# Conservation of frequencies and applications to the well-posedness problem of the Navier-Stokes equations

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CUNY GC Harmonic Analysis and PDE Seminar

Dec 10, 2021

# Outline

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\text{(NSE)} : \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^d \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in} \quad \mathbb{R}^d \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^d. \end{cases}
$$

Outline:

- Brief overview on the global regularity and blowup.
- Conservation of frequencies and applications to well-posedness problems.
- Conservation of frequencies through the lens of stochastic branching processes.

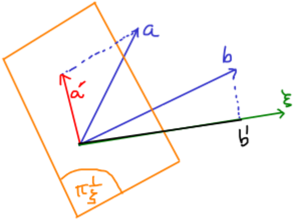
# Global regularity and blowup

- Mild solutions are solutions obtained by Picard's iteration $u_{n+1} = e^{\Delta t} u_0 + B(u_n, u_n)$. Regularity of $u$ can be as good as $U$, e.g. smooth, analytic, etc.
- Local existence if $u_0$ is in scale-subcritical spaces. Global existence if $u_0$ is in scale-critical spaces.
- Finite time blowup is unknown. Finite time blowup, if exists, happens simultaneously in many spaces: $L_t^p L_x^q$, homogeneous Sobolev, Besov, etc.
- The energy identity plays the role of a mechanism to prevent blowup.
- Model equations that exhibit blowup: Montgomery-Smith (2001), Gallagher-Paicu (2009), Sinai-Li (2008), Tao (2015).

# Fourier-transformed Navier-Stokes equations (FNS)

$$v(\xi,t) = e^{-|\xi|^2 t} v_0(\xi) + c_0 \int_0^t e^{-|\xi|^2 s} |\xi| \int_{\mathbb{R}^d} v(\eta, t-s) \odot_\xi v(\xi-\eta, t-s)\, d\eta\, ds$$

where $v = \hat{u}$ and $a \odot_\xi b = -i(e_\xi \cdot b)(\pi_{\xi^\perp} a)$.




$a \odot_\xi b = -i\, b' a'$

# Global regularity and blowup

- Li-Ozawa-Wang (2018): if $\mathrm{supp}\,\hat{u}_0 \subset \{(\xi_1, \ldots, \xi_d) : \xi_1 \geq \ell > 0\}$ and $\|u_0\|_{L^\infty} \ll \ell$ then the solution is global-in-time.
- Feichtinger-Gröchenig-Li-Wang (2021): if $\mathrm{supp}\,\hat{u}_0 \subset \{(\xi_1, \ldots, \xi_d) : \xi_k \geq 0 \,\forall\, k\}$ and $u_0 \in L^2$ then the solution is global-in-time.

**Theorem (P., 2021)**

If $\mathrm{supp}\,\hat{u}_0 \subset \mathbb{R}^d_+$ and $\hat{u}_0$ belongs to a Herz space $\dot{K}^\alpha_{p,q}$, where $\alpha, p, q$ are in a suitable range, then the solution is global-in-time. [If $d = 2$ or 3, $L^2$ is included in this family.]

$$\dot{K}^\alpha_{p,q} = \{f : |\xi|^\alpha f \mathbb{1}_{A_k} \in l^q L^p\}, \quad A_k = \{\xi : \ 2^k \leq |\xi| \leq 2^{k+1}\}$$

# Conservation of frequencies

$$v(\xi,t) = e^{-|\xi|^2 t} v_0(\xi) + c_0 \int_0^t e^{-|\xi|^2 s} |\xi| \int_{\mathbb{R}^d} v(\eta, t-s) \odot_\xi v(\xi-\eta, t-s) d\eta ds$$

**Conservation of frequencies:** $\boxed{v_0 \to e^{\xi \cdot a} v_0, \quad v \to e^{\xi \cdot a} v}$

# Applications of the conservation of frequencies

## Frequency trap

If $v_0$ is supported in a cone $\mathcal{C}_{a,\theta} = \{\xi : \ \xi \cdot a \geq |\xi||a| \cos \theta\}$ with an opening angle $\theta \in [0, \pi/2]$ then $v$ is supported in the same cone for all $t$.

*Proof.*

$$\begin{aligned} v^{(n+1)}(\xi, t) &= e^{-|\xi|^2 t} v_0(\xi) \\ &+ c_0 \int_0^t e^{-|\xi|^2 s} |\xi| \int_{\mathbb{R}^d} v^{(n)}(\eta, t-s) \odot_\xi v^{(n)}(\xi - \eta, t-s) d\eta ds \\ v^{(0)} &= 0 \end{aligned}$$

By induction on $n$: $v^{(n)}$ is supported in the cone.

# Applications of the conservation of frequencies

## Global solution for supercritical/critical initial data

If $\operatorname{supp} v_0 \subset \mathbb{R}^d_+$ and $v_0 \in L^2$, where $d = 2$ or 3, then the solution is global-in-time.

*Proof.* By Hölder's inequality,

$$\|e^{-\xi_d} v_0\|_{\dot{K}^{-1}_{1,2}} \le \|v_0\|_{\dot{K}^{0}_{2,2}} \|e^{-\xi_d} \mathbb{1}_{\xi_d \ge 0}\|_{\dot{K}^{-1}_{2,\infty}} < \infty.$$

Thus, $e^{-n\xi_d} v_0$ is small in $\dot{K}^{-1}_{1,2}$ if $n$ is large enough. $\dot{K}^{-1}_{1,2}$ is a scale-critical space for which global well-posedness is known (Cannone-Wu, 2012). Scale back to get the solution.

# Applications of the conservation of frequencies

**Global solution for subcritical initial data**

If $\text{supp} v_0 \subset \mathbb{R}^d_+$ and $v_0 \in L^p$, where $1 \leq p < \frac{d}{d-1}$, then the solution is global-in-time.

*Proof.* The solution exists at least until time $T \sim \|v_0\|_p^{-\delta}$ where $\delta = 2p/(d-(d-1)p)$.
The scaled initial data $v_{0n} = e^{-n\xi_d} v_0$ gives a solution at least until time $T_n \sim \|v_{0n}\|_p^{-\delta} \to \infty$ as $n \to \infty$.
Scale back to get a global solution.

# The half-space is optimal

All aforementioned arguments also work for the simplified equation:

$$v(\xi,t) = e^{-|\xi|^2 t} v_0(\xi) + c_0 \int_0^t e^{-|\xi|^2 s} |\xi| \int_{\mathbb{R}^d} v(\eta, t-s) v(\xi-\eta, t-s) d\eta ds$$

Observation: $u = \check{v}$ solves the Montgomery-Smith equation

$$\partial_t u - \Delta u = \sqrt{-\Delta}(u^2)$$

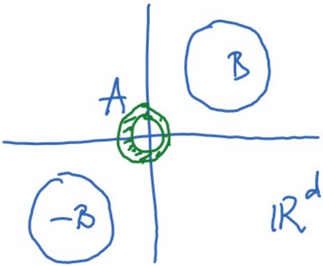
**Definition:** open set $D \subset \mathbb{R}^d$ has *Property (P)* if any bounded initial data, compactly supported in $D$ yields a global solution.

**Theorem (P., 2021)**

A half-space is a maximal set that has Property (P).

# The half-space is optimal - Proof

Any open set larger than a half-space contains two balls symmetric to the origin.




Put $\Omega = B \cup (-B)$.
*Key observation*: $|(\xi - \Omega) \cap \Omega| > c > 0$ for every $\xi \in A$.
Put $\phi = \mathbb{1}_\Omega$ and choose the initial data $v_0 = M\phi$. Oseen's representation:

$$v = \sum_{k=1}^{\infty} M^k v_k \quad \text{where} \quad v_1(\xi, t) = e^{-t|\xi|^2}\phi,$$

# The half-space is optimal - Proof

$$v_n(\xi, t) = \int_0^t \int_{\mathbb{R}^d} |\xi| e^{-(t-s)|\xi|^2} \sum_{k=1}^{n} v_k(\eta, s) v_{n-k}(\xi - \eta, s) d\eta ds$$

Put $w_k(t) = \inf_{\xi \in A} v_{2^k}(\xi, t)$. By induction,

$$w_k(t) \gtrsim e^{-\alpha 2^k} 2^k (te^{-t})^{2^k - 1} \quad \forall\, k$$

For $\xi \in A$,

$$v(\xi, 1) \geq \sum M^{2^k} v_{2^k}(\xi, 1) \gtrsim \sum M^{2^k} e^{-\alpha 2^k} 2^k e^{-(2^k - 1)} \to \infty. \quad \square$$

# Through the lens of stochastic branching processes

Le Jan-Sznitman 1997, Bhattacharya et al 2003: $\chi = c_0 \dfrac{\hat{u}}{h}$ satisfies

$$\begin{aligned}\chi(\xi, t) &= e^{-t|\xi|^2}\chi_0(\xi) \\ &+ \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^d} \chi(\eta, t-s) \odot_\xi \chi(\xi-\eta, t-s) H(\eta|\xi) d\eta ds\end{aligned}$$

where $H(\eta|\xi) = \dfrac{h(\eta)h(\xi-\eta)}{|\xi| h(\xi)}$.

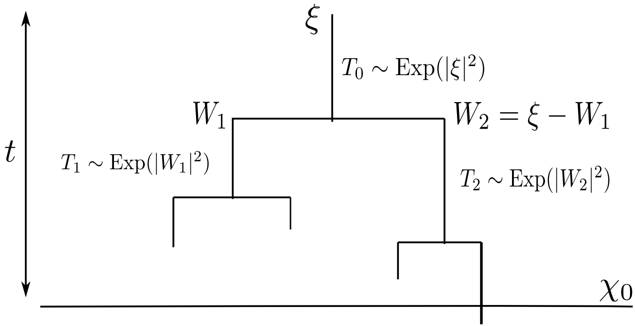
$h$ satisfies $h * h = |\xi| h$, called *majorizing kernel*.

$$\boxed{\chi = \mathbb{E}\mathbf{X}}$$

where the stochastic process $\mathbf{X}$ is defined recursively by...

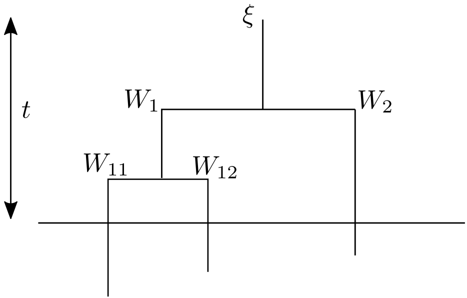
# The stochastic process **X**

$$\mathbf{X}(\xi,t)=\begin{cases}\chi_0(\xi) & \text{if} \quad T_0>t,\\ \mathbf{X}^{(1)}(W_1,t-T_0)\odot_\xi \mathbf{X}^{(2)}(\xi-W_1,t-T_0) & \text{if} \quad T_0\le t.\end{cases}$$



# An example of $\mathbf{X}$

Consider the following event:




On this event,

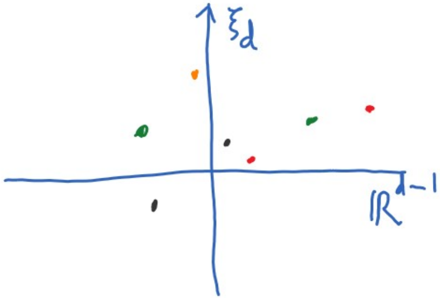
$$\mathbf{X}(\xi, t) = (\chi_0(W_{11}) \odot_{W_1} \chi_0(W_{12})) \odot_\xi \chi_0(W_2).$$

# Half-space is optimal: probabilistic explanation

**Revisit**

Any bounded $v_0$, $\mathrm{supp} v_0 \subset \mathbb{R}^d_+$, yields a global solution $v$.

With $h(\xi) = c_d|\xi|^{1-d}$, branching distribution is $H(\eta|\xi) = \frac{c_d|\xi|^{d-2}}{|\eta|^{d-1}|\xi-\eta|^{d-1}}$




$$p_k(\xi, t) = \mathbb{P}_\xi(\text{by time } t, \text{ all } k \text{ paths terminate in } \mathbb{R}^d_+)$$

# Half-space is optimal: probabilistic explanation

Only need to deal with the scalar equation:

$$v(\xi,t) = e^{-|\xi|^2 t} v_0(\xi) + c_0 \int_0^t e^{-|\xi|^2 s} |\xi| \int_{\mathbb{R}^d} v(\eta, t-s) v(\xi-\eta, t-s) d\eta ds$$

Can assume $v_0 = M|\xi|^{1-d} \mathbb{1}_{\xi_d \geq 0}$. Equivalently, $\chi_0(\xi) \sim v/h \sim M\mathbb{1}_{\xi_d \geq 0}$.

$$\chi \sim \sum M^k p_k$$

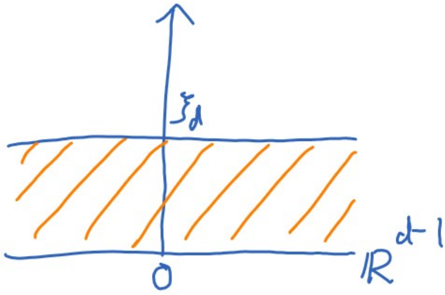
To quantify the rate of decay of $p_k$ as $k \to \infty$, one needs to look more closely at the geometry of the half-space.

$$p_n(\xi,t) = \int_0^t \int_{\mathbb{R}^d} |\xi|^2 e^{-(t-s)|\xi|^2} \sum_{k=1}^{n} p_k(\eta, s) p_{n-k}(\xi-\eta, s) H(\eta|\xi) d\eta ds$$

# Half-space is optimal: probabilistic explanation

The key idea is the following:

$$\boxed{\int_{(\xi-\mathbb{R}^d_+)\cap\mathbb{R}^d_+} H(\eta|\xi) \le c_{\delta,d}\left(\frac{\xi_d}{|\xi|}\right)^{\delta} \quad \forall\delta\in(0,1)}.$$



# Half-space is optimal: probabilistic explanation

**Theorem (P., 2021)**

$$p_k(\xi, t) \le c_d^{k-1} \frac{(\xi_d \sqrt{t})^{\frac{1}{2}(k-1)}}{\{(k-1)!\}^{1/8}} e^{-|\xi|\sqrt{t}}.$$

The following lemma is needed for the induction procedure: for $\phi(k) = (k!)^\gamma$, where $\gamma \in (0,1)$, one has

$$\sum_{k=1}^{n-1} \frac{a^k}{\phi(k)} \frac{b^{n-k}}{\phi(n-k)} \le \frac{(a+b)^n}{\phi(n)} n^{\gamma-1} \quad \forall\, a, b > 0,\ n \in \mathbb{N}.$$

Thank You!