A global regularity criterion for the Navier-Stokes equations based on approximate solutions

Tuan Pham

Oregon State University

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NSE, classical solutions

$$
\text{(NSE)}: \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \text{div } u = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^3. \end{cases}
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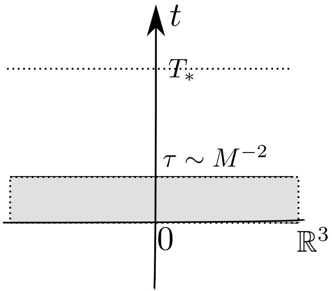
Mild solutions (∼ classical sols):

Perturbation methods:

- Leray (1934): $u_0 \in L^q$, $q > 3$
- Kato (1984): $u_0 \in L^3$

✓ Local existence, uniqueness, smoothness

? Global existence



$$\|u_0\|_{L^\infty} = M$$

Weak Solutions

Energy methods:

- Leray-Hopf (1934, '51): $u_0 \in L^2$

$$\int_{\mathbb{R}^3} \frac{|u(x,t)|^2}{2} dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx ds \le \int_{\mathbb{R}^3} \frac{|u_0(x)|^2}{2} dx$$

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- Local energy solutions (Scheffer '77, C-K-N '82, Lemarié-Rieusset 2002,...): $u_0 \in L^2_{\mathrm{uloc}}$

$$\int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2 \phi dxdt \leq \int_0^\infty \int_{\mathbb{R}^3} \left[\frac{|u|^2}{2} (\partial_t \phi + \Delta \phi) + \left(\frac{|u|^2}{2} + p \right) u \nabla \phi \right] dxdt$$

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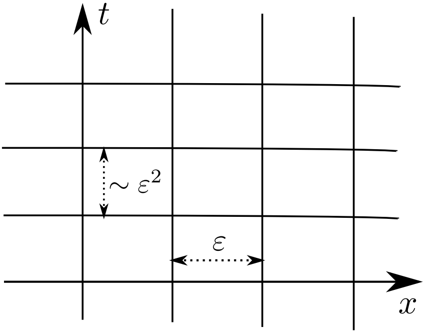
✓ Global existence ? Uniqueness, smoothness

Motivating questions



- $\varepsilon \ldots$ mesh size (resolution)
- $u_\varepsilon \ldots$ approximate solution
- $u \ldots$ exact classical solution
- $|u_\varepsilon| \leq M \quad \forall x,\ t$
- What can we say about u?

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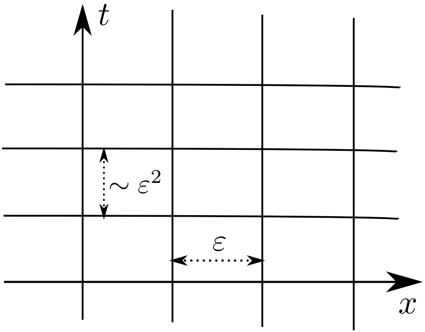

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- What can we say about u?

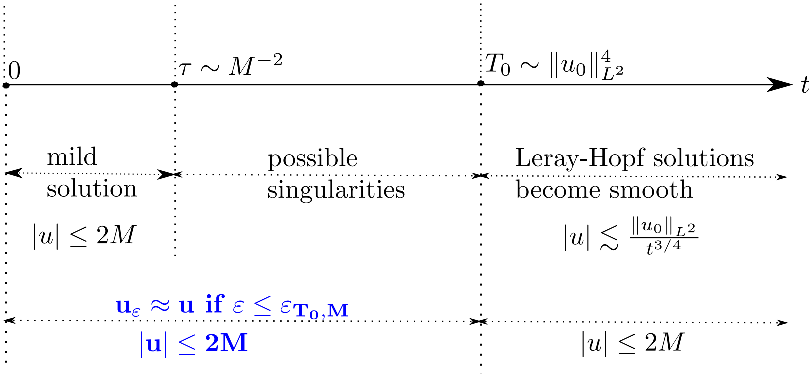
Theorem (Buyang Li 2014)

If $\varepsilon \lesssim \exp(-(\|u_0\|_{H_0^1 \cap H^2} + 1)^\alpha M^\alpha)$ then u exists globally and $|u| \leq 2M$.

α... large number (~ 225)

Heuristics

For $u_0 \in L^2 \cap L^\infty$ and $\|u_0\|_{L^\infty} = M$,



Motivating questions

- Is there a scale-invariant relation of ε, M, $\|u_0\|_{L^2}$?
  Scaling symmetry:

$$
\begin{aligned}
u(x,t) &\to u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t) \\
p(x,t) &\to p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t) \\
u_0(x) &\to u_{\lambda 0}(x) = \lambda u_0(\lambda x) \\
&\dots
\end{aligned}
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  Dimensions (C-K-N 1982):

$$[\textbf{Length}] = [x] = 1, \quad [\textbf{Pressure}] = [p] = -2,$$

$$[\textbf{Time}] = [t] = 2, \quad [\textbf{Energy}] = [\|u_0\|_{L^2}^2] = 1,$$

$$[\textbf{Velocity}] = [u] = -1, \quad [\varepsilon] = 1,\ [M] = -1, \dots$$

Motivating questions - Main result in global picture

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  Not very reasonable:

  $$\varepsilon \lesssim M^{-\alpha} \|u_0\|_{L^2}^{\beta} \quad \text{where} \quad \alpha + \frac{\beta}{2} = 1$$

  Somewhat more reasonable:

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Theorem (P.–Sverák 2019)

Let $f_\varepsilon = f_\varepsilon(x,t)$ be a function such that $\|Ff_\varepsilon\|_{L^\infty} \lesssim \varepsilon M^2$. Suppose the approximate Navier-Stokes system $(\text{NSE})_\varepsilon$ has a solution on $(0, T)$ with $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^3\times(0,T))} \leq M$. Then there exists absolute constants δ_1, $\delta_2 > 0$ such that if

$$\varepsilon \leq \frac{\delta_1}{M} \exp(-\delta_2 T M^2)$$

then (NSE) has a mild solution on $(0, T)$ with $\|u\|_{L^\infty(\mathbb{R}^3\times(0,T))} \leq 2M$.

Approximate Navier-Stokes systems

where

$$(\mathrm{NSE})_\varepsilon : \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = \operatorname{div} f_\varepsilon, \\ \operatorname{div} u = 0, \\ u(\cdot, 0) = u_0. \end{cases}$$

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Exact solution:

$$u = U + B(u, u)$$

Approximate solution:

$$u_\varepsilon = U + Ff_\varepsilon + B(u_\varepsilon, u_\varepsilon)$$

$$Ff_\varepsilon(x, t) = \int_0^t \Gamma(t - s) * \mathbb{P}\,\mathrm{div}\; f_\varepsilon(s) ds$$

Examples of approximate NSE

- Leray's mollified NSE:

$$\partial_t u - \Delta u + (u * \eta_\varepsilon) \cdot \nabla u + \nabla p = 0$$

$\eta_\varepsilon \dots$ standard mollifiers in $\mathbb{R}^3$
$f_\varepsilon = (u * \eta_\varepsilon - u) \otimes u$

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$f_\varepsilon = (u * \eta_\varepsilon - u) \otimes u$

- Galerkin-type approximate NSE:

$$\partial_t u - \Delta u + P_\varepsilon(u \cdot \nabla u) + \nabla p = 0$$

$P_\varepsilon \ldots$ low-pass Fourier filter with threshold ε^{-1}
$f_\varepsilon = (Id - P_\varepsilon)(u \otimes u)$

Global picture - Sketch of proof

$v = u - u_\varepsilon, \quad v(t) = \Gamma(t) * v(0) + Ff_\varepsilon(t) + B(v, u_\varepsilon) + B(u_\varepsilon, v) + B(v, v)$

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Bilinear form:

$$\|B\|_{L^\infty \times L^\infty(\mathbb{R}^3 \times (0,t))} \lesssim \sqrt{t}$$

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Put: $\varphi(t) = \sup_{0<s<t} \|v(t)\|_{L^\infty}$. Then

$$\varphi(\tau) \leq \varphi(0) + \alpha + \beta\varphi(\tau) + \gamma\varphi(\tau)^2$$

$$\tau = \frac{\theta}{M^2}, \qquad \beta \sim \theta$$

$$\alpha \sim \varepsilon M^2, \qquad \gamma \sim \frac{\theta}{M}$$

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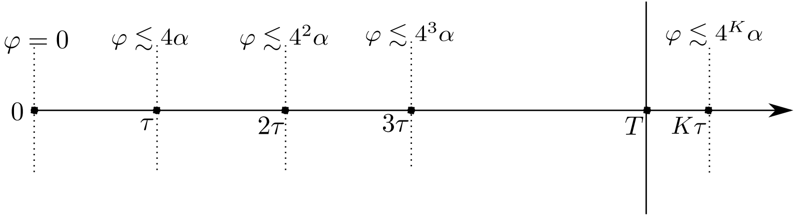
Lemma: If $\beta < \frac{1}{2}$ and $\varphi(0) + \alpha < \frac{1}{16\gamma}$ then $\varphi(\tau) < 4(\varphi(0) + \alpha)$.

Global picture - Sketch of proof

Growth of $\varphi = \varphi(t)$:



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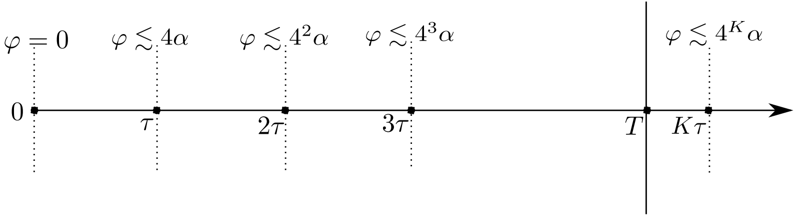

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Condition for the process to work:

$$4^K\alpha \lesssim \frac{1}{16\gamma} \quad \Leftrightarrow \quad 4^{TM^2}\varepsilon M \lesssim \frac{1}{16}$$

Main result in local picture

Theorem (P.–Sverák 2019)

Let $f_\varepsilon = f_\varepsilon(x,t)$ be a function such that

$$\|f_\varepsilon\|_{L^q(Q_{R,\rho}(z_0))} \lesssim \varepsilon^{\sigma_1} R^{\sigma_2} \rho^{\sigma_3} M^{\sigma_4} \quad \forall\, R,\ \rho > 0,\ z_0 \in \mathbb{R}^3 \times \mathbb{R}$$

for some constants $\sigma_i \geq 0$, $\sigma_1 > 0$, $q > 5$ satisfying

$$\sigma_1 + \sigma_2 + 2\sigma_3 - \sigma_4 = -2 + \frac{5}{q}.$$

Suppose $(\mathrm{NSE})_\varepsilon$ has a solution with $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^3\times(0,T))} \leq M$. Then there exist constants δ_1, $\delta_2 > 0$ depending on σ_1 such that if

$$\varepsilon \leq \frac{\delta_1}{M} \exp(-\delta_2 T M^2)$$

then (NSE) has a mild solution on $(0, T)$ with $\|u\|_{L^\infty(\mathbb{R}^3\times(0,T))} \lesssim M$.

Local picture - Sketch of proof

$v = u - u_\varepsilon$ is **local energy solution** to generalized NSE:

$$\partial_t v - \Delta v + \operatorname{div}(u_\varepsilon \otimes v + v \otimes u_\varepsilon + v \otimes v) + \nabla \pi = \operatorname{div} f_\varepsilon$$

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Regularity criterion (∼ C-K-N 1982, ∼ Jia- Sverák 2012)

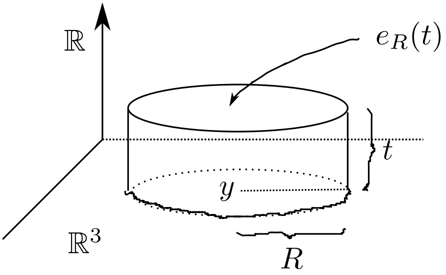
There exists δ, $C > 0$ only depending on m and q such that if

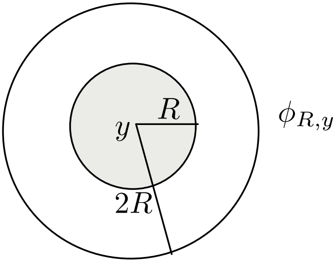
$$\frac{1}{R^2} \int_{Q_R(z_0)} \left(|v|^3 + |\pi|^{\frac{3}{2}}\right) dz + R^{m-5} \int_{Q_R(z_0)} |a|^m dz + R^{2q-5} \int_{Q_R(z_0)} |f|^q dz < \delta$$

then $|v| \leq CR^{-1}$ on $Q_{R/2}(z_0)$.

Local picture - Sketch of proof

$$e_R(t) = \sup_{s\in(0,t),\, y\in\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{|v(x,s)|^2}{2}\phi_{R,y}\,dx + \int_0^s \int_{\mathbb{R}^3} |\nabla v(x,\tau)|^2 \phi_{R,y}\,dx\,d\tau \right)$$





Local picture - Sketch of proof

Based on scaling, choose

$$R = \kappa M^{-1}, \quad \tau = \theta R^2$$

where θ, $\kappa > 0$ are small absolute constants.

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Local energy estimate:

$$e(\tau) \leq e(0) + \alpha e(\tau)^{1/2} + \beta e(\tau) + \gamma e(\tau)^{3/2}$$

where

$$\alpha = \varepsilon^{\sigma_1} M^{\sigma_1 - 1/2}, \qquad \beta = \theta^{1/5}, \qquad \gamma = M^{1/2}\kappa^{-1/2}$$

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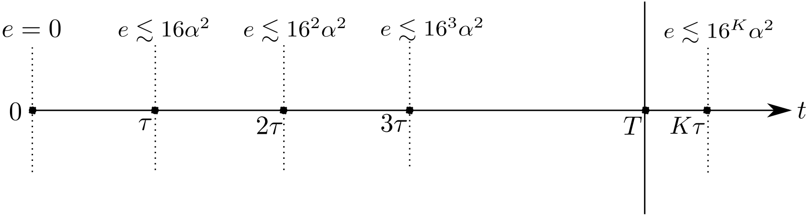
Lemma: If $\beta < 1/2$ and $\alpha\gamma < 1/64$ then

- If $e(0) = 0$ then $e(\tau) < 16\alpha^2$.
- If $0 < e(0) < 1/(256\gamma^2)$ then $e(\tau) < \max\{4e(0), 64\alpha^2\}$.

Local picture - Sketch of proof

Growth of local energy $e = e(t)$:



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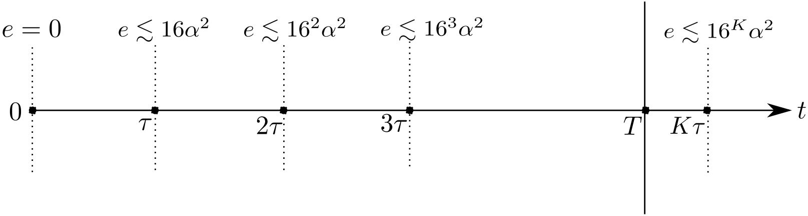

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Condition for the process to work:

$$16^K\alpha^2 \lesssim \frac{1}{256\gamma^2} \quad \Leftrightarrow \quad 16^{TM^2}(\varepsilon M)^{2\sigma_1} \lesssim \frac{1}{256}$$

Local picture - Sketch of proof

By Sobolev embeddings,

$$\frac{1}{R^2}\int_{Q_R(z_0)} |v|^3 dz \lesssim \left(\frac{e(\tau)}{R}\right)^{3/2} \lesssim (\varepsilon M)^{3\sigma_1}$$

$$\frac{1}{R^2}\int_{Q_R(z_0)} |\pi|^{\frac{3}{2}} dz \lesssim (\varepsilon M)^{3\sigma_1/2} + \left(\frac{e(\tau)}{R}\right)^{3/2} \lesssim (\varepsilon M)^{3\sigma_1/2}$$

Both quantities are small.

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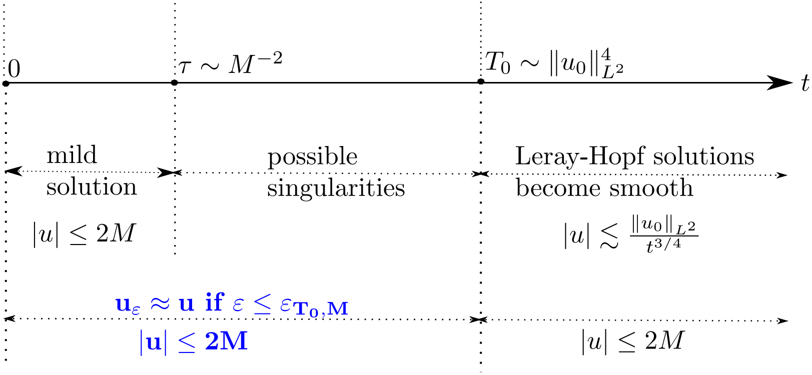
Both quantities are small.
v **is regular on each subinterval!**



✓ $\varepsilon \lesssim M^{-1}\exp(-\|u_0\|_{L^2}^4 M^2)$

? $\varepsilon \lesssim M^{-1}$

Thank You!