \sum \beta_i f_i \in V_0$. We have

$$\phi_1(f \otimes f) = \phi_1\left(\sum_{ij} \beta_i \beta_j f_i \otimes f_j\right) = \sum_{ij} \beta_i \beta_j \phi_1(f_i \otimes f_j) = \sum_{ij} \beta_i \beta_j h_{ij} = h$$

where $h(x,y) = \sum \beta_i \beta_j h_{ij}(x,y) = \sum \beta_i \beta_j f_i(x) f_j(y) = \left(\sum \beta_i f_i(x)\right)\left(\sum \beta_j f_j(y)\right)$

$$= f(x) f(y)$$

Thus $h \in T$ and $\phi_1(S_0) \subset T$. Hence $T = \phi_1(S_0)$. Then

$$V_0 \wedge V_0 = (V_0 \otimes V_0)/S_0 \simeq \phi_1(V_0 \otimes V_0)/\phi_1(S_0) = Y/T$$

Since the set $\{f_i \otimes f_j + S_0 \,/\, i<j\}$ is a basis of $(V_0 \otimes V_0)/S_0$, the set $\{\phi_1(f_i \otimes f_j) + \phi(S_0) \,/\, i<j\} = \{h_{ij} + T \,/\, i<j\}$ is a basis of Y/T.

② Let $n \in \mathbb{N}$. An element $\tilde{f} \in L^2(X^n)$ is infact an equivalence class of functions, not a single function.

Definition: $\tilde{f}$ is called alternating if it has an alternating representative.

In other words,

$$\tilde{f} \text{ is alternating} \Leftrightarrow \text{“}\exists f \in \tilde{f} : \; f(x_1,\dots,x_n) = (-1)^{\sigma} f(R^{\sigma}(x_1,\dots,x_n)) \quad \forall \sigma \in S_n \quad \forall (x_1,\dots,x_n) \in X^n\text{”}$$

Here S_n is the set of all n-permutations.

③ $\tilde{f} \in L^2(X^n)$ is alternating if and only if for every $g \in \tilde{f}$,

$$g(x_1,\dots,x_n) = (-1)^{\sigma} g(R^{\sigma}(x_1,\dots,x_n)) \quad \forall \sigma \in S_n \text{ for almost every } (x_1,\dots,x_n) \in X^n.$$

Proof The forward part:

There exists $f \in \tilde{f}$ such that $f(x_1, \dots, x_n) = (-1)^\sigma f(R^\sigma(x_1 \dots x_n))$ $\forall \sigma \in S_n$, $\forall (x_1, \dots, x_n) \in X^n$. Let $g \in \tilde{f}$ be another reprentative. Then $g(x_1 \dots x_n) = f(x_1 \dots x_n)$ for almost every $(x_1, \dots x_n) \in X^n$. For each $\sigma \in S_n$, we put

$$A_\sigma = \{(x_1, \dots, x_n) / f(R^\sigma(x_1, \dots, x_n)) = g(R^\sigma(x_1, \dots, x_n))\}$$

Let id be the identity permutation, we have

$$(x_1, \dots, x_n) \in A_\sigma \iff R^\sigma(x_1, \dots, x_n) \in A_{id},$$

i.e. $A_{id} = R^\sigma A_\sigma$. Since R^σ is a bijection from X^n to itself, we have $X^n \setminus A_{id} = R^\sigma(X^n \setminus A_\sigma)$. Thus,

$$0 = \mu(X^n \setminus A_{id}) = \mu(R^\sigma(X^n \setminus A_\sigma))$$
$$= \mu(X^n \setminus A_\sigma) \quad \text{(because } R^\sigma \text{ is a rotation)}$$

Put $A = \bigcap_{\sigma \in S_n} A_\sigma$. Then $\mu(X^n \setminus A) = 0$. For each $x = (x_1, \dots, x_n) \in A$, we have

$$g(R^\sigma(x)) \overset{x \in A_\sigma}{=\!=} f(R^\sigma(x)) = (-1)^\sigma f(x) \overset{x \in A_{id}}{=\!=} (-1)^\sigma g(x)$$

Thus, $g(R^\sigma(x)) = (-1)^\sigma g(x)$ $\forall \sigma \in S_n$ for almost every $x \in X^n$.

The backward part:

Suppose that $g \in \tilde{f}$ satisfies $\mu(X^n \setminus B) = 0$ where

$$B = \{ x \in X^n \,/\, g(R^\sigma(x)) = (-1)^\sigma g(x) \quad \forall \sigma \in S_n \}$$

We define $f(y) = \begin{cases} g(y) & \text{if } y \in B \\ 0 & \text{otherwise} \end{cases}$

Then $f \in \tilde{f}$. For each $y \in X^n$, we will show that $f(\sigma(y)) = (-1)^\sigma f(y)$ for every $\sigma \in S_n$. If $y \in B$ then $R^\sigma y \in B$. Thus,

$$f(\sigma(y)) = g(\sigma(y)) = (-1)^\sigma g(y) = \cancel{f(y)} (-1)^\sigma f(y).$$

If $y \notin B$ then $R^\sigma y \notin B$. Then

$$f(\sigma(y)) = 0 = (-1)^\sigma f(y).$$

4 Define a map $\text{Alt}: L^2(X^2) \longrightarrow L^2(X^2)$

$$f(z) \longmapsto g(z) = C \sum_{\sigma \in S_2} (-1)^\sigma f(R^\sigma(z))$$

where $C > 0$ is a coefficient that is chosen for the purpose of normalization. More explicitly,

$$\text{Alt}(f)(x,y) = C\,[f(x,y) - f(y,x)] \qquad (5)$$

Then Alt is well-defined, linear, and $\text{Alt}(f)$ is alternating.

Proof To show that Alt is well-defined, we show that $f_\sigma = f \circ R^\sigma \in L^2(X^2)$ for every $\sigma \in S_2$.

Because R^σ is a rotation,

$$\int_{X^2} |f^\sigma|^2 = \int_{X^2} |f\circ R^\sigma|^2 = \int_{X^2} |f|^2 = \|f\|_{L^2}^2 < \infty$$

Thus, $f \in L^2(X^2)$. By (5),

$$Alt(\alpha f_1 + f_2) = C[(\alpha f_1 + f_2)(x,y) - (\alpha f_1 + f_2)(y,x)]$$

$$= C[\alpha(f_1(x,y) - f_1(y,x)) + f_2(x,y) - f_2(y,x)]$$

$$= \alpha\, Alt(f_1) + Alt(f_2)$$

thus Alt is linear. We have

$$Alt(f)(y,x) = C[f(y,x) - f(x,y)] = -C[f(x,y) - f(y,x)] = -Alt(f)(x,y)$$

Hence $Alt(f)$ is alternating.

(5) Define a map $\xi : Y/T \longrightarrow Alt(Y)$

$$f+T \longmapsto Alt(f) \quad \forall f \in Y$$

Then ξ is well-defined, linear and bijective.

Proof To show that ξ is well-defined means to show that $Alt(f)$ is independent of the choice of representative in $f+T$. Since Alt is linear, it is sufficient to show that if $h \in T$ then $Alt(h) = 0$.

By the definition of T,

$$h = \sum \beta_i v_i(x) v_i(y) \qquad \text{where each } v_i \in V_0$$

(finite sum)

Then $\mathrm{Alt}(h) = \sum \beta_i \mathrm{Alt}(v_i(x) v_i(y))$

$$= \sum \beta_i C[v_i(x) v_i(y) - v_i(y) v_i(x)]$$

$$= 0.$$

Since Alt is linear, ξ is also linear. Moreover, by its definition, ξ is surjective. To show that ξ is injective, we show that if $\mathrm{Alt}(h) = 0$ for some $h \in Y$ then $h \in T$. Since $h \in Y$, we can write

$$h = \sum \alpha_{ij} h_{ij} \quad \text{where } (\alpha_{ij}) \in \ell_{fin} \text{ and}$$

$$h_{ij}(x, y) = f_i(x) f_j(y)$$

We have

$$0 = \mathrm{Alt}(h) = C[h(x,y) - h(y,x)] = C\left[\sum \alpha_{ij} f_i(x) f_j(y) - \sum \alpha_{ij} f_i(y) f_j(x)\right]$$

$$= C\left[\sum \alpha_{ij} f_i(x) f_j(y) - \sum \alpha_{ji} f_j(y) f_i(x)\right]$$

$$= C \sum (\alpha_{ij} - \alpha_{ji}) f_i(x) f_j(y)$$

$$= C \sum (\alpha_{ij} - \alpha_{ji}) h_{ij}$$

Since $\{h_{ij} / i, j \in \mathbb{N}\}$ is a basis of Y, we must have $\alpha_{ij} = \alpha_{ji}$ $\forall i, j$.

Then $\quad h = \sum \alpha_{ij} h_{ij} = \frac{1}{2} \sum \alpha_{ij} (h_{ij} + h_{ji})$

$$= \frac{1}{2} \sum \alpha_{ij} (f_i(x) f_j(y) + f_j(x) f_i(y))$$

Put $\tilde{f}_{ij}^{\pm}(z) = f_i(z) \pm f_j(z) \quad \forall z \in X$.

we have
$$f_i(x) f_j(y) + f_j(x) f_i(y) = \frac{1}{2}\Big[\big(f_i(x) + f_j(x)\big)\big(f_i(y) + f_j(y)\big) - \big(f_i(x) - f_j(x)\big)\big(f_i(y) - f_j(y)\big)\Big]$$
$$= \frac{1}{2}\left[\tilde{f}_{ij}^{+}(x)\,\tilde{f}_{ij}^{+}(y) - \tilde{f}_{ij}^{-}(x)\,\tilde{f}_{ij}^{-}(y)\right]$$

Moreover, since $\tilde{f}_{ij}^{\pm} \in V_0$, the above expression belongs to T.

Thus $f_i(x) f_j(y) + f_j(x) f_i(y) \in T$. Therefore, $h \in T$. ∎

Since ξ is a linear isomorphism,
$$\mathrm{Alt}(Y) \simeq Y/T = V_0 \wedge V_0$$

By Point 1, Y/T has a basis $\{h_{ij} + T \,/\, i<j\}$. Hence $\mathrm{Alt}(Y)$ has a basis $\{k_{ij} \,/\, i<j\}$ where $k_{ij} = \mathrm{Alt}(h_{ij})$. More explicitly,
$$k_{ij}(x,y) = C\left[h_{ij}(x,y) - h_{ij}(y,x)\right] = C\left[f_i(x) f_j(y) - f_i(y) f_j(x)\right]$$

⑥ Put $H = \left\{ \sum_{i<j} \alpha_{ij} k_{ij} \,/\, (\alpha_{ij}) \in \ell^2 \right\}$

Then $H \subset L^2(X^2)$ and is a Hilbert space with the inner product
$$\left\langle \sum_{i<j} \alpha_{ij} k_{ij} \,,\, \sum_{r<s} \beta_{rs} k_{rs} \right\rangle := \sum_{i<j} \alpha_{ij} \overline{\beta_{ij}} \qquad (6)$$

Proof

For each $i<j$, we have $\int_{X^2} |k_{ij}|^2 \,dx\,dy = C^2 \int_{X^2} |h_{ij}(x,y) - h_{ij}(y,x)|^2 \,dx\,dy$

$$\le 2C^2 \int_{X^2} \left(|h_{ij}(x,y)|^2 + |h_{ij}(y,x)|^2 \right) dx\,dy$$

$$= 2C^2 \left(\|h_{ij}\|^2 + \|h_{ij}\|^2 \right) = 4C^2$$

Thus, $\|k_{ij}\|_{L^2} \le 2C$. For each $(\alpha_{ij}) \in \ell^2$, we have

$$\sum_{i<j} \alpha_{ij} k_{ij} = \sum \alpha_{ij} C[h_{ij}(x,y) - h_{ij}(y,x)]$$

$$= C \underbrace{\sum \alpha_{ij} h_{ij}(x,y)}_{\in L^2(X^2)} - C \underbrace{\sum \alpha_{ij} h_{ij}(y,x)}_{\in L^2(X^2)}$$

Hence $\sum_{i<j} \alpha_{ij} k_{ij} \in L^2(X^2)$ and $H \subset L^2(X^2)$.

To show that the mapping $\langle \cdot , \cdot \rangle$ at (6) is actually an inner product, we only need to show that each element in H corresponds uniquely to a sequence $(\alpha_{ij}) \in \ell^2$. To do so, supposing that

$$\sum_{i<j} \alpha_{ij} k_{ij} = 0,$$

we will show that $\alpha_{ij} \equiv 0$. For each $r<s$, we have

$$0 = \left\langle \sum_{i<j} \alpha_{ij} k_{ij} , h_{rs} \right\rangle_{L^2(X^2)} = \sum_{i<j} \alpha_{ij} \langle k_{ij}, h_{rs} \rangle$$

$$= C \sum_{i<j} \alpha_{ij} \langle h_{ij}(x,y) - h_{ij}(y,x), h_{rs} \rangle$$

$$= C \sum_{i<j} \alpha_{ij} \left[\int_{X^2} h_{ij}(x,y) h_{rs}(x,y)\,dx\,dy - \int_{X^2} h_{ij}(y,x) h_{rs}(x,y)\,dx\,dy \right]$$

$$= C\sum_{i<j} \alpha_{ij}\left(\int_X f_i f_r \int_X f_j f_s - \int_X f_j f_r \int_X f_i f_s \right)$$

$$= C\sum_{i<j} \alpha_{ij}\left(\delta_{ir}\delta_{js} - \delta_{jr}\delta_{is} \right) = C\alpha_{rs}$$

Thus $\alpha_{rs} = 0 \quad \forall r<s$. Since

$$\left\langle \sum_{i<j} \alpha_{ij} k_{ij}, \sum_{r<s} \beta_{rs} k_{rs} \right\rangle = \left\langle (\alpha_{ij}), (\beta_{rs}) \right\rangle_{\ell^2}$$

and ℓ^2 is complete, H is also complete. Therefore, H is a Hilbert space.

(7) $Alt(Y)$ is dense in $(H, \|.\|)$.

Proof By Point 5, we know that

$$Alt(Y) = \left\{ \sum_{i<j} \alpha_{ij} k_{ij} \;/\; (\alpha_{ij}) \in \ell_{fin} \right\}$$

Let $f = \sum_{i<j} \alpha_{ij} k_{ij}$, where $(\alpha_{ij}) \in \ell^2$, be an element of H.

For each $n \in \mathbb{N}$, we put $f_n = \sum_{i<j<n} \alpha_{ij} k_{ij} \in Alt(Y)$.

Then
$$\|f - f_n\|^2 = \Big\| \sum_{\substack{j \geq n \\ i<j}} \alpha_{ij} k_{ij} \Big\|^2 = \sum_{\substack{i<j \\ j \geq n}} |\alpha_{ij}|^2 \xrightarrow{n\to\infty} 0$$

Thus $f_n \longrightarrow f$. ∎

Consequently, $H = V \hat{\wedge} V$.

(8) $H = \mathrm{Alt}(L^2(X^2))$

Proof First, we show that $H \subset \mathrm{Alt}(L^2(X^2))$.

For each $(\alpha_{ij})_{i<j} \in \ell^2$,

$$f = \sum_{i<j} \alpha_{ij} k_{ij} = \sum \alpha_{ij} \mathrm{Alt}(h_{ij})$$

By the definition of Alt in Point 4, we see that Alt is continuous.

because

$$\|\mathrm{Alt}(f)\|_{L^2(X^2)} = \|C \sum_{\sigma \in S_2} (-1)^\sigma f(R^\sigma(z))\|_{L^2(X^2)}$$

$$\leq C \sum_{\sigma \in S_2} \|f \circ R^\sigma\|_{L^2(X^2)} = \sigma \sum_{\sigma \in S_2} \|f\|_{L^2(X^2)}$$

$$= \sigma(2!) \|f\|_{L^2(X^2)}$$

Put $\quad h = \sum_{i<j} \alpha_{ij} h_{ij} \in L^2(X^2)$.

Since Alt is linear and continuous, we have

$$f = \sum_{i<j} \alpha_{ij} \mathrm{Alt}(h_{ij}) = \mathrm{Alt}\Big(\sum_{i<j} \alpha_{ij} h_{ij}\Big) = \mathrm{Alt}(h) \in \mathrm{Alt}(L^2(X^2)).$$

Next, we show that $\mathrm{Alt}(L^2(X^2)) \subset H$. Take $g \in L^2(X^2)$.

By Point 2, Part IV, report on "Hilbert Schmidt operator....", g is the limit of a sequence in Y, namely

$$g_n = \sum_{i,j} \alpha_{ij}^n h_{ij} \quad \text{where } (\alpha_{ij}^n) \in \ell_{fin}$$

Then $\{(\alpha_{ij}^n)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in ℓ^2 and thus converges to $(\alpha_{ij}) \in \ell^2$. Then $$g = \sum \alpha_{ij} h_{ij}$$

Since Alt is linear and continuous, we have

$$\begin{aligned}
Alt(g) &= \sum \alpha_{ij} Alt(h_{ij}) = \sum \alpha_{ij} k_{ij} \\
&= \sum_{i<j} \alpha_{ij} k_{ij} + \underbrace{\sum \alpha_{ii} k_{ii}}_{=0} + \sum_{i>j} \alpha_{ij} k_{ij} \\
&= \sum_{i<j} \alpha_{ij} k_{ij} + \sum_{i<j} \alpha_{ji} \underbrace{k_{ji}}_{=-k_{ij}} \\
&= \sum_{i<j} \underbrace{(\alpha_{ij} - \alpha_{ji})}_{\in \ell^2} k_{ij}
\end{aligned}$$

Thus $Alt(g) \in H$. ∎

In conclusion, $Alt(L^2(X^2)) = L^2(X) \wedge L^2(X)$ and a basis of $Alt(L^2(X^2))$ is $\{ k_{ij} = Alt(h_{ij}) \,/\, i<j \}$. (orthonormal basis). The inner product on $Alt(L^2(X^2))$ is

$$\left\langle \sum_{i<j} \alpha_{ij} k_{ij} , \sum_{r<s} \beta_{rs} k_{rs} \right\rangle = \sum_{i<j} \alpha_{ij} \overline{\beta_{ij}} . \qquad (7)$$

We can easily generalize this conclusion to n-fold wedge product of $L^2(X)$ as follow:

$$L^2(X) \wedge \cdots \wedge L^2(X) = Alt(L^2(X^n)),$$

with an orthonormal basis $\{ k_{i_1 \dots i_n} = Alt(h_{i_1 \dots i_n}) \,/\, i_1 < \cdots < i_n \}$ where

$h_{i_1 \cdots i_n}(x_1, \dots, x_n) = f_{i_1}(x_1) \cdots f_{i_n}(x_n)$. and $Alt(f) = C \sum_{\sigma \in S_n} (-1)^\sigma f \circ R^\sigma$

The inner product on $Alt(L^2(X^n))$ is of the same fashion as (7).

$$\left\langle \sum_{i_1<\dots<i_n} \alpha_{i_1\dots i_n} k_{i_1\dots i_n}, \sum_{j_1<\dots<j_n} \beta_{j_1\dots j_n} k_{j_1\dots j_n} \right\rangle = \sum_{i_1<\dots<i_n} \alpha_{i_1\dots i_n} \overline{\beta_{i_1\dots i_n}}$$

One more thing important to note is that

$$\begin{aligned} k_{i_1\dots i_n}(x_1,\dots,x_n) &= Alt(h_{i_1\dots i_n})(x_1,\dots,x_n) \\ &= C\sum_{\sigma\in S_n} (-1)^\sigma h_{i_1\dots i_n}(R^\sigma(x_1\dots x_n)) \\ &= C\sum_{\sigma\in S_n} (-1)^\sigma h_{i_1\dots i_n}(x_{\sigma(1)},\dots,x_{\sigma(n)}) \\ &= C\sum_{\sigma\in S_n} (-1)^\sigma f_{i_1}(x_{\sigma(1)})\dots f_{i_n}(x_{\sigma(n)}) \qquad (8) \\ &= C\det(M_{i_1\dots i_n}) \end{aligned}$$

where $\quad [M_{i_1\dots i_n}]_{jk} = f_{i_j}(x_k)$

this determinant is called Slater determinant.

(9) If we choose $C = \frac{1}{\sqrt{N!}}$ then the inner product on $L^2(X)\wedge\dots\wedge L^2(X)$ $=\wedge^N L^2(X)$ is exactly a restriction of the ~~inner~~ usual inner product on $L^2(X^n)$. Consequently, $\wedge^N L^2(X)$ is simply a sub Hilbert space of $L^2(X^n)$ with an orthonormal basis $\{k_{i_1\dots i_N} / i_1<\dots<i_N\}$.

Proof Denote $\langle\cdot,\cdot\rangle_W$ the inner product on the wedge product $\wedge^N L^2(X)$, $\langle\cdot,\cdot\rangle_{L^2}$ the usual inner product on $L^2(X^n)$.

We know that $\langle k_{i_1\cdots i_N}, k_{j_1\cdots j_N}\rangle_W = \delta_{i_1 j_1}\cdots\delta_{i_N j_N}$

Thus all we have to do is to show that if $C = 1/\sqrt{N!}$ then

$$\langle k_{i_1\cdots i_N}, k_{j_1\cdots j_N}\rangle_{L^2} = \delta_{i_1 j_1}\cdots\delta_{i_N, j_N}$$

By (8), we have

$$k_{i_1\cdots i_N}(x_1,\dots,x_N) = C\sum_{\sigma\in S_N}(-1)^\sigma f_{i_1}(x_{\sigma(1)})\cdots f_{i_N}(x_{\sigma(N)})$$

$$= C\sum_{\sigma\in S_N}(-1)^\sigma f_{i\sigma(1)}(x_1)\cdots f_{i\sigma(N)}(x_N)$$

Similarly,

$$k_{j_1\cdots j_N}(x_1,\dots,x_N) = C\sum_{\sigma'\in S_N}(-1)^{\sigma'} f_{j\sigma'(1)}(x_1)\cdots f_{j\sigma'(N)}(x_N)$$

Thus,

$$k_{i_1\cdots i_N}(x_1,\dots,x_N)\,\overline{k_{j_1\cdots j_N}}(x_1,\dots,x_N) = C^2\sum_{\sigma,\sigma'\in S_N}(-1)^\sigma(-1)^{\sigma'} f_{i\sigma(1)}(x_1)\, \overline{f_{j\sigma'(1)}}(x_1)\cdots$$
$$f_{i\sigma(N)}(x_N)\,\overline{f_{j\sigma'(N)}}(x_N)$$

Hence

$$\langle k_{i_1\cdots i_N}, k_{j_1\cdots j_N}\rangle_{L^2} = \int k_{i_1\cdots i_N}(x_1,\dots,x_N)\,\overline{k_{j_1\cdots j_N}}(x_1,\dots,x_N)\,dx_1\cdots dx_N$$

$$= C^2\sum_{\sigma,\sigma'\in S_N}(-1)^\sigma(-1)^{\sigma'}\int f_{i\sigma(1)}(x_1)\,\overline{f_{j\sigma'(1)}}(x_1)\cdots f_{i\sigma(N)}(x_N)\,\overline{f_{j\sigma'(N)}}(x_N)\,dx_1\cdots dx_N$$

$$= C^2\sum_{\sigma,\sigma'\in S_N}(-1)^\sigma(-1)^{\sigma'}\left(\int f_{i\sigma(1)}(x_1)\,\overline{f_{j\sigma'(1)}}(x_1)\,dx_1\right)\cdots\left(\int f_{i\sigma(N)}(x_N)\,\overline{f_{j\sigma'(N)}}(x_N)\,dx_N\right)$$

$$= C^2 \sum_{\sigma,\sigma' \in S_N} (-1)^{\sigma}(-1)^{\sigma'} \delta_{i_{\sigma(1)}, j_{\sigma'(1)}} \cdots \delta_{i_{\sigma(N)} j_{\sigma'(N)}}$$

The summand is nonzero if and only if $(i_{\sigma(1)}, \dots, i_{\sigma(N)}) = (j_{\sigma'(1)}, \dots, j_{\sigma'(N)})$ which implies $\{i_1, \dots, i_N\} = \{j_1, \dots, j_N\}$. Because $i_1 < \dots < i_N$ and $j_1 < \dots < j_N$, it is imposible for a summand to be nonzero if $\{i_1, \dots, i_N\} \neq \{j_1, \dots, j_N\}$. ~~There~~ If $\{i_1, \dots, i_N\} = \{j_1, \dots, j_N\}$, i.e. $i_k = j_k \, \forall k$, then $\delta_{i_{\sigma(1)} j_{\sigma'(1)}} \cdots \delta_{i_{\sigma(N)} j_{\sigma'(N)}} \neq 0$ only for $\sigma = \sigma'$. Therefore,

- If $\{i_1, \dots, i_N\} \neq \{j_1 \dots, j_N\}$, $\quad \langle k_{i_1 \dots i_N}, k_{j_1 \dots j_N} \rangle_{L^2} = 0$
- If $\{i_1 \dots, i_N\} = \{j_1, \dots, j_N\}$,

$$\langle k_{i_1 \dots i_N}, k_{j_1 \dots j_N} \rangle_{L^2} = C^2 \sum_{\substack{\sigma,\sigma' \in S_N \\ \sigma = \sigma'}} (-1)^{\sigma}(-1)^{\sigma'} \delta_{i_{\sigma(1)}, i_{\sigma(1)}} \cdots \delta_{i_{\sigma(N)} i_{\sigma(N)}}$$

$$= C^2 (N!) = \left(\frac{1}{\sqrt{N!}}\right)^2 N! = 1$$

That means. $\langle k_{i_1 \dots i_N}, k_{j_1 \dots j_N} \rangle_W = \langle k_{i_1 \dots i_N}, k_{j_1 \dots j_N} \rangle_{L^2}$.