Stochastic cascade methods for differential equations

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Equation $u' + u = f$, $u(0) = u_0$

$$\begin{cases} u' + u &= f, \\ u(0) &= u_0. \end{cases}$$

Integral form:

$$u(t) = e^{-t}u_0 + \int_0^t e^{-s} f(t-s)ds$$

Probabilistically, $u(t) = \mathbb{E}[X(t)]$ where

$$X(t) = \begin{cases} u_0 & \text{if} \quad T \geq t, \\ f(t-T) & \text{if} \quad T < t \end{cases}$$

and $T \sim Exp(1)$.

Logistic equation $u' + u = u^2$, $u(0) = u_0$

$$\begin{cases} u' + u & = & u^2, \\ u(0) & = & u_0. \end{cases}$$

Integral form:

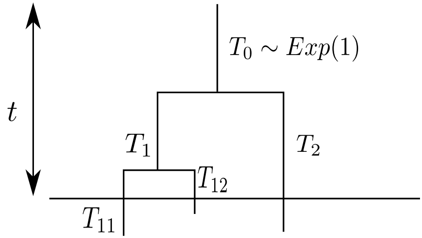
$$u(t) = e^{-t}u_0 + \int_0^t e^{-s}u^2(t-s)ds$$

Probabilistically, $u(t) = \mathbb{E}[X(t)]$ where

$$X(t) = \begin{cases} u_0 & \text{if} \quad T \geq t, \\ X^{(1)}(t-T)X^{(2)}(t-T) & \text{if} \quad T < t. \end{cases}$$

$T \sim Exp(1)$
$X^{(1)}$ and $X^{(2)}$ are i.i.d. copies of X.

Logistic equation $u' + u = u^2$, $u(0) = u_0$



In this event, $X(t) = u_0^3$. In general, $X(t) = u_0^{N(t)}$.

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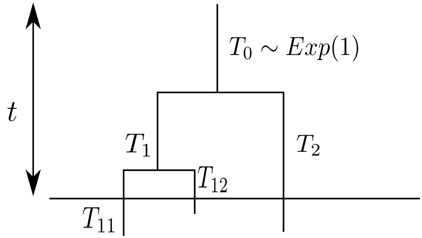

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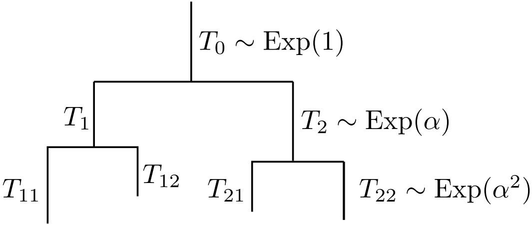
Explicit solution: $u(t) = \dfrac{u_0}{u_0 - (u_0 - 1)e^t}$

- If $-1 \leq u_0 \leq 1$, global solution
- If $u_0 > 1$, blowup solution

α-Riccati equation (Athreya 1985, Dascaliuc et al. 2018)

$$u' + u = u^2(\alpha t), \ u(0) = u_0$$



S = shortest path (random)

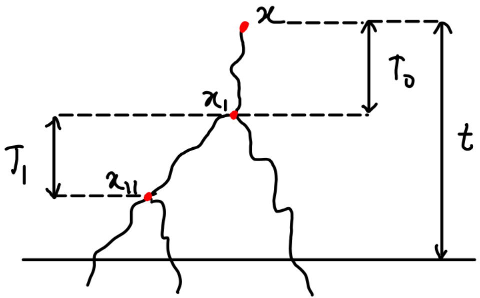
- If $0 < \alpha \leq 1$ then $S = \infty$ a.s. $\rightsquigarrow$ non-explosion
- If $\alpha > 1$ then $S < \infty$ a.s. $\rightsquigarrow$ explosion
- $u(t) = \mathbb{P}(S < t)$ solves the equation with $u_0 = 0$ $\rightsquigarrow$ non-uniqueness of solutions for $\alpha > 1$.

KPP-Fisher equation (1930s)

$$u_t - \frac{1}{2} u_{xx} = u^2 - u, \quad u(x,0) = u_0(x)$$

McKean (1975) observed that $u(x,t) = \mathbb{E}[X(x,t)]$ where

$$X(x,t) = \begin{cases} u_0(B_t^x) & \text{if} \quad T \geq t, \\ X^{(1)}(B_T^x, t-T)\, X^{(2)}(B_T^x, t-T) & \text{if} \quad T < t. \end{cases}$$

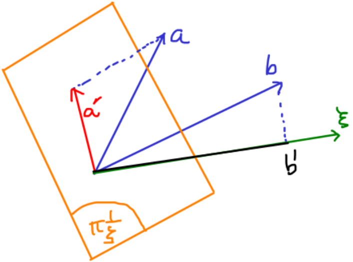


Navier-Stokes equations

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \operatorname{div}\, u = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^3. \end{cases}$$

In Fourier domain:

$$\hat{u}(\xi, t) = e^{-|\xi|^2 t} \hat{u}_0(\xi) + c \int_0^t e^{-|\xi|^2 s} |\xi| \int_{\mathbb{R}^3} \hat{u}(\eta, t-s) \odot_\xi \hat{u}(\xi - \eta, t - s) d\eta ds$$



$a \odot_\xi b = -i\, b' a'$

Normalized Navier-Stokes equations

Normalization (LeJan-Sznitman 1997): $v = c\hat{u}/h$

$$v(\xi,t) = e^{-t|\xi|^2} v_0(\xi) + \int_0^t e^{-s|\xi|^2} |\xi|^2 \int_{\mathbb{R}^3} v(\eta, t-s) \odot_\xi v(\xi-\eta, t-s) H(\eta|\xi) d\eta ds$$

where $H(\eta|\xi) = \frac{h(\eta)h(\xi-\eta)}{|\xi|h(\xi)}$ and $h * h = |\xi| h$.

Normalized Navier-Stokes equations

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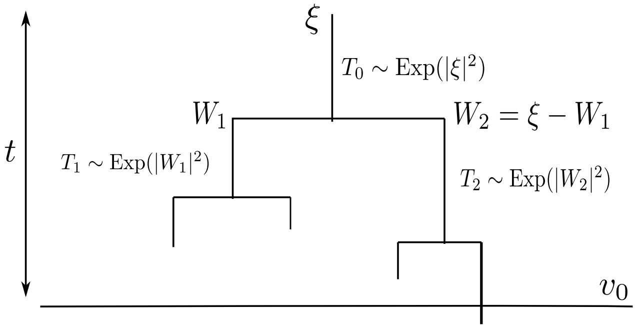
where $H(\eta|\xi) = \frac{h(\eta)h(\xi-\eta)}{|\xi|h(\xi)}$ and $h * h = |\xi|h$.

$v(\xi,t) = \mathbb{E}[X(\xi,t)]$ where

$$X(\xi,t) = \begin{cases} v_0(\xi) & \text{if} \quad T_0 \geq t, \\ X^{(1)}(W_1, t-T_0) \odot_\xi X^{(2)}(W_2, t-T_0) & \text{if} \quad T_0 < t. \end{cases}$$

$W_1 \sim H(\cdot|\xi)$ and $W_2 = \xi - W_1 \sim H(\cdot|\xi)$.

Stochastic cascade



- $w = \mathbb{P}_\xi(S > t)$ solves

$$w(\xi, t) = e^{-t|\xi|^2} + \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^3} w(\eta, t-s)w(\xi-\eta, t-s)H(\eta|\xi)d\eta ds$$

- Bessel kernel $h(\xi) = c\frac{e^{-|\xi|}}{|\xi|} \rightsquigarrow$ non-explosion.
- Self-similar kernel $h(\xi) = c|\xi|^{-2} \rightsquigarrow$ explosion

Non-explosion of Bessel cascade - Analytic approach

$$h(\xi) = c\frac{e^{-|\xi|}}{|\xi|}$$

Dascaliuc, Pham, Thomann (2021): for all ξ, $S = \infty$ a.s.

Sketched proof: we will show that $w \equiv 1$ is a unique solution.
Reverse the normalization: $w = c\hat{u}/h$ where

$$\text{(MS)}: \;\; u_t - \Delta u = \sqrt{-\Delta}(u^2), \;\; u_0(x) = \frac{2}{|x|^2 + 1}$$

$$u = e^{\Delta t} u_0 + \int_0^t \sqrt{-\Delta} e^{\Delta(t-s)} u^2(s) ds$$

Kernel $G(t)$ satisfies $\|G(t)\|_{L^q} \lesssim t^{\frac{3}{2q}-2}$ for all $q \in [1, \infty]$.
By fixed-point argument, (MS) has a unique solution $u(x,t) = u_0(x)$.

Non-explosion of Bessel cascade - Applications

$$(\mathrm{MS})_a\colon\ u_t - \Delta u = \sqrt{-\Delta}(u^2),\ u_0(x) = \frac{2a}{|x|^2+1}$$

Dascaliuc, Pham, Thomann (2021):

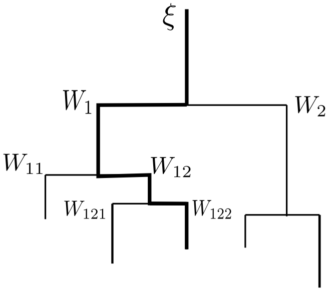
- $a > 1$*: finite-time blowup solution*
- $-1 \leq a \leq 1$*: global solution*
- $-1 < a < 1$*: solution exponentially decays in time*

Under certain assumption, NSE has a minimal blowup initial data (Rusin-Sverak 2011, Jia-Sverak 2013, Gallagher et al 2016, Pham 2018,...), but MS doesn't have minimal blowup data.

Dascaliuc, Pham, Thomann, Waymire (2021): proof of non-explosion using probabilistic argument

Explosion of self-similar cascade

$$h(\xi) = c|\xi|^{-2}$$



Dascaliuc, Pham, Thomann, Waymire (2021):

$$\textit{(MS)}: \; u_t - \Delta u = \sqrt{-\Delta}(u^2), \; u_0(x) = \frac{2}{\pi}\frac{1}{|x|}$$

has at least two solutions: $u_1 = u_0(x)$ *and* $u_2 = c\mathcal{F}^{-1}\{|\xi|^{-2}\mathbb{P}_\xi(S > t)\}$.

Proposed research

1. Monte Carlo simulation
   - solution in Fourier domain,
   - suitable to study energy cascade,
   - can apply to many differential equations,
   - very costly,
   - Decoupling Principle can be used to reduce the cost,
   - explosion issue.
2. Stochastic cascade and mean-field models for turbulence
   - make precise the notion of averaging commonly used in empirical theories of energy cascade (Kolmogorov 5/3, Large Eddy Simulation,. . .)
   - depletion of nonlinearity ($\odot$-product) needs to be better understood,
   - Mean-field models that preserve the energy (dyadic shell model, Burgers equation) are good starting points.