Blowup solutions of a Navier-Stokes-like equation from a probabilistic perspective

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The ODE $u' + u = f$

$$\begin{cases} u' + u &= f, \\ u(0) &= u_0. \end{cases}$$

Integral form:

$$u(t) = e^{-t}u_0 + \int_0^t e^{-s} f(t-s)ds$$

In terms of probability:

$$u(t) = \mathbb{E}[u_0 I_{T\geq t}] + \mathbb{E}[f(t-T)I_{T<t}] = \mathbb{E}[u_0 I_{T\geq t} + f(t-T)I_{T<t}]$$

where T is a random variable exponentially distributed $T \sim Exp(1)$.

The ODE $u' + u = f$

Consider the random variable:

$$U(t) = \begin{cases} u_0 & \text{if} \quad T \geq t, \\ f(t-T) & \text{if} \quad T < t. \end{cases}$$

U can be approximated by sampling and taking average.

$$u(t) = \mathbb{E}[U(t)].$$

The ODE $u' + u = u^2$

$$\begin{cases} u' + u &= u^2, \\ u(0) &= u_0. \end{cases}$$

Integral form:

$$u(t) = e^{-t}u_0 + \int_0^t e^{-s}u^2(t-s)ds$$

In terms of probability:

$$u(t) = \mathbb{E}[u_0 I_{T\geq t}] + \mathbb{E}[u^2(t-T)I_{T<t}] = \mathbb{E}[u_0 I_{T\geq t} + u^2(t-T)I_{T<t}]$$

where $T \sim Exp(1)$.

The ODE $u' + u = u^2$

Define (recursively) the random variable

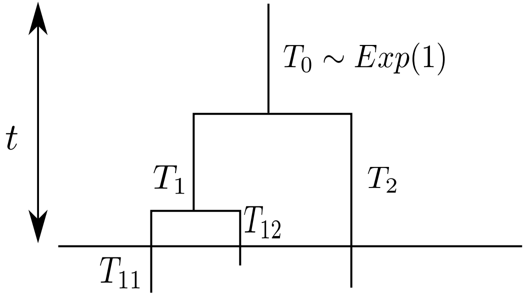
$$U(t) = \begin{cases} u_0 & \text{if} \quad T \geq t, \\ U^{(1)}(t-T)U^{(2)}(t-T) & \text{if} \quad T < t. \end{cases}$$

where $U^{(1)}$ and $U^{(2)}$ are independent copies of U.

$$u(t) = \mathbb{E}[U(t)].$$

We can describe U by the following branching process.

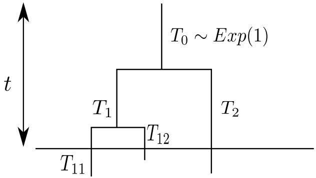
The ODE $u' + u = u^2$



In this event,

$$U(t) = u_0^3$$

The ODE $u' + u = u^2$



Comment:

- If $0 < u_0 < 1$, global solution (i.e. solution exists for all $t > 0$).

$$u(t) = \frac{ke^t}{ke^t + 1}$$

- If $u_0 > 1$, solution blows up after finite time.

$$u(t) = \frac{ke^t}{ke^t - 1}$$

Navier-Stokes equations

$$(\mathrm{NSE}): \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ \mathrm{div}\ u = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^d. \end{cases}$$

Integro-differential equation:

$$u(x,t) = e^{\Delta t} u_0 - \int_0^t e^{\Delta s} \mathbf{P} \mathrm{div}[u(t-s) \otimes u(t-s)] ds.$$

Navier-Stokes equations

$$\text{(NSE)}: \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^d \times (0, \infty), \\ \text{div } u = 0 & \text{in} \quad \mathbb{R}^d \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^d. \end{cases}$$

Integro-differential equation:

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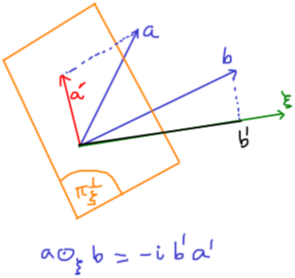
In Fourier domain:

$$\hat{u}(\xi, t) = e^{-|\xi|^2 t} \hat{u}_0(\xi) + c_0 \int_0^t e^{-|\xi|^2 s} |\xi| \int_{\mathbb{R}^d} \hat{u}(\eta, t-s) \odot_\xi \hat{u}(\xi - \eta, t-s) d\eta ds$$

where $a \odot_\xi b = -i(e_\xi \cdot b)(\pi_{\xi^\perp} a)$.

Fourier-transformed Navier-Stokes equations (FNS)



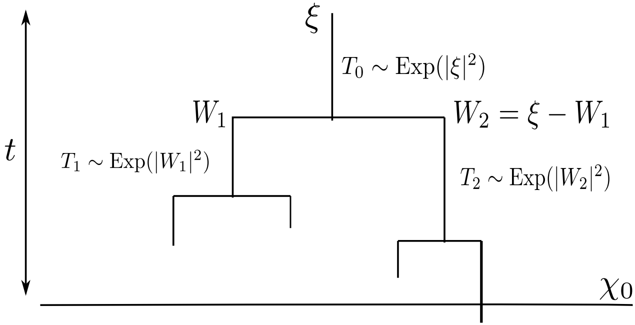
Normalization to (FNS): LJS 1997, Bhattacharya et al 2003.

$$
\begin{aligned}
\chi(\xi, t) &= e^{-t|\xi|^2}\chi_0(\xi) \\
&+ \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^d} \chi(\eta, t-s) \odot_\xi \chi(\xi-\eta, t-s) H(\eta|\xi) d\eta ds
\end{aligned}
$$

where $\chi = c_0 \hat{u}/h$ and $H(\eta|\xi) = \frac{h(\eta)h(\xi-\eta)}{|\xi| h(\xi)}$.

h: *majorizing kernel*, i.e. $h * h = |\xi| h$.

Cascade structure of FNS

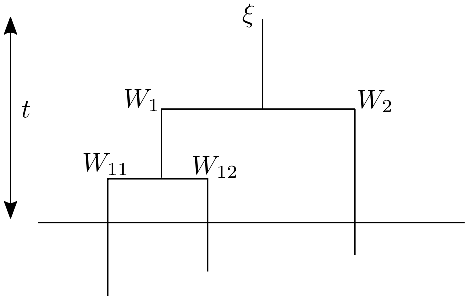


Define a stochastic multiplicative functional recursively as

$$\mathbf{X}_{\mathrm{FNS}}(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T_0 > t, \\ \mathbf{X}^{(1)}_{\mathrm{FNS}}(W_1, t - T_0) \odot_\xi \mathbf{X}^{(2)}_{\mathrm{FNS}}(\xi - W_1, t - T_0) & \text{if} \quad T_0 \le t. \end{cases}$$

An example of $\mathbf{X}_{\mathrm{FNS}}$

Consider the following event:



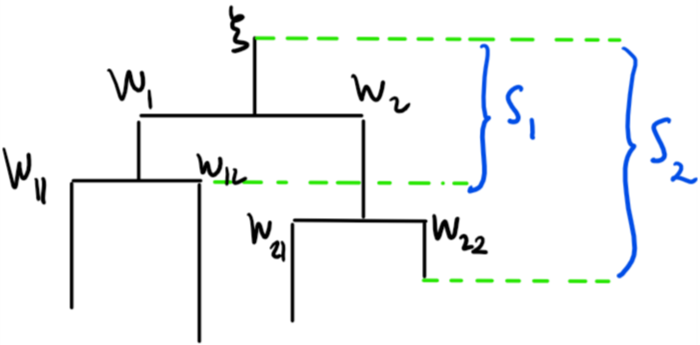
On this event,

$$\mathbf{X}_{\mathrm{FNS}}(\xi, t) = (\chi_0(W_{11}) \odot_{W_1} \chi_0(W_{12})) \odot_{\xi} \chi_0(W_2).$$

Three ingredients: clocks, branching process, product.
Cascade structure = clocks + branching process.

Stochastic explosion

$$S_n = \min_{|\nu|=n} \sum_{j=0}^{n} T_{\nu|j}, \qquad S = \lim_{n\to\infty} S_n = \sup_{n\in\mathbb{N}} S_n$$



Explosion event: $\{S < \infty\}$.

Non-explosion event : $\{S = \infty\}$.

Stochastic explosion

Branching process may never stop, potentially making $\mathbf{X}_{FNS}$ not well-defined.

- Property of cascade structure, not of product.
- Depending on the majorizing kernel h.
- 3D self-similar cascade $h_{\mathrm{dilog}}(\xi) = C|\xi|^{-2}$: stochastic explosion a.s. (Dascaliuc, Pham, Thomann, Waymire 2019)
- 3D Bessel cascade $h_{\mathrm{b}}(\xi) = C|\xi|^{-1}e^{-|\xi|}$: non-explosive a.s. (Orum, Pham 2019)

Cheap Navier-Stokes equation

$$\text{(cNSE)}: \begin{cases} \partial_t u - \Delta u = \sqrt{-\Delta}(u^2) & \text{in } \mathbb{R}^d \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^d \end{cases}$$

With $\chi = c_0 \hat{u}/h$, we have

$$\begin{aligned} \chi(\xi,t) &= e^{-t|\xi|^2}\chi_0(\xi) \\ &+ \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^d} \chi(\eta, t-s)\chi(\xi-\eta, t-s)H(\eta|\xi)d\eta ds \\ &= \mathbb{E}[\mathbf{X}(\xi,t)] \end{aligned}$$

where

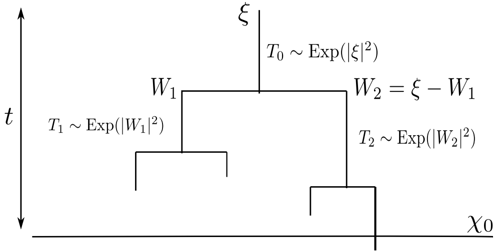
$$\mathbf{X}(\xi,t) = \begin{cases} \chi_0(\xi) & \text{if } T_0 > t, \\ \mathbf{X}^{(1)}(W_1, t-T_0)\mathbf{X}^{(2)}(\xi - W_1, t - T_0) & \text{if } T_0 \le t. \end{cases}$$

Cascade solutions

Consider $\chi_0(\xi) \equiv \gamma > 0$. In case of non-explosion (e.g. the Bessel cascade),

$$\boxed{\chi(\xi, t) = \sum_{n=1}^{\infty} \gamma^n p_n(\xi, t)}$$

$p_n(\xi, t) = \mathbb{P}(S_\xi > t$, exactly n branches cross).



Cheap NSE in 3D

Bessel majorizing kernel: $h = h_{\mathrm{b}}(\xi) = \frac{1}{2\pi}\frac{e^{-|\xi|}}{|\xi|}$.

$$\text{(cNSE)} : \begin{cases} \partial_t u - \Delta u = \sqrt{-\Delta}(u^2) & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(\cdot,0) = \frac{2\gamma}{1+|x|^2} & \text{in } \mathbb{R}^3 \end{cases}$$

Dascaliuc, Orum, Pham (2020)

- If $0 \le \gamma < 1$, (cNSE) has a unique solution in $L^5(\mathbb{R}^3 \times (0,\infty))$.
- If $\gamma = 1$, (cNSE) has a time-independent solution in $L^5(\mathbb{R}^3 \times (0,T))$ for any $T < \infty$.
- If $\gamma > 1$, u blows up after finite time.

Cheap NSE in 3D when $\gamma < 1$

$p_n(\xi, t) = \mathbb{P}(S_\xi > t$, exactly n branches cross).
By conditioning on the first time of branching, we get

$$p_n(\xi, t) = \int_0^t |\xi|^2 e^{-s|\xi|^2} \int_{\mathbb{R}^3} \sum_{k=1}^{n-1} p_k(\eta, t-s) p_{n-k}(\xi - \eta, t-s) H(\eta|\xi) d\eta ds$$

By induction, one can prove

$$p_n(\xi, t) \leq \min\{1, \theta \lambda^{n-1} C_n e^{-|\xi|\sqrt{t}}\},$$

If $\gamma < 1$, choose κ small such that $4^\kappa \lambda^\kappa \gamma < 1$.

$$\chi(\xi, t) = \sum_{n=1}^{\infty} \gamma^n p_n(\xi, t) \lesssim \sum_{n=1}^{\infty} (\underbrace{4^\kappa \lambda^\kappa \gamma}_{<1})^n e^{-\kappa|\xi|\sqrt{t}}.$$

Cheap NSE in 3D when $\gamma > 1$

$$\gamma \rightsquigarrow \mathbf{X}^{(1)} \rightsquigarrow \chi_1 = \mathbb{E}[\mathbf{X}^{(1)}],$$
$$\frac{1}{\gamma} \rightsquigarrow \mathbf{X}^{(2)} \rightsquigarrow \chi_2 = \mathbb{E}[\mathbf{X}^{(2)}],$$
$$\sqrt{\gamma} \rightsquigarrow \mathbf{X}^{(3)} \rightsquigarrow \chi_3 = \mathbb{E}[\mathbf{X}^{(3)}],$$
$$\frac{1}{\sqrt{\gamma}} \rightsquigarrow \mathbf{X}^{(4)} \rightsquigarrow \chi_4 = \mathbb{E}[\mathbf{X}^{(4)}].$$

Observation: $\mathbf{X}^{(1)}\mathbf{X}^{(2)} = \mathbf{X}^{(3)}\mathbf{X}^{(4)} = 1$, $\mathbf{X}^{(3)} = \sqrt{\mathbf{X}^{(1)}}$, $\mathbf{X}^{(2)} = \sqrt{\mathbf{X}^{(4)}}$.

$$\chi = \chi_1 \geq \frac{1}{\chi_2} \gtrsim e^{\kappa|\xi|\sqrt{t}} \quad \text{(Schwarz's inequality)}$$

Thus, $\hat{u} \sim \chi h \gtrsim e^{(\kappa\sqrt{t}-1)|\xi|} \Rightarrow u$ blows up by time $t = \kappa^{-2}$.

Thank You!