Stochastic cascade, symmetry and comparison method for the Navier-Stokes equations

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MSRI - Applied Fluids seminar

May 7, 2021

The equation

$$
\text{(NSE)}: \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^d \times (0, \infty), \\ \text{div } u = 0 & \text{in} \quad \mathbb{R}^d \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^d. \end{cases}
$$

Goals:

- Brief overview on the global regularity and blowup.
- Le Jan-Sznitman's method for NSE on the Fourier side. Two problems: *stochastic explosion* and *integrability issue*.
- Conservation of frequency and applications to global regularity and finite time blowup of a model equation.

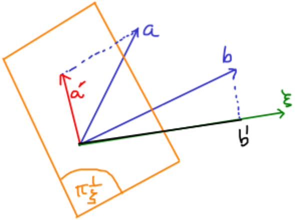
Global regularity and blowup

- Mild solutions are solutions obtained by Picard's iteration $u_{n+1} = e^{\Delta t} u_0 + B(u_n, u_n)$. Regularity of u can be as good as U, e.g. smooth, analytic, etc.
- Local existence if u_0 is in scale-subcritical spaces. Global existence if u_0 is in scale-critical spaces.
- Finite time blowup is unknown. Finite time blowup, if exists, happens simultaneously in many spaces: $L_t^p L_x^q$, homogeneous Sobolev, Besov, etc.
- The energy identity plays the role of a mechanism to prevent blowup.
- Model equations that exhibit blowup: Montgomery-Smith (2001), Gallagher-Paicu (2009), Sinai-Li (2008), Tao (2015).

Fourier-transformed Navier-Stokes equations (FNS)

$$\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u}_0(\xi) + c_0 \int_0^t e^{-|\xi|^2 s}|\xi| \int_{\mathbb{R}^d} \hat{u}(\eta,t-s) \odot_\xi \hat{u}(\xi-\eta,t-s)\, d\eta\, ds$$

where $a \odot_\xi b = -i(e_\xi \cdot b)(\pi_{\xi^\perp} a)$.



$a \odot_\xi b = -i\, b' a'$

Global regularity and blowup

- Li-Ozawa-Wang (2018): if $\text{supp}\hat{u}_0 \subset \{(\xi_1, \ldots, \xi_d) : \xi_k \geq \ell \gg 1 \,\forall k\}$ and $\|\hat{u}_0\|_{L^\infty} \leq \ell$ then the solution is global-in-time.
- Feichtinger-Gröchenig-Li-Wang (2021): if $\text{supp}\hat{u}_0 \subset \{(\xi_1, \ldots, \xi_d) : \xi_k \geq 0 \,\forall\, k\}$ and $u_0 \in L^2$ then the solution is global-in-time.

Theorem (P., 2021)

If $\text{supp}\hat{u}_0 \subset \mathbb{R}^d_+$ and $\hat{u}_0$ belongs to a Herz space $\dot{K}^\alpha_{p,q}$, where α, p, q are in a suitable range, then the solution is global-in-time. [If $d = 2$ or 3, L^2 is included in this family.]

$$\dot{K}^\alpha_{p,q} = \{f : |\xi|^\alpha f \mathbb{1}_{A_k} \in l^q L^p\}$$

Le Jan-Sznitman's construction of solutions

Normalization: (LJS 1997, Bhattacharya et al 2003)

$$\begin{aligned}\chi(\xi,t) &= e^{-t|\xi|^2}\chi_0(\xi) \\ &+ \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^d} \chi(\eta,t-s) \odot_\xi \chi(\xi-\eta,t-s) H(\eta|\xi) d\eta ds\end{aligned}$$

where $\chi = c_0 \dfrac{\hat{u}}{h}$ and $H(\eta|\xi) = \dfrac{h(\eta)h(\xi-\eta)}{|\xi|h(\xi)}$.

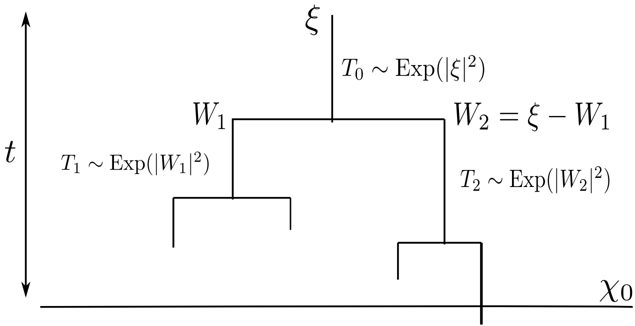
h satisfies $h * h = |\xi| h$, called *majorizing kernel*.

$$\boxed{\chi = \mathbb{E}[\mathbf{X}_{FNS}]}$$

where the stochastic process $\mathbf{X}_{FNS}$ is defined recursively by. . .

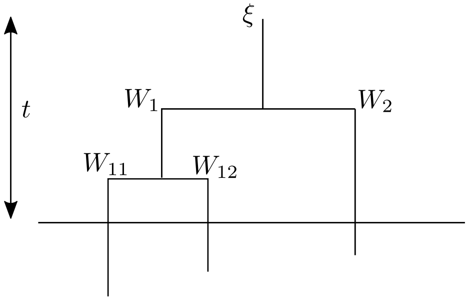
Le Jan-Sznitman's construction of solutions

$$\mathbf{X}_{\text{FNS}}(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T_0 > t, \\ \mathbf{X}^{(1)}_{\text{FNS}}(W_1, t - T_0) \odot_\xi \mathbf{X}^{(2)}_{\text{FNS}}(\xi - W_1, t - T_0) & \text{if} \quad T_0 \leq t. \end{cases}$$



An example of $\mathbf{X}_{\mathrm{FNS}}$

Consider the following event:



On this event,

$$\mathbf{X}_{\mathsf{FNS}}(\xi, t) = (\chi_0(W_{11}) \odot_{W_1} \chi_0(W_{12})) \odot_{\xi} \chi_0(W_2).$$

Three ingredients: clocks, branching distribution, product.

Two issues of the construction

For $\chi = \mathbb{E}[\mathbf{X}_{FNS}]$ to be well-defined, one needs:

- *Stochastic non-explosion*: after finitely many branchings on the tree, all branches must terminate.
- *Integrability*: $\mathbb{E}|\mathbf{X}_{FNS}| < \infty$.

Can stochastic explosion happen?

- $h(\xi) = C_d|\xi|^{1-d}$: yes for $d = 3$, no for $d \geq 12$.
- $h(\xi) = C|\xi|^{-1}e^{-|\xi|}$ $(d = 3)$: no
  (Dascaliuc, P., Thomann, Waymire '21)

Overcome explosion issue: $\boxed{\chi = \mathbb{E}[\mathbf{X}_{FNS}\mathbb{1}_{no.exp}]}$

Simplifying the product

How to deal with the integrability issue?

$$\begin{aligned}\chi(\xi,t) &= e^{-t|\xi|^2}\chi_0(\xi) \\ &+ \int_0^t e^{-s|\xi|^2}|\xi|^2\int_{\mathbb{R}^d}\chi(\eta,t-s)\chi(\xi-\eta,t-s)H(\eta|\xi)d\eta ds \\ &= \mathbb{E}[\mathbf{X}(\xi,t)]\end{aligned}$$

where

$$\mathbf{X}(\xi,t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T_0 > t, \\ \mathbf{X}^{(1)}(W_1, t-T_0)\mathbf{X}^{(2)}(\xi - W_1, t-T_0) & \text{if} \quad T_0 \le t. \end{cases}$$

$$(\mathrm{MS}): \begin{cases} \partial_t u - \Delta u &= \sqrt{-\Delta}(u^2) \quad \text{in} \quad \mathbb{R}^d \times (0,\infty), \\ u(\cdot,0) &= u_0 \quad \text{in} \quad \mathbb{R}^d \end{cases}$$

Majorizing principle and symmetries

χ_0 is a vectorial value function, yielding $\mathbf{X}_{FNS}$.
$|\chi_0|$ is a scalar function, yielding $\mathbf{X}$.

- *Majorizing principle*:

$$|\mathbf{X}_{FNS}| \leq \mathbf{X}. \quad (\text{Thus, } \mathbb{E}|\mathbf{X}_{FNS}| \leq \mathbb{E}\mathbf{X}.)$$

- *Power symmetry*:

$$|\chi_0| \to |\chi_0|^\alpha, \ \mathbf{X} \to \mathbf{X}^\alpha.$$

- *Conservation of frequency*:

$$\chi_0 \to e^{\xi \cdot a}\chi_0, \quad \mathbf{X}_{FNS} \to e^{\xi \cdot a}\mathbf{X}_{FNS}, \quad \mathbf{X} \to e^{\xi \cdot a}\mathbf{X}.$$

Majorizing principle and symmetries

Generalized majorizing principle:

- For a positive submultiplicative function ϕ,

$$|\chi_0| \to \zeta_0 = \phi(|\chi_0|),\ \phi(\mathbf{X}) \le \mathbf{Z}$$

- If, in addition, ϕ is convex then

$$|\chi_0| \to \zeta_0 = \phi(|\chi_0|),\ \phi(\chi) \le \zeta$$

Here $\mathbf{Z}$ and ζ are the stochastic process and solution yielded from ζ_0.

Rule of thumb: the (Fourier) Montgomery-Smith equation controls the pointwise values of the solution to (Fourier) Navier-Stokes equations.

Applications of the conservation of frequency

Frequency trap

If $\hat{u}_0$ is supported in a cone $\mathcal{C}_{a,\theta} = \{\xi : \ \xi \cdot a \geq |\xi||a| \cos\theta\}$ with an opening angle $\theta \in [0, \pi/2]$ then $\hat{u}$ is supported in the same cone for all t.

Proof. $\chi_0 = \hat{u}/h$ is supported in the cone. For $\xi \in \mathbb{R}^d$ outside of the cone and with a realization of the cascade,

$$\mathbf{X} = \chi_0(w_1)\ldots\chi_0(w_m).$$

Since $w_1 + \ldots + w_m = \xi$, at least one w_k has to lie outside of the cone. Thus, $\mathbf{X} = 0$.

Applications of the conservation of frequency

Conservation of frequency can be seen from the equation (Fourier side):

$$\hat{u}_0 \to e^{\xi \cdot a}\hat{u}_0, \quad \hat{u} \to e^{\xi \cdot a}\hat{u}$$

Global solution for supercritical/critical initial data

If supp$\hat{u}_0 \subset \mathbb{R}^d_+$ and $\hat{u}_0 \in L^2$, where $d = 2$ or 3, then the solution is global-in-time.

Proof. By Hölder's inequality,

$$\|e^{-\xi_d}\hat{u}_0\|_{\dot{K}^{-1}_{1,2}} \le \|\hat{u}_0\|_{\dot{K}^0_{2,2}} \|e^{-\xi_d}\mathbb{1}_{\xi_d \ge 0}\|_{\dot{K}^{-1}_{2,\infty}} < \infty.$$

Thus, $e^{-n\xi_d}\hat{u}_0$ is small in $\dot{K}^{-1}_{1,2}$ if n is large enough. $\dot{K}^{-1}_{1,2}$ is a scale-critical space for which global well-posedness is known (Cannone-Wu, 2012). Scale back to get the solution.

Applications of the conservation of frequency

Global solution for subcritical initial data

If supp$\hat{u}_0 \subset \mathbb{R}^d_+$ and $\hat{u}_0 \in L^p$, where $1 \leq p < \frac{d}{d-1}$, then the solution is global-in-time.

Proof. The solution exists at least until time $T \sim \|\hat{u}_0\|_p^{-\delta}$ where $\delta = 2p/(d-(d-1)p)$.
The scaled initial data $\hat{u}_{0n} = e^{-n\xi_d}\hat{u}_0$ gives a solution at least until time $T_n \sim \|\hat{u}_{0n}\|_p^{-\delta} \to \infty$ as $n \to \infty$.
Scale back to get a global solution.

The half-space is optimal

Blowup: solution $v \geq 0$ to the (Fourier) Montgomery-Smith equation is said to blow up after finite time if there exists $T > 0$ such that the set $\{\xi : \; v(\xi, T) = \infty\}$ has positive Lebesgue measure.

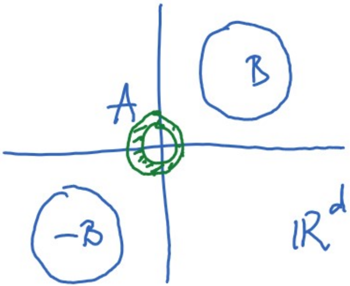
"Good" domains: a set $\bar{D}$ (closure of an open set $D \subset \mathbb{R}^d$) is said to have Property (P) if any bounded function, compactly supported on $\bar{D}$, gives a global solution.

Theorem (P., 2021)

A half-space is a maximal set that has Property (P).

Proof. An open set strictly larger than a half-space must contain two balls symmetric w.r.t. the origin.

The half-space is optimal



Put $\Omega = B \cup (-B)$.
Key observation: $|(\xi - \Omega) \cap \Omega| > c > 0$ for every $\xi \in A$.
Put $\phi = \mathbb{1}_\Omega$ and choose the initial condition $v_0 = M\phi$. Oseen's representation of the solution says that

$$v = \sum_{k=1}^{\infty} M^k v_k \quad \text{where} \quad v_1(\xi, t) = e^{-t|\xi|^2} \phi,$$

The half-space is optimal

$$v_n(\xi,t) = \int_0^t \int_{\mathbb{R}^d} |\xi| e^{-(t-s)|\xi|^2} \sum_{k=1}^{n} v_k(\eta,s) v_{n-k}(\xi-\eta,s) d\eta ds$$

Put $w_k(t) = \inf_{\xi \in A} v_{2^k}(\xi,t)$. By induction,

$$w_k(t) \gtrsim e^{-\alpha 2^k} 2^k (te^{-t})^{2^k-1} \ \ \forall\, k$$

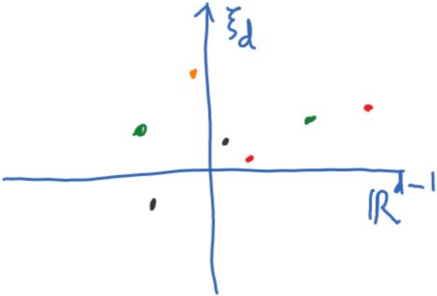
For $\xi \in A$,

$$v(\xi,1) \geq \sum M^{2^k} v_{2^k}(\xi,1) \gtrsim \sum M^{2^k} e^{-\alpha 2^k} 2^k e^{-(2^k-1)} \to \infty.$$

Probability for all paths to end up in the half-space

Branching distribution:

$$H(\eta|\xi) \sim \frac{|\xi|^{d-2}}{|\eta|^{d-1}|\xi-\eta|^{d-1}}$$



$$p_k(\xi, t) = \mathbb{P}_\xi(\text{by time } t, \text{ all paths ending up in } \mathbb{R}^d_+)$$

Probability for all paths to end up in the half-space

Take $\chi_0(\xi) = M\mathbb{1}_{\xi_d \geq 0}$, or equivalently $\hat{u}_0 \sim M|\xi|^{1-d}\mathbb{1}_{\xi_d \geq 0}$.
Note that $e^{-\xi_d}\hat{u}_0 \in L^p$ for any $1 < p < d/(d-1)$.
Global solution $\hat{u} \sim |\xi|^{1-d}\chi$ exists. Thus,

$$\chi = \sum M^k p_k < \infty \quad \forall\, M > 0.$$

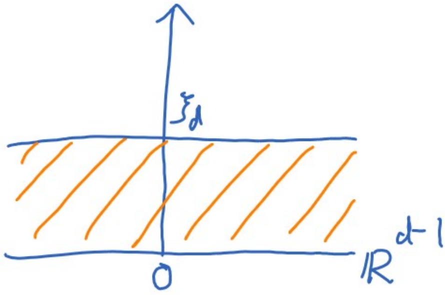
p_k must decay faster than any exponential function of k. To quantify the rate of decay, one needs to understand more the geometry of the half-space.

$$p_n(\xi, t) = \int_0^t \int_{\mathbb{R}^d} |\xi|^2 e^{-(t-s)|\xi|^2} \sum_{k=1}^{n} p_k(\eta, s) p_{n-k}(\xi - \eta, s) H(\eta|\xi) d\eta ds$$

Probability for all paths to end up in the half-space

The key idea is the following:

$$\boxed{\int_{(\xi-\mathbb{R}^d_+)\cap\mathbb{R}^d_+} H(\eta|\xi) \le c_{\delta,d}\left(\frac{\xi_d}{|\xi|}\right)^{\delta} \quad \forall \delta\in(0,1)}.$$



Probability for all paths to end up in the half-space

The following lemma is needed for the induction procedure: for $\phi(k) = (k!)^{\gamma}$, where $\gamma \in (0,1)$, one has

$$\sum_{k=1}^{n-1} \frac{a^k}{\phi(k)} \frac{b^{n-k}}{\phi(n-k)} \leq \frac{(a+b)^n}{\phi(n)} n^{\gamma-1} \quad \forall\, a, b > 0,\ n \in \mathbb{N}.$$

Theorem (P., 2021)

$$p_k(\xi, t) \leq c_d^{k-1} \frac{(\xi_d \sqrt{t})^{\frac{1}{2}(k-1)}}{\{(k-1)!\}^{1/8}} e^{-|\xi|\sqrt{t}}.$$

Thank You!