Stochastic cascade method for differential equations

Tuan Pham

Brigham Young University

February 28, 2022

Differential equations are everywhere!

$$y' = ay + b$$

- Population growth
- Mixing
- Radioactive decay
- Compound interest
- Newton's law of cooling
- ...

$$y' = ay^2 + by + c$$

- Population growth
- Chemical reaction
- Falling object
- Learning curve
- ...

Differential equations are everywhere!

Diffusion equation:

$$u_t - au_{xx} = f(x,t)$$

Wave equation:

$$u_{tt} - au_{xx} = f(x,t)$$

Minimal surface:

$$(1+u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1+u_y^2)u_{xx} = 0$$

Navier-Stokes equations:

$$u_t - \Delta u + u\nabla u + \nabla p = 0, \quad \operatorname{div} u = 0$$

Methods to solve a differential equation

- Integrating factor
- Separation of variables
- Power series
- Laplace transform
- Iteration method
- Discretization methods (finite difference/volume/element methods)
- ...

Methods to solve a differential equation

- Integrating factor
- Separation of variables
- Power series
- Laplace transform
- Iteration method
- Discretization methods (finite difference/volume/element methods)
- . . .
- Stochastic cascade method

History: Feynman, Kac, Itô (1940s, 1950s), McKean (1970s), Le Jan, Sznitman (1990s)

Some probability background

- Random variable: $X \in \mathbb{R}$

Figure 1: Samplings of X

- Expected value: $\mathbb{E}[X] = \int_{-\infty}^{\infty} x p(x) dx$

Some probability background

- Random variable: $X \in \mathbb{R}$

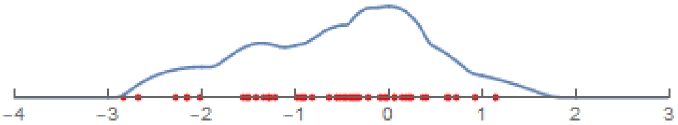


Figure 2: Probability density function $p(x)$ of X

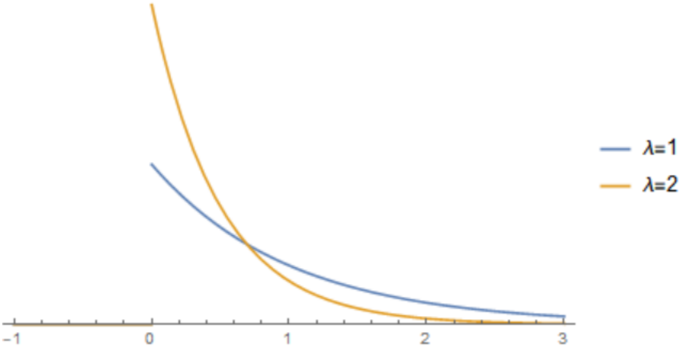
- Expected value: $\mathbb{E}[X] = \int_{-\infty}^{\infty} xp(x)dx$

Some probability background

Waiting time (with intensity λ): $T \sim \text{Exp}(\lambda)$

$$p(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

$$\mathbb{E}[T] = 1/\lambda$$



$$y' + y = f, \quad y(0) = y_0$$

Solution (integrating factor method):

$$y(t) = e^{-t} y_0 + \int_0^t e^{-s} f(t-s) ds$$

Equivalently, $y(t) = \mathbb{E}[X(t)]$ where

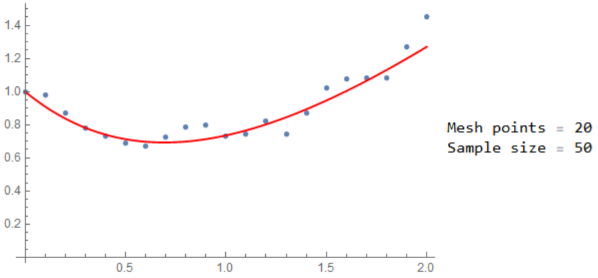
$$X(t) = \begin{cases} y_0 & \text{if} \quad T \geq t, \\ f(t-T) & \text{if} \quad T < t \end{cases}$$

and $T \sim \text{Exp}(1)$.

$$y' + y = t, \quad y(0) = 1$$

Exact solution: $y(t) = t - 1 + 2e^{-t}$
Stochastic cascade method: $y(t) = \mathbb{E}[X(t)]$

$$X(t) = \begin{cases} 1 & \text{if} \quad T \geq t, \\ t - T & \text{if} \quad T < t \end{cases}$$



Logistic equation $y' + y = y^2$, $y(0) = y_0$

$$y' + y = y^2, \quad y(0) = y_0$$

Using integrating factor, we get

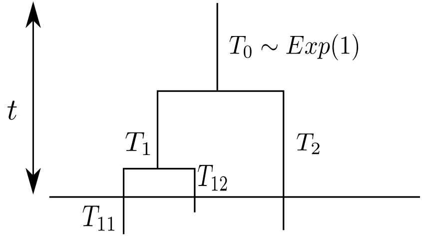
$$y(t) = e^{-t} y_0 + \int_0^t e^{-s} y^2(t-s) ds$$

Equivalently, $y(t) = \mathbb{E}[X(t)]$ where

$$X(t) = \begin{cases} y_0 & \text{if} \quad T \geq t, \\ X^{(1)}(t-T) X^{(2)}(t-T) & \text{if} \quad T < t. \end{cases}$$

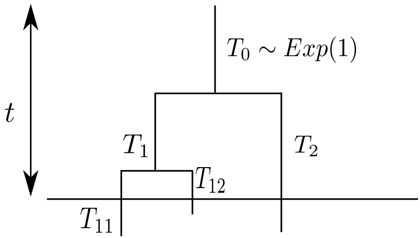
$X^{(1)}$ and $X^{(2)}$ are independent copies of X.

Logistic equation $y' + y = y^2$, $y(0) = y_0$



In this event, $X(t) = y_0^3$. In general, $X(t) = y_0^{N(t)}$.

Logistic equation $y' + y = y^2$, $y(0) = y_0$

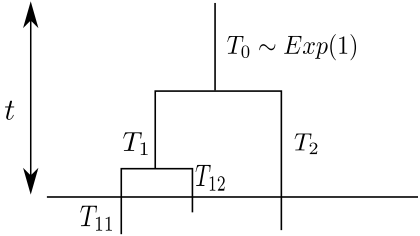


In this event, $X(t) = y_0^3$. In general, $X(t) = y_0^{N(t)}$.

Observations:

- If $-1 \leq y_0 \leq 1$, global solution
- If $y_0 > 1$, solution might blow up after finite time

Logistic equation $y' + y = y^2$, $y(0) = y_0$



In this event, $X(t) = y_0^3$. In general, $X(t) = y_0^{N(t)}$.

Observations:

- If $-1 \leq y_0 \leq 1$, global solution
- If $y_0 > 1$, solution might blow up after finite time

Compare with explicit solution: $y(t) = \dfrac{y_0}{y_0 - (y_0 - 1)e^t}$

α-Riccati equation (Athreya 1985, Dascaliuc et al. 2018)

$$y' + y = y^2(\alpha t), \quad y(0) = y_0$$

α-Riccati equation (Athreya 1985, Dascaliuc et al. 2018)

$$y' + y = y^2(\alpha t), \quad y(0) = y_0$$

Power series: $y = a_0 + a_1 t^2 + a_2 t^2 + a_3 t^3 + \dots$

$$\begin{cases} a_0 = y_0 \\ a_0 + a_1 = a_0^2 \\ a_1 + 2a_2 = 2a_0 a_1 \alpha \\ a_2 + 3a_3 = a_1^2 \alpha^2 + 2a_0 a_2 \\ \dots \end{cases}$$

α-Riccati equation (Athreya 1985, Dascaliuc et al. 2018)

$$y' + y = y^2(\alpha t), \quad y(0) = y_0$$

Power series: $y = a_0 + a_1 t^2 + a_2 t^2 + a_3 t^3 + \dots$

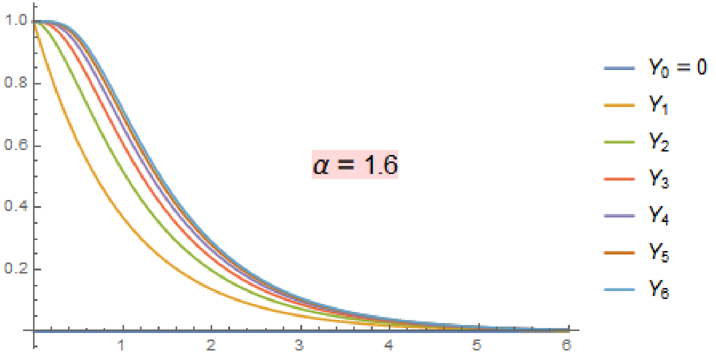
$$\begin{cases} a_0 = y_0 \\ a_0 + a_1 = a_0^2 \\ a_1 + 2a_2 = 2a_0 a_1 \alpha \\ a_2 + 3a_3 = a_1^2 \alpha^2 + 2a_0 a_2 \\ \dots \end{cases}$$

Is this power-series solution the only solution? Are there other solutions?

$$y' + y = y^2(\alpha t), \quad y(0) = 1$$

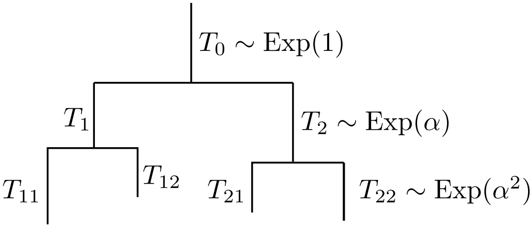
Integral form: $$y(t) = e^{-t} + \int_0^t e^{t-s} y^2(\alpha s) ds$$

Iteration: $$Y_n(t) = e^{-t} + \int_0^t e^{t-s} Y_{n-1}^2(\alpha s) ds, \quad Y_0(t) = 0$$



α-Riccati equation, stochastic cascade method

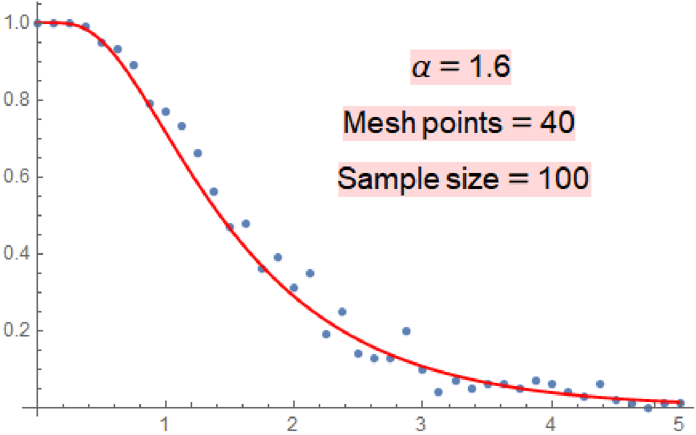
$$y' + y = y^2(\alpha t), \ \ y(0) = y_0$$



- $0 < \alpha \leq 1$: non-explosion
- $\alpha > 1$: explosion $\rightsquigarrow$ nonuniqueness of solutions

α-Riccati equation, Monte Carlo simulation

$$y(t) = \mathbb{E}[X(t)]$$



Other equations

- Reaction-diffusion equation:

$$u_t - a u_{xx} = b(x)u$$

- KPP-Fisher equation (1930s):

$$u_t - \frac{1}{2}u_{xx} = u^2 - u, \quad u(x,0) = u_0(x)$$

- Navier-Stokes equations:

$$u_t - \nu\Delta u + u\nabla u + \nabla p = 0, \quad \operatorname{div} u = 0$$

- Euler equation:

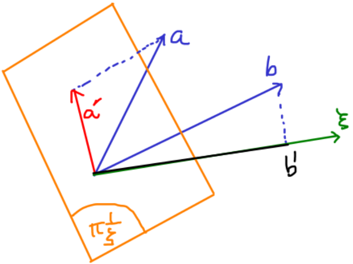
$$u_t + u\nabla u + \nabla p = 0, \quad \operatorname{div} u = 0$$

Navier-Stokes equations

$$\begin{cases} u_t - \Delta u + u\nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ \operatorname{div}\ u = 0 & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in} \quad \mathbb{R}^3. \end{cases}$$

In Fourier domain:

$$\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u}_0(\xi) + c\int_0^t e^{-|\xi|^2 s}|\xi| \int_{\mathbb{R}^3} \hat{u}(\eta, t-s) \odot_\xi \hat{u}(\xi-\eta, t-s)\, d\eta\, ds$$



$$a \odot_\xi b = -i\, b' a'$$

Normalized Navier-Stokes equations

Normalization (LeJan-Sznitman 1997): $v = c\hat{u}/h$

$$v(\xi,t) = e^{-t|\xi|^2} v_0(\xi) + \int_0^t e^{-s|\xi|^2} |\xi|^2 \int_{\mathbb{R}^3} v(\eta, t-s) \odot_\xi v(\xi-\eta, t-s) H(\eta|\xi) d\eta ds$$

where $H(\eta|\xi) = \frac{h(\eta)h(\xi-\eta)}{|\xi|h(\xi)}$ and $h * h = |\xi|h$.

Normalized Navier-Stokes equations

Normalization (LeJan-Sznitman 1997): $v = c\hat{u}/h$

$$v(\xi,t) = e^{-t|\xi|^2} v_0(\xi) + \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^3} v(\eta, t-s) \odot_\xi v(\xi-\eta, t-s) H(\eta|\xi) d\eta ds$$

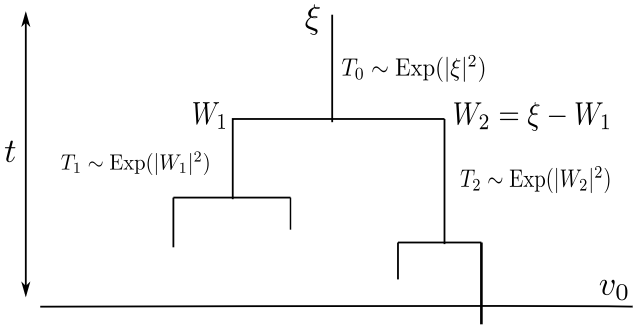
where $H(\eta|\xi) = \frac{h(\eta)h(\xi-\eta)}{|\xi|h(\xi)}$ and $h * h = |\xi|h$.

$v(\xi,t) = \mathbb{E}[X(\xi,t)]$ where

$$X(\xi,t) = \begin{cases} v_0(\xi) & \text{if} \quad T_0 \geq t, \\ X^{(1)}(W_1, t-T_0) \odot_\xi X^{(2)}(W_2, t-T_0) & \text{if} \quad T_0 < t. \end{cases}$$

$W_1 \sim H(\cdot|\xi)$ and $W_2 = \xi - W_1 \sim H(\cdot|\xi)$.

Stochastic cascade



- Bessel kernel $h(\xi) = c\,\frac{e^{-|\xi|}}{|\xi|} \rightsquigarrow$ non-explosion
- Self-similar kernel $h(\xi) = c|\xi|^{-2} \rightsquigarrow$ explosion

Dascaliuc, Pham, Thomann, Waymire (2021)

Non-explosion of Bessel cascade - Analytic approach

$$h(\xi) = c\frac{e^{-|\xi|}}{|\xi|}$$

Sketched proof: $w(\xi, t) = \mathbb{P}_\xi(\text{all paths cross horizon } t)$ solves

$$w(\xi, t) = e^{-t|\xi|^2} + \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^3} w(\eta, t-s)w(\xi-\eta, t-s)H(\eta|\xi)d\eta ds$$

We show that $w \equiv 1$ is a unique solution. Note that $w = c\hat{u}/h$ solves

$$\text{(MS)}: \;\; u_t - \Delta u = \sqrt{-\Delta}(u^2), \;\; u_0(x) = \frac{2}{|x|^2+1}$$

$$u = e^{\Delta t}u_0 + \int_0^t \sqrt{-\Delta}e^{\Delta(t-s)}u^2(s)ds$$

Kernel $G(t)$ satisfies $\|G(t)\|_{L^q} \lesssim t^{\frac{3}{2q}-2}$ for all $q \in [1, \infty]$.
By fixed-point argument, (MS) has a unique solution $u(x, t) = u_0(x)$.

Non-explosion of Bessel cascade - Applications

$$(\mathrm{MS})_a\colon\ \ u_t - \Delta u = \sqrt{-\Delta}(u^2),\ \ u_0(x) = \frac{2a}{|x|^2+1}$$

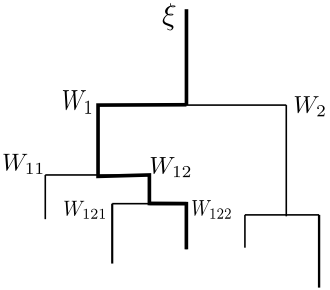
Dascaliuc, Pham, Thomann (2021):

- $a > 1$*: finite-time blowup solution*
- $-1 \leq a \leq 1$*: global solution*
- $-1 < a < 1$*: solution exponentially decays in time*

Under certain assumption, NSE has a minimal blowup initial data (Rusin-Sverak 2011, Jia-Sverak 2013, Gallagher et al 2016, Pham 2018,...), but MS doesn't have minimal blowup data.

Explosion of self-similar cascade, nonuniqueness

$$h(\xi) = c|\xi|^{-2}$$



Dascaliuc, Pham, Thomann, Waymire (2021):

$$\text{(MS): } u_t - \Delta u = \sqrt{-\Delta}(u^2), \; u_0(x) = \frac{2}{\pi}\frac{1}{|x|}$$

has at least two solutions: $u_1 = u_0(x)$ *and* $u_2 = c\mathcal{F}^{-1}\{|\xi|^{-2}\mathbb{P}_\xi(S > t)\}$.

Proposed research

1. Monte Carlo simulation
    - solution in Fourier domain,
    - suitable to study energy cascade,
    - can apply to various differential equations,
    - very costly,
    - Decoupling Principle can be used to reduce the cost,
    - explosion issue.
2. Stochastic cascade and mean-field models for turbulence
    - make precise the notion of averaging commonly used in empirical theories of energy cascade (Kolmogorov 5/3, Large Eddy Simulation,...)
    - depletion of nonlinearity ($\odot$-product) needs to be better understood,
    - Mean-field models that preserve the energy (dyadic shell model, Burgers equation) are good starting points.