Stochastic methods for problems arising in Fluid Dynamics

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Collaborative work

- Radu Dascaliuc, Tuan Pham, Enrique Thomann: "*On Le Jan-Sznitman's stochastic approach to the Navier-Stokes equations*". **Trans. Amer. Math. Soc.**, Vol 377, No. 4, 2335-2365, April 2024.
- Radu Dascaliuc, Tuan Pham, Enrique Thomann, Edward Waymire: "*Erratum to Stochastic explosion and non-uniqueness for α-Riccati equation*". **J. Math. Anal. Appl.**, Vol 527, Issue 2, November 2023.
- Radu Dascaliuc, Tuan Pham, Enrique Thomann, Edward Waymire: "*Doubly Stochastic Yule Cascades (Part II): The explosion problem in the non-reversible case*". **Ann. Inst. Henri Poincaré Probab. Stat.**, No. 4, Vol 59, 1904-1933, 2023.
- Radu Dascaliuc, Tuan Pham, Enrique Thomann, and Edward Waymire: "*Doubly Stochastic Yule Cascades (Part I): The explosion problem in the time-reversible case*". **J. Funct. Anal.**, Vol 284, Issue 1, 2023.
- Tuan Pham and Jared Whitehead: "*Hydrodynamic stability of Couette flows with stochastically moving boundary*" (in preparation)

Birth process of a population

Simple birth model (Yule process): each individual gives birth repeatedly and independently at rate λ.

$$\mathbb{P}(\text{birth occurs between t and t+dt}) = \lambda dt$$

Simulation with $\lambda = 1$

Equivalently, each individual gives *two births at once* at rate λ and then dies immediately after giving birth.

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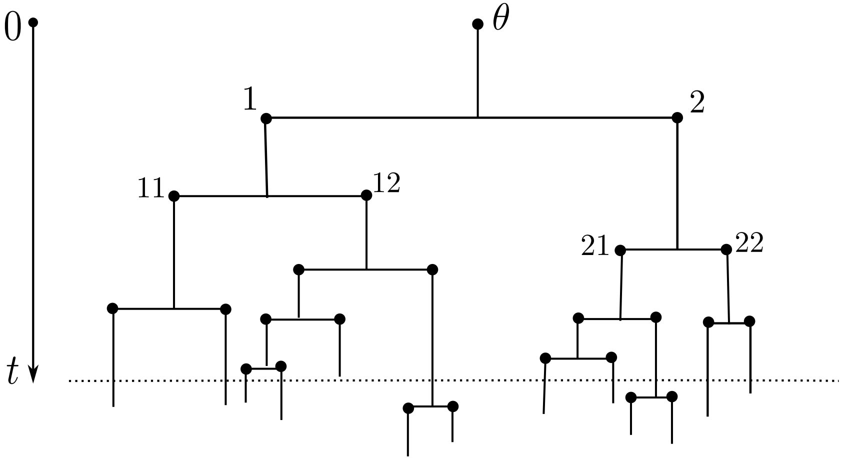
Aldous-Shields model: each individual of generation n gives *two births at once* at rate α^n and then dies immediately after giving births.

$$\mathbb{P}(\text{birth occurs between t and t+dt}) = \alpha^n dt$$

Comparison: $\alpha = 0.8$, $\alpha = 1$, $\alpha = 1.1$

Tree representation



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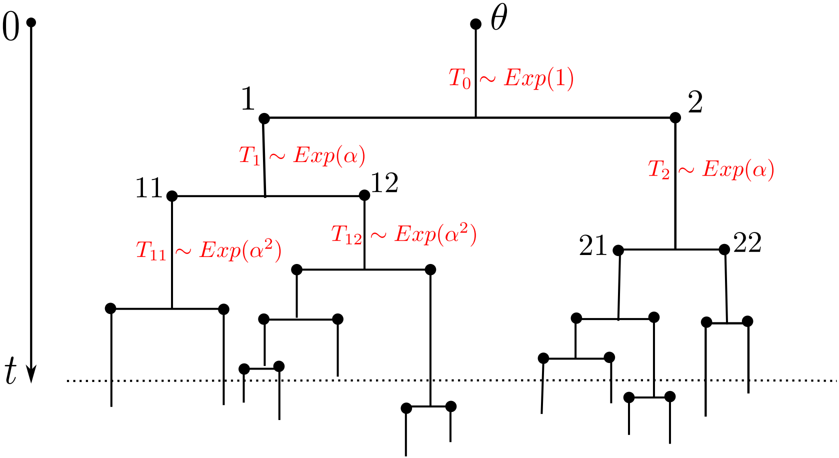


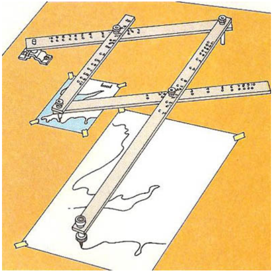
Applications of Aldous-Shields model

- *Lempel-Ziv data compression:*
  The bit sequence 22112121112111211... parses into 2, 21, 1, 212, 11, 12, 111, 211,... These form vertices of a tree! ($\alpha = 1/2$)
- *Growth of cancer cells* ($\alpha > 1$): Levy-Allsopp-Futcher-Greider-Harley (1992), Arkus (2005).
- *Cellular ageing* ($\alpha < 1$): Best-Pfaffelhuber (2010) studied the distribution of

$$\frac{\#\text{proliferating cells}}{\#\text{senescence cells}}$$

- *Pantograph equation and α-Riccati equation*

Pantograph equation



$$y'(t) + y(t) = 2y(\alpha t)$$

Denote $N(t)$ = number of vertices crossing the horizon t.

$$N(t) = \begin{cases} 1 & \text{if} \quad T_0 > t, \\ N^{(1)}(\alpha(t - T_0)) + N^{(2)}(\alpha(t - T_0)) & \text{if} \quad T_0 < t \end{cases}$$

Then $y(t) = \mathbb{E}[N(t)]$ satisfies the pantograph equation with $y(0) = 1$.

α-Riccati equation

$$y'(t) + y(t) = y^2(\alpha t),\ y(0) = y_0$$

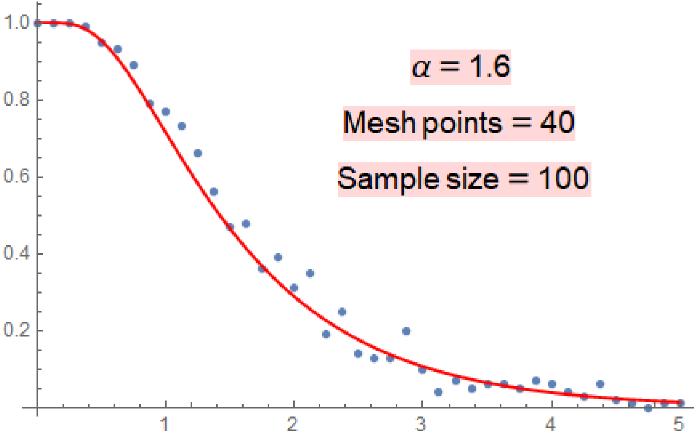
Let $X(t) = y_0^{N(t)}$. One can see that

$$X(t) = \begin{cases} y_0 & \text{if} \quad T_0 > t, \\ X^{(1)}(\alpha(t - T_0))X^{(2)}(\alpha(t - T_0)) & \text{if} \quad T_0 < t \end{cases}$$

$y(t) = \mathbb{E}[X(t)]$ satisfies the α-Riccati equation with $y(0) = y_0$.

α-Riccati equation, Monte Carlo simulation

$$y(t) = \mathbb{E}[X(t)]$$



Navier-Stokes equations

$$\begin{cases} u_t - \Delta u + u \nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \text{div } u = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^3. \end{cases}$$

In Fourier domain:

$$\hat{u}(\xi, t) = e^{-|\xi|^2 t} \hat{u}_0(\xi) + c \int_0^t e^{-|\xi|^2 s} |\xi| \int_{\mathbb{R}^3} \hat{u}(\eta, t-s) \odot_\xi \hat{u}(\xi - \eta, t - s) d\eta ds$$

Normalization (LeJan-Sznitman 1997): $v = c\hat{u}/h$

$$v(\xi, t) = e^{-t|\xi|^2} v_0(\xi) + \int_0^t e^{-s|\xi|^2} |\xi|^2 \int_{\mathbb{R}^3} v(\eta, t-s) \odot_\xi v(\xi - \eta, t-s) H(\eta|\xi) d\eta ds$$

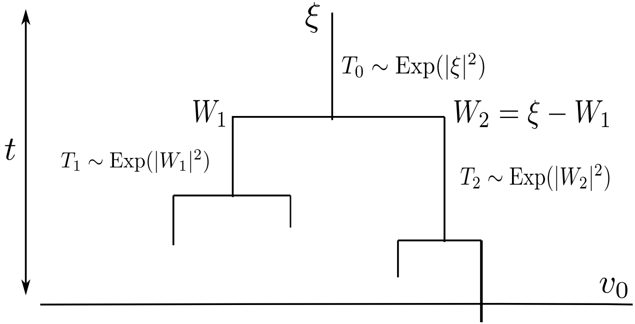
where $H(\eta|\xi) = \frac{h(\eta)h(\xi-\eta)}{|\xi|h(\xi)}$.

Navier-Stokes equations

$v(\xi, t) = \mathbb{E}[X(\xi, t)]$ where

$$X(\xi, t) = \begin{cases} v_0(\xi) & \text{if} \quad T_0 \geq t, \\ X^{(1)}(W_1, t - T_0) \odot_\xi X^{(2)}(W_2, t - T_0) & \text{if} \quad T_0 < t. \end{cases}$$

$W_1 \sim H(\cdot|\xi)$ and $W_2 = \xi - W_1 \sim H(\cdot|\xi)$.



What can we do?

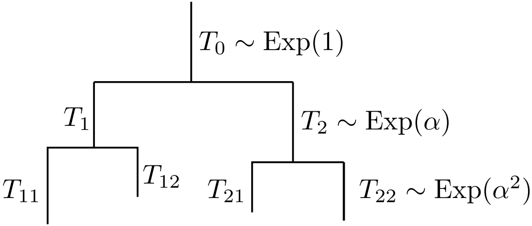
Open problems:

- Does the NSE have a physical solution for all time?
- Is the solution unique?
- How to simulate the NSE effectively?
- When does turbulence occur?
- ...

We use the stochastic cascade method to approach the uniqueness (or the lack of) problem, starting with some simplified models of NSE.

α-Riccati equation, stochastic cascade method

$$y' + y = y^2(\alpha t)$$



- $0 < \alpha \leq 1$: non-explosion
- $1 < \alpha < 2$: explosion, infinitely many vertices crossing horizon
- $2 \leq \alpha$: hyper-explosion, finitely many vertices crossing horizon

Know: explosion $\implies$ nonuniqueness of solutions

Navier-Stokes equations

Dascaliuc, Pham, Thomann, Waymire (2023)

- Bessel kernel $h(\xi) = c\frac{e^{-|\xi|}}{|\xi|} \rightsquigarrow$ non-explosion
- Self-similar kernel $h(\xi) = c|\xi|^{-2} \rightsquigarrow$ explosion

Dascaliuc, Pham, Thomann (2023): Montgomery-Smith equation

$$u_t - \Delta u = \sqrt{-\Delta}(u^2),\ u_0(x) = \frac{2a}{|x|^2 + 1}$$

- $a > 1$: finite-time blowup solution
- $-1 \leq a \leq 1$: global solution
- $-1 < a < 1$: solution exponentially decays in time

Proposed research

1. Monte Carlo simulation
   - very costly,
   - explosion issue.
2. Develop a general theory on the stochastic cascades
   - quantify the cancellation property of the product
   - continue the solution while the stochastic cascade fails to be L^1

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Thank You!