Hydrodynamic stability in the presence of stochastic boundary forcing: two case studies in convection and shear flow

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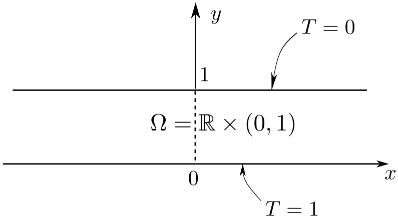
joint work with J. Foldes and J. Whitehead

AMS Contributed Paper

January 9, 2025

Rayleigh-Bénard convection



$$(Boussinesq): \begin{cases} \frac{1}{Pr}(u_t + u\nabla u) + \nabla p &= Ra(0, T) + \Delta u, \\ \mathrm{div}\ u &= 0, \\ T_t + u\nabla T &= \Delta T. \end{cases}$$

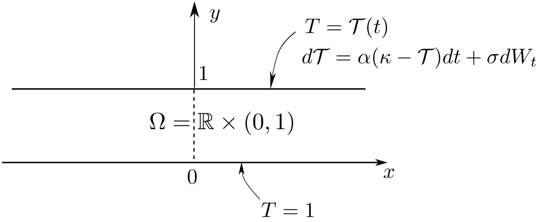
- Conductive state solution: $u(x, y, t) = 0,\ T(x, y, t) = 1 - y$
- If $Ra \le Ra_c$: conductive state is a global attractor (Rayleigh 1916)
  - Rigid-rigid boundaries: $Ra_c \approx 1708$ (Drazin-Reid, pp. 50-52)
  - Free-free boundaries: $Ra_c = \frac{27\pi^4}{4} \approx 657.5$ (Rayleigh 1916)
  - Rigid-free boundaries: $Ra_c \approx 1101$ (Drazin-Reid, pp. 50-52)

RB convection - Stochastic forcing

Question: How does stochastic forcing affect the hydrodynamic stability?
Some related works:

- *Venturi, Choi, Karniadakis* (2012): Boussinesq with boundary stochastic forcing (no conductive state)
- *Hairer, Mattingly* (2006, 2008): Navier-Stokes with bulk stochastic forcing
- *Foldes, Glatt-Holtz, Richards, Whitehead* (2024): Boussinesq with bulk stochastic forcing

RB convection - Stochastic boundary forcing



- $T(x, y = 1, t) = \mathcal{T}(t)$ is an Ornstein-Uhlenbeck process:

$$d\mathcal{T} = \alpha(\kappa - \mathcal{T})dt + \sigma dW_t$$

- $\alpha, \sigma > 0$ are the drift term and strength of the noise
- As $t \to \infty$, $\mathcal{T}(t) \xrightarrow{d} \mathcal{N}(\kappa, \frac{\sigma^2}{2\alpha})$
- Full RB layer: $\kappa = 0$
- Boundary RB layer: $\kappa = 1/2$

RB convection - Stochastic boundary forcing

Theorem (Foldes, Pham, Whitehead (2025))

The conductive state $u = 0$, $T(x, y, t) = \tau(y, t)$ *where*

$$\tau_t - \tau_{yy} = 0,\ \tau(0, t) = 1,\ \tau(1, t) = \mathcal{T}(t)$$

is ergodic and $\tau(y, t) \xrightarrow{d} \tau^S(y)$ *as* $t \to \infty$, *where*

$$\tau^S(y) = 1 + (\gamma_0 + \kappa - 1)\, y + \sum_{k=1}^{\infty} \gamma_k \sin(k\pi(1 - y))$$

Here, $(\gamma_0, \gamma_1, \gamma_2, \ldots)$ *is a Gaussian vector* $\mathcal{N}(0, \Sigma)$ *where*

$$\begin{aligned} \Sigma_{km} &= \frac{2\sigma^2 \left(8\pi^2 k^2 m^2 + \alpha(k^2 + m^2)\right)}{\pi^2 km(k^2 + m^2)(\pi^2 k^2 + \alpha)(\pi^2 m^2 + \alpha)}, \\ \Sigma_{k0} = \Sigma_{0k} &= -\frac{\sigma^2}{\pi k(\pi^2 k^2 + \alpha)}, \quad \Sigma_{00} = \frac{\sigma^2}{2\alpha} \end{aligned}$$

RB convection - Energy stability

Let $\theta = T - \tau$. Then

$$\frac{1}{2}\frac{d}{dt}\left(\|\theta\|^2 + \frac{1}{Pr\,Ra}\|u\|^2\right) = -\mathcal{Q}(u,\theta,\tau)$$

where

$$\mathcal{Q}(u,\theta,\tau) = \|\nabla\theta\|^2 + \frac{1}{Ra}\|\nabla u\|^2 + \int_\Omega u_2\theta\,(\tau_y - 1)\,dxdy$$

Let

$$\lambda(\tau) = \min_{u,\theta} \frac{\mathcal{Q}(u,\theta,\tau)}{\|\theta\|^2 + (Pr\,Ra)^{-1}\|u\|^2}$$

The conductive state τ is *exponentially asymptotically stable* if

$$\mathbb{E}[\lambda(\tau^S)] = \liminf_{t\to\infty} \frac{1}{t}\int_0^t \lambda(\tau(s))ds > 0$$

RB convection - critical Rayleigh number

Note: $\lambda(\tau)$ is the smallest eigenvalue of the Euler-Lagrange equations

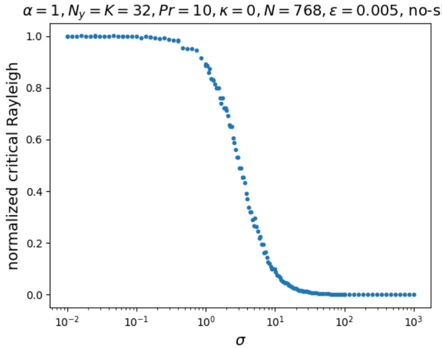
$$\begin{aligned} \frac{\lambda}{Pr\,Ra}u &= -\frac{1}{Ra}\Delta u + \nabla q + \kappa\,(\tau_y - 1)\,(0,\theta), \quad \nabla \cdot u = 0, \\ \lambda\theta &= -\Delta\theta + \kappa\,(\tau_y - 1)\,u_2. \end{aligned}$$

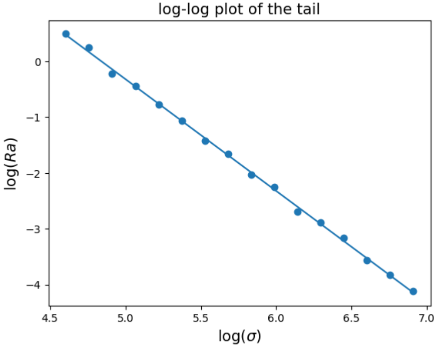
Step 1: Reduction to an ODE eigenvalue problem:

$$\begin{aligned} -\frac{\lambda}{Pr\,Ra}(\partial_{yy} - \xi^2)\hat{u}_2 &= \frac{1}{Ra}(\partial_{yyyy} - 2\xi^2\partial_{yy} + \xi^4)\hat{u}_2 + \frac{1}{2}(\tau_y^S - 1)\xi^2\hat{\theta} \\ \lambda\hat{\theta} &= -(\partial_{yy} - \xi^2)\hat{\theta} + \frac{1}{2}(\tau_y^S - 1)\hat{u}_2 \end{aligned}$$

Step 2: For $\xi \in [a, b]$, we use Monte Carlo generated samples of τ^S to find the smallest eigenvalue numerically via the Dedalus software package.
Step 3: Find $\lambda_* = \min\{\lambda : \ \xi \in [a, b]\}$ via *scipy.optimize* of Python.
Step 4: Since we want $\mathbb{E}\lambda_* = 0$, we adjust the interval $[a, b]$ via bisection root-finding method and go back to Step 2.

RB convection - Numerical results

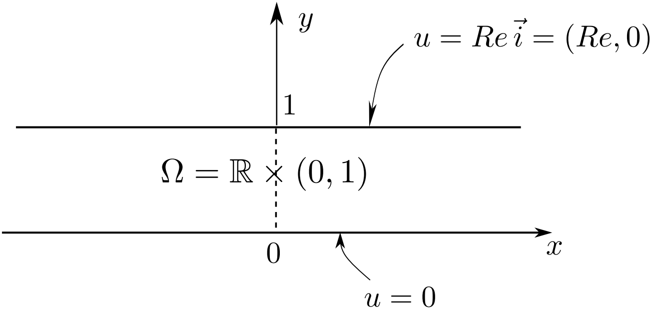




Observations:

- Stochastic forcing at small magnitude has little to no effect, while strong stochastic forcing has a destabilizing effect.
- Algebraic decay rate $Ra \sim \sigma^{-2}$ as $\sigma \to \infty$.

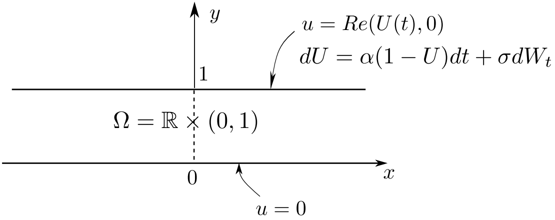
Couette flow



$$\begin{cases} u_t + u\nabla u + \nabla p &= \Delta u, \\ \operatorname{div} u &= 0. \end{cases}$$

- Laminar flow solution: $u(x, y, t) = y\vec{i} = (y, 0)$
- If $Re \leq Re_c \approx 177.2$: laminar flow is a global attractor (Orr 1907)

Couette flow - Stochastic boundary forcing



Theorem (Foldes, Pham, Whitehead (2025))

The laminar flow $v(x, y, t) = Re(\chi(y, t), 0)$ *where*

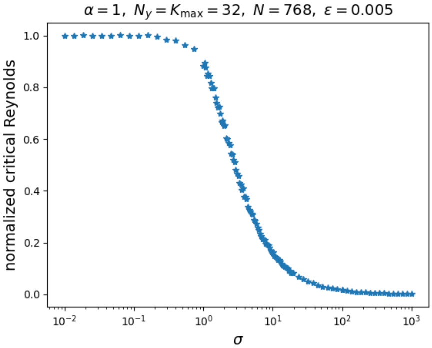
$$\chi_t - \chi_{yy} = 0, \ \chi(0, t) = 0, \ \chi(1, t) = U(t)$$

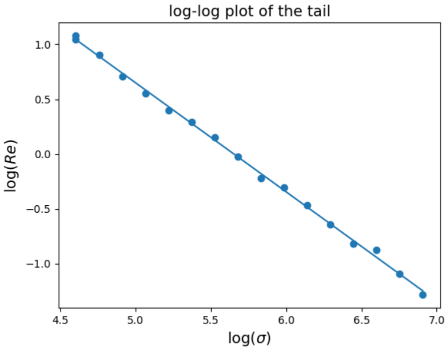
is ergodic and $\chi(y, t) \xrightarrow{d} \chi^S(y)$ *as* $t \to \infty$, *where*

$$\chi^S(y) = (1 + \gamma_0)\, y + \sum_{k=1}^{\infty} \gamma_k \sin(k\pi(1 - y))$$

Here, $(\gamma_0, \gamma_1, \gamma_2, \ldots)$ *is a Gaussian vector distributed as in RB convection case.*

Couette flow - Numerical results





Observations:

- Stochastic forcing at small magnitude has little to no effect, while strong stochastic forcing has a destabilizing effect.
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Thank You!