Analysis seminar (11/26/2018)

* Blowup phenomenon in PDE:

$$\begin{cases} \partial_t u - \Delta u = u^\sigma & \text{in } \mathbb{R}^n \\ u(0) = u_0 \geq 0 \end{cases}$$

Note that the initial condition guarantees that the solution must stay nonnegative over time. Why?

$$u = \underbrace{U}_{\Gamma(t) * u_0} + F(u)$$

Iteration: $u_{n+1} = U + F(u_n) \geq 0 \quad \forall n$

This phenomenon can be seen from stochastic cascade point of view.

Fujita's exponent (can be seen via scaling)

$1 \qquad\qquad 1+\frac{2}{n}$

blowup if $u_0 \not\equiv 0$ — global solution if u_0 is small $\left(u_0(x) \leq \varepsilon e^{-|x|^2}\right)$; blow up if u_0 is large (in some sense)

(See [1, chapter 5] for a survey)

By blowup in finite time, we mean $\|u(t)\|_{L^\infty_x} \to \infty$ as $t \to T_*^-$.

same scaling as NSE in 1D, which doesn't blow up.

For $n=1$: $\sigma = 3$. A simple proof is due to Kaplan (1963)

$$(\mathrm{I}): \begin{cases} u_t - u_{xx} = u^3 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Let $c(t)$ be the $\frac{c_1}{2}$, 1^{st} coefficient of $u(x,t)$ in the Fourier series:

$$u(x,t) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$$

$$c(t) = \int_0^1 u(x,t) \sin(\pi x)\, dx$$

$$u_{xx} = \sum_{k=1}^{\infty} -c_k \pi^2 \sin(k\pi x)$$

First coefficient of u_{xx} is $-c_1\pi^2 = -2c\pi^2$.

" u^3 is $2\int_0^1 u^3 \sin(\pi x)\,dx = 2\gamma$

Estimate: $$c(t) = \int_0^1 u \sin(\pi x)\,dx \le \left\{\int_0^1 u^3 \sin(\pi x)\,dx\right\}^{1/3} \left\{\int_0^1 \sin(\pi x)\,dx\right\}^{2/3}$$
$$= c_0 \gamma^{1/3}$$

Equating the first Fourier coefficients:
$$c' + c\pi^2 = \gamma \ge c_0^{-3} c^3$$
$$\Rightarrow c' \ge c_0^{-3} c^3 - c\pi^2$$

This ODE blows up if $c(0)$ is large. (this condition is diff. from Fujita's condition)
(Proof's version taken from [2])

* Stationary solutions can be computed explicitly as follows:
(0 is not the only stationary solution) .... whereas stationary NSE in 1D has unique stationary sol.

$$\begin{cases} u'' = -u^3 \\ u(0) = u(1) = 0 \end{cases}$$

Multiply both sides by u':
$$\frac{d}{dt}\left(\frac{1}{2}(u')^2\right) = -\frac{d}{dt}\left(\frac{1}{4}u^2\right)$$

$$\Rightarrow \frac{1}{2}(u')^2 + \frac{1}{4}u^2 = C^2$$

$$\Rightarrow \begin{cases} u' = C\sqrt{2}\cos\theta \\ u^2 = 2C\sin\theta \end{cases}$$

$\theta(0) = 0, \quad \theta(1) = \pi.$

$$u = \sqrt{2C}\sqrt{\sin\theta}$$

$$\Rightarrow u' = \sqrt{2C}\,\frac{\theta'\cos\theta}{2\sqrt{\sin\theta}} = \frac{1}{2\sqrt{C}}\,\frac{\theta' u'}{\sqrt{\sin\theta}}$$

$$\Rightarrow \theta' = 2\sqrt{C}\sqrt{\sin\theta}$$

$$\int_0^{\theta} \frac{ds}{\sqrt{\sin s}} = 2\sqrt{C}x$$

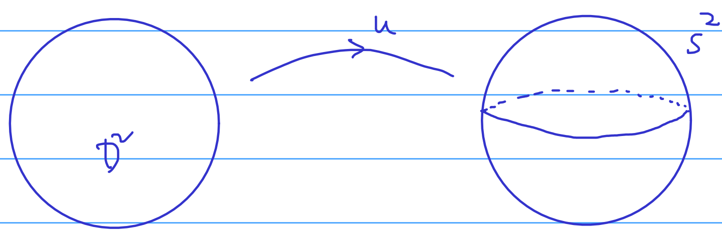
For $x = \pi$:
$$\int_0^{\pi} \frac{ds}{\sqrt{\sin s}} = 2\sqrt{C}\,\pi$$

$\rightarrow$ get value of C.

* Harmonic map heat flow:

$u: D^2 \longrightarrow S^2$

$$\begin{cases} \partial_t u - \Delta u = |\nabla u|^2 u \\ u|_{\partial D^2} = u_0|_{\partial D^2} \\ u(0) = u_0 \end{cases}$$



Consider solutions of the form:

$$u(x,t) = \left(\frac{x}{r}\sinh(r,t),\ \cosh(r,t)\right)$$

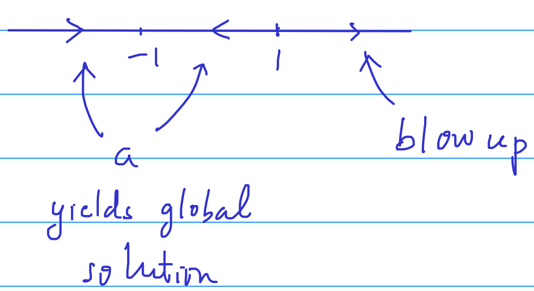
Result by Chang - Ding - Ye (1992):

- If $|h_0(r)| \leq \pi$ $\forall r \leq 1$ then $\exists$ global solution
- If $|h_0(1)| > \pi$ then the solution blows up.

$\cdots\rightarrow$ This is the case when minimal data for blow up doesn't exist.

In fact, it is quite hard to find an ODE that has minimal blow up data.

$$u' = u^2 - 1, \quad u(0) = a$$



* Local regularity of NSE:

$$\begin{cases} \partial_t u - \Delta u + u\nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u(0) = u_0 \end{cases}$$

We know that if u blows up after finite time, $\|u\|_{L^5_{t,x}} \to 0$.
What can we say about local regularity?
For example, can $u \in L^5_{t,x}(Q_r(x_0, T_*))$ for every $r>0$, $x_0 \in \mathbb{R}^3$?

Def:

$z = (x,t) \in \mathbb{R}^3 \times \mathbb{R}$ is called a singular point of u if

$$\|u\|_{L^\infty(Q_r(z))} = \infty \quad \forall\, r>0$$

$Q_r(z) = B_r(x) \times (t - r^2, t)$ .... parabolic cylinder.

Known: $u \in L^r_t L^q_x (Q_1(z)) \Rightarrow z$ is regular point.

$\frac{2}{r} + \frac{n}{q} < 1$, $q > n$: Serrin (1962-63)

$\frac{2}{r} + \frac{n}{q} = 1$, $q > n$: Struwe (1988), Takahashi (1990)

$r = \infty$, $q = 3$, $n = 3$: Sverak et. al. (2003)

Serrin's result is quite classical. The pressure term causes difficulty in regularity theory because it is non-local:

$$\Delta p = -\operatorname{div}(u\nabla u)$$

$$\Rightarrow p = -\Delta^{-1} \operatorname{div}\operatorname{div}(u \otimes u) = k * (u \otimes u)$$

where $k(x) = C\, \nabla^2\left(\frac{1}{|x|^{n-2}}\right) \sim |x|^{-n}$ as $x \to \infty$.

Pressure in a local region is influenced by the velocity field from far away. A usual way to overcome this difficulty is to introduce $\omega = \operatorname{curl} u$ (vorticity).

Take curl of both sides of NSE:

$$\partial_t \omega - \Delta \omega = \operatorname{div}(u \otimes \omega - \omega \otimes u)$$

How to get back u?

$$\operatorname{curl} \underbrace{\operatorname{curl} u}_{\omega} = \nabla(\operatorname{div} u) - \Delta u = -\Delta u$$

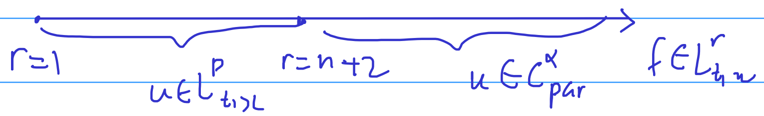
$$\Rightarrow \quad u = (-\Delta)^{-1} \operatorname{curl} \omega \qquad \text{(Biot-Savart's law)}$$

Recall: (see [4])

$$\partial_t \omega - \Delta \omega = \operatorname{div} f \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

$$f \in L^r_{t,x}, \quad \omega \in L^p_{t,x}$$

Scaling:
$$\frac{1}{p} = \frac{1}{r} - \frac{1}{n+2}$$



If $u \in L^m_{t,x}$ with $m > n+2$ and $\omega \in L^{q_1}_{t,x}$ then $u \otimes \omega \in L^q$ with

$$\frac{1}{q} = \frac{1}{m} + \frac{1}{q_1}$$

By the above embedding result, $\omega \in L^{q_2}$ with

$$\frac{1}{q_2} = \frac{1}{q} - \frac{1}{n+2} = \frac{1}{q_1} + \underbrace{\left(\frac{1}{m} - \frac{1}{n+2}\right)}_{<0} < \frac{1}{q_1}$$

Doing this bootstrapping k times:

$$\frac{1}{q_k} = \frac{1}{q_1} + (k-1)\left(\frac{1}{m} - \frac{1}{n+2}\right)$$

until $q_k > n+2$. Then ω is in fact Hölder continuous.

The techniques in [5] are inspired by Lin's 1998. To see the idea, let's consider a simplied model of NSE:

Time-dependent NSE in 3D has the same scaling property as stationary NSE in 5D (time is "absorbed" into space, and is counted as 2 spatial variable).

$$-\Delta u + u\nabla u + \nabla p = 0$$

Struwe (1995) showed that NSE in 5D is globally well-posed.
$\rightsquigarrow$ not very good toy model in the sense that pressure is still present.

$$i\partial_t u + (1 - i\sigma)\Delta u + |u|^2 u = 0 \quad \text{(in } \mathbb{R}^3\text{)} \quad \ldots\text{ Complex Ginzburg-Landau equation.}$$

This equation satisfies energy identity:

$$\int_{\mathbb{R}^3} \frac{1}{2}|u|^2 dx + \sigma \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx ds = \int_{\mathbb{R}^3} \frac{1}{2} |u_0|^2$$

and has the same scaling property as 3D NSE.
This equation has blow up solutions.

For the purpose of investigating local theory, let's consider the case $\sigma = 0$, and u is time-independent:

$$\Delta u + |u|^2 u = 0 \qquad \text{in } \mathbb{R}^n$$

This equation has a singular solution: $u(x) = \frac{\sqrt{n-3}}{r} \vec{e} \quad (r = |x|)$

$\vec{e}$: const. unit vect.

The quantity

$$I(r) = \frac{1}{r^{n-3}} \int_{B_r} |u - (u)_{B_r}|^3 dx$$

cannot be made arbitrarily small by shrinking r.

⊛ Theorem: For $n=3$, fix $\tau = \frac{1}{2}$.

There exist $\varepsilon > 0$, $C > 0$ such that if

$$I(x,r) < \varepsilon$$

then $\quad I(x,\tau r) \le C\tau^6 I(x,r)$

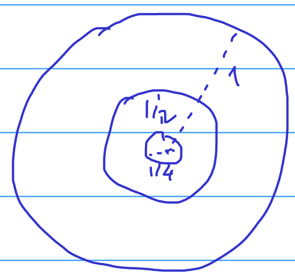
Here $\quad I(x,r) = \int_{B_{x,r}} |u-(u)_{B_{x,r}}|^3 dx$ ..... scaling-invariant

Campanato's integral characterization of Hölder spaces:

If $\quad \int_{B_{x,r}} |u-(u)_{B_{x,r}}|^p dy \le C r^{p\alpha} \quad \forall r<r_0, \ \forall x \in B_1$

Then $\quad u \in C^\alpha(\overline{B}_1)$.

Take $x=0$ and $r=1$ for simplicity.



For $1/2 \le r \le 1$,

$$I_r = \int_{B_r} |u-(u)_{B_r}|^3 dx$$

Control $(u)_{B_r}$ in terms of $(u)_{B_1}$:

$$\begin{aligned}
|(u)_{B_r}| - |(u)_{B_1}| &\le |(u)_{B_r} - (u)_{B_1}| \\
&= \left| \frac{1}{|B_r|} \int_{B_r} (u-(u)_{B_1})\, dy \right| \\
&\le \frac{1}{|B_r|} \int_{B_r} |u-(u)_{B_1}|\, dy \\
&\le r^{-3} \frac{1}{|B_1|} \int_{B_1} |u-(u)_{B_1}|\, dy \\
&\le C_0 r^{-3} I_1^{1/3} \\
&\le C_0 I_1^{1/3}
\end{aligned}$$

It's then easy to show that $I_r \leq C_0 I_1$ for some abs. constant C_0, for all $r \in (\frac{1}{2}, 1)$.

Then by the theorem above:

$$I_{\tau^k r} \leq (C\tau^{6})^k I_1 \qquad \forall r \in (\tau, 1)$$

Then

$$I_\rho \leq C\rho^6 I_1 \qquad \forall \rho \in (0, 1)$$

This means:

$$\int_{B_{x,\rho}} |u - (u)_{B_{x,\rho}}|^3 \, dy \leq C\rho^3 \qquad \forall x \in B_{1/2}$$

This u is α- Hölder continuous $\forall\, 0 < \alpha < 1$.

<u>Theorem</u>: (C-K-N 1982, Lin 1998)

$\exists\, \varepsilon > 0$, $C > 0$ such that if

$$\frac{1}{r^2} \int_{Q_r} (|u|^3 + |p|^{3/2}) < \varepsilon$$

then u is Hölder cont. in $Q_{r/2}$ and $\|u\|_{L^\infty(Q_{r/2})} \leq \frac{C}{r}$.

* Consequences:

If $u \in L^5(Q_1)$ and $p \in L^{5/2}(Q_1)$ then

$$\frac{1}{r^2} \int_{Q_r} |u|^3 \overset{\text{Holder}}{\leq} C \left\{ \int_{Q_r} |u|^5 \right\}^{3/5} \longrightarrow 0 \quad \text{as } r \to 0$$

$$\frac{1}{r^2} \int_{Q_r} |p|^{3/2} \overset{\text{Holder}}{\leq} C \left\{ \int_{Q_r} |p|^{5/2} \right\}^{2/5} \longrightarrow 0 \quad \text{as } r \to 0$$

Thus, u is regular in $Q_{1/2}$.

* sw_solution: $u = v + w$

v solves Stokes with data f, u_0

w: right-hand-side $= -u\cdot\nabla u$

$\sqrt{t}\, v \in L^\infty_{t,x}$

$$\int_{\mathbb{R}^3} \frac{1}{2}|w|^2 + \int_0^t\int_{\mathbb{R}^3} |\nabla w|^2 \,dx\,ds \le \Phi\left(\|f\|_{L^{5/3}}, \|u_0\|_{L^3}\right)$$

$$\int_0^t\int_{\mathbb{R}^3} |\pi|^{3/2} \,dx\,ds \le \Psi\left(\|f\|_{L^{5/3}}, \|u_0\|_{L^3}\right)$$

Then

$$\int_{Q_1(x_0,T_*)} \left(|w|^3 + |\pi|^{3/2}\right) \longrightarrow 0 \quad \text{as} \quad x_0 \to \infty.$$

Thus, w is bounded on $(\mathbb{R}^3 \setminus B_R) \times (0, T_*)$.

* If $u \in L^5_{loc}(\mathbb{R}^3 \times (0,T_*))$ then $u \in L^5_{uloc}(\mathbb{R}^3 \times (0,T_*))$.

Then on each $Q_1(x_0,T_*)$, one can adjust the pressure (add a quantity only depending on t) so that $p \in L^{5/2}(Q_1(x_0,T_*))$.

Then u is bounded in Q_1.

Then $u \in L^\infty(\mathbb{R}^3 \times (0,T_*))$.

Then $w \in L^\infty(\mathbb{R}^3 \times (\tfrac{1}{2}T_*, T_*)) \cap L^2$

Then $w \in L^5(\_\_\_\_)$

Then $u \in L^5(\_\_\_\_) \rightsquigarrow$ contradiction.

One cannot do such shrinking procedure for the case $u \in L^\infty_t L^3_x(Q_1)$.
This case needs more analysis. [5]

Seregin shows the same result for half space (boundary regularity) [8]

Albritton-Barker ..... domain with curved boundary [9]

How to prove theorem (*)? Let consider $n=3$.

Suppose it is not true.

$$\exists\, \varepsilon_k \downarrow 0 : \qquad I_1^{(k)} = \int_{B_1} |u^k - (u^k)_{B_1}|^3 dy = \varepsilon_k$$

$$\text{and} \quad I_\tau^{(k)} \geq C\tau^6 \varepsilon_k$$

constant to be determined

We'll denote $\varepsilon_k^{1/3}$ by ε for simplicity

u^k by u

$$v = \frac{u - (u)_{B_1}}{\varepsilon}$$

Then
$$\boxed{\int_{B_1} |v|^3 dy = 1, \qquad (v)_{B_1} = 0}$$

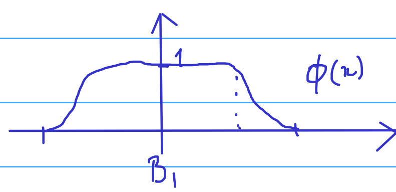
$$I_\tau^{(k)} = \int_{B_\tau} |v - (v)_\tau|^3 dy \geq C\tau^6$$

$$\Rightarrow \quad \boxed{\fint_{B_\tau} |v - (v)_\tau|^3 dy \geq C\tau^3}$$

The eq. $\Delta u + |u|^2 u = 0$ becomes

$$0 = \varepsilon \Delta v + |a + \varepsilon v|^2 (a + \varepsilon v)$$
$$= \varepsilon \Delta v + \varepsilon^3 |v|^2 v + 2\varepsilon (a\cdot v) a + 2\varepsilon^2 (a\cdot v) v + \varepsilon^2 |v|^2 a$$
$$+ \varepsilon |a|^2 v + |a|^2 a \qquad (**)$$

Choose test function

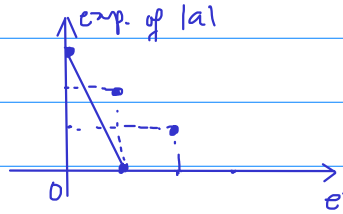


Then $$\varepsilon\int_{B_1} v\Delta\phi + \varepsilon^3\int_{B_1}|v|^2 v\phi + \cdots + |a|^2 a\int_{B_1}\phi = 0$$

Use the fact that $\int_{B_1}|v|^3 = 1$:

$$C_0|a|^3 \lesssim \varepsilon C_1|a|^2 + \varepsilon^2 C_2|a| + \varepsilon$$

Newton's polygon:



We get $|a| \lesssim \varepsilon^{1/3}$

Write $a = \varepsilon^{1/3} b$, where $|b| \le C$. Then (**) becomes

$$\varepsilon\Delta v + \varepsilon\,|b+\varepsilon^{2/3}v|^2(b+\varepsilon^{2/3}v) = 0$$

then $\Delta v = -|b|^2 b + o(\varepsilon)$

There is a sequence $\varepsilon_n \downarrow 0$ such that v^{ε_n} converges weakly to $L^3(B_1)$ (call the limit function v_0).

Then $\Delta v^{\varepsilon_n} \longrightarrow \Delta v$ and thus $\Delta v_0 = -|b_0|^2 b_0$ in B_1

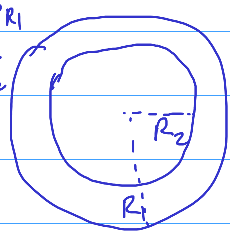
We want to have strong convergence in $L^3(B_1)$.

Caccipoli's inequality for harmonic function:

$$\Delta u = b \Rightarrow \int_{B_{R_2}}|\nabla u|^2 \le \frac{C}{(R_1-R_2)^2}\int_{B_{R_1}}|u|^2 + \int_{B_{R_1}} bu$$

$$\eta = \begin{cases} 1 & \dots B_{R_1} \\ 0 & \dots B_{R_2}^c \end{cases}$$

$$|\nabla\eta| \lesssim \frac{C}{R_1-R_2}$$



why? $$0 = \int_{B_1}(\Delta u)u\eta^2 = -\int_{B_1}\nabla u\cdot\nabla(u\eta^2) + \int_{B_1} bu\eta^2$$

$$= -\int|\nabla u|^2\eta^2 - 2\int \eta u\nabla u\,\nabla\eta + \int bu\eta^2$$

$$\Rightarrow \int_{B_1}|\nabla u|^2\eta^2 = -2\int_{B_1}\eta u\nabla u\,\nabla\eta + \int_{B_1} bu\eta^2$$

$$\leq 2\int_{B_1}\left(4|u|^2|\nabla\eta|^2+\frac{1}{4}|\nabla u|^2\eta^2\right)+\int_{B_1} bu\eta^2$$

Then $\displaystyle\int_{B_1}|\nabla u|^2\eta^2 \leq 16\int_{B_1}|u|^2|\nabla\eta|^2 \leq \frac{C}{(R_1-R_2)^2}\int_{B_1}|u|^2+\int_{B_1} bu$

We will use a perturbation of this technique:

$$\underbrace{-\int_{B_1}(\Delta v)\, v\eta^2}_{\displaystyle\int_{B_1}|\nabla v|^2\eta^2+\underbrace{\int_{B_1}2v\eta\nabla v\nabla\eta}_{\displaystyle 1.1\lesssim\frac{1}{(R_1-R_2)^2}\underbrace{\int_{B_1}v^2}_{\leq C}+\frac{1}{2}\int|\nabla v|^2\eta^2}} = \int_{B_1}|b+\varepsilon^{2/3}v|^2(b+\varepsilon^{2/3}v)\,v\eta^2 \qquad \begin{pmatrix}R_1=1\\ R_2<1\end{pmatrix}$$

Then $\displaystyle\int_{B_1}|\nabla v|^2\eta^2 \leq \frac{C}{(R_1-R_2)^2}+\int_{B_1}|b+\varepsilon^{2/3}v|^2(b+\varepsilon^{2/3}v)\,v\eta^2$

The only term we worry about on the right hand side is

$$\int_{B_1}v^4\eta^2=\int_{B_1}v^2(v\eta)^2\overset{\text{Hölder}}{\leq}\underbrace{\left\{\int_{B_1}|v|^3\right\}^{2/3}}_{\leq 1}\underbrace{\left\{\int_{B_1}(v\eta)^6\right\}^{1/3}}_{\leq\|\nabla(v\eta)\|_{L^2}^2}$$

$$\leq 2\int_{B_1}|\nabla v|^2\eta^2+2\underbrace{\int_{B_1}|v|^2}_{\leq C}$$

∇v is bounded in $L^2(B_{R_2})$

Compact embedding: $W^{1,2}\hookrightarrow L^3$

Convergent: $v^{\varepsilon_n}\to v_0$ in $L^3(B_{R_2})$

$v_0+|b|^2 b_0(1-|b|^2)$ is harmonic in B_1, so

$$\fint_{B_r}|v_0-(v_0)_{B_r}|^3 \lesssim C_0 r^3$$

If $n=4$ or 5, one needs to consider the complex Ginzburg-Landau eqs:

$$(1-\sigma i)\Delta u + |u|^2 u = 0 \qquad (***)$$

because $\begin{cases} W^{1,2} \hookrightarrow L^4 & \text{for } n=4 \\ W^{1,2} \hookrightarrow L^{10/3} & \text{for } n=5 \end{cases}$

The term $\int_{B_1} v^4$ can't be bounded by $\|v\|_{W^{1,2}}$ if $n=5$.

The complex version ($\sigma \neq 0$) will help us eliminate this term: multiply both sides of $(***)$ by $\bar{u}$ and take the imaginary part.

$\rightarrow$ this will get rid of $\int |u|^4$.

References:

[1] Bei Hu "Blow up theories for semilinear parabolic eqs." 2011

[2] John Hunter's lecture notes. UC Davis.

[3] Vladimir Sverak's lecture notes. University of Minnesota.

[4] Ladyzhenskaiya, Solonnikov, Ural'ceva book, 1968.

[5] Escauriaza - Seregin - Sverak 2003

[6] F. Lin "A new proof for partial regularity of NSE" 1998

[7] Struwe "Regular solution of NSE in $\mathbb{R}^5$" 1995.

[8] Seregin "On smoothness of $L^{3,\infty}$-sol. of NSE up to the boundary" 2005

[9] Albritton - Barker 2018