Non-explosion criterion for branching Markov chains and applications

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KPP equation in Fourier domain

$$u_t - Du_{xx} = ru(1-u) \quad \forall t > 0,\, x \in \mathbb{R}$$

By rescaling the time and space variables, and introducing $v = 1 - u$, we get

$$v_t - v_{xx} = v^2 - v \quad \forall t > 0,\, x \in \mathbb{R}.$$

In Fourier domain (integral form),

$$\hat{v}(\xi, t) = e^{-(1+\xi^2)t}\hat{v}_0(\xi) + \int_0^t \int_{-\infty}^{\infty} e^{-(1+\xi^2)s}\hat{v}(\eta, t-s)\hat{v}(\xi-\eta, t-s)d\eta ds.$$

Normalization: $\chi(\xi, t) = \frac{\hat{v}(\xi,t)}{h(\xi)}$

KPP equation in Fourier domain

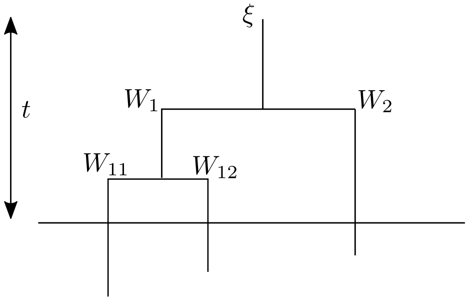
$$
\begin{aligned}
\chi(\xi, t) &= e^{-(1+\xi^2)t}\chi_0(\xi) \\
&+ \int_0^t \int_{-\infty}^{\infty} (1+\xi^2)e^{-(1+\xi^2)s}\chi(\eta, t-s)\chi(\xi-\eta, t-s)H(\eta|\xi)d\eta ds.
\end{aligned}
$$

where $H(\eta|\xi) = \frac{h(\eta)h(\xi-\eta)}{(1+\xi^2)h(\xi)}$.

$$
\mathbf{X}(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T_0 > t, \\ \mathbf{X}^{(1)}(W_1, t)\mathbf{X}^{(2)}(\xi - W_1, t) & \text{if} \quad T_0 \le t. \end{cases}
$$

$$
\chi(\xi, t) = \mathbb{E}[\mathbf{X}(\xi, t)]
$$

Cascade of Fourier-transformed KPP equation



On this event,

$$\mathbf{X}(\xi, t) = \chi_0(W_{11})\chi_0(W_{12})\chi_0(W_2)$$

Explosion time

The explosion time is

$$\zeta = \zeta(\xi) = \lim_{n\to\infty} \min_{|v|=n} \sum_{j=0}^{n} \frac{T_{v|j}}{1 + W_{v|j}^2}.$$

The event $[\zeta = \infty]$ is called the non-explosion event. If $\zeta = \infty$ a.s then the cascade is called non-explosive.

Navier-Stokes equations

$$
(\mathrm{NSE}):\ \begin{cases} \partial_t u - \Delta u + u\cdot\nabla u + \nabla p = 0 & \text{in } \mathbb{R}^d\times(0,\infty),\\ \operatorname{div}\ u = 0 & \text{in } \mathbb{R}^d\times(0,\infty),\\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^d. \end{cases}
$$

Integro-differential equation:

$$
u(x,t) = e^{\Delta t}u_0 - \int_0^t e^{\Delta s}\mathbf{P}\mathrm{div}[u(t-s)\otimes u(t-s)]ds.
$$

Navier-Stokes equations

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Integro-differential equation:

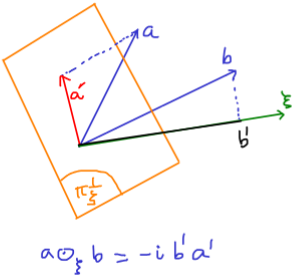
$$
u(x,t) = e^{\Delta t} u_0 - \int_0^t e^{\Delta s} \mathbf{P} \text{div}[u(t-s) \otimes u(t-s)] ds.
$$

In Fourier domain:

$$
\hat{u}(\xi,t) = e^{-|\xi|^2 t} \hat{u}_0(\xi) + c_0 \int_0^t e^{-|\xi|^2 s} |\xi| \int_{\mathbb{R}^d} \hat{u}(\eta, t-s) \odot_\xi \hat{u}(\xi-\eta, t-s) d\eta ds
$$

where $a \odot_\xi b = -i(e_\xi \cdot b)(\pi_{\xi^\perp} a)$.

Fourier-transformed Navier-Stokes equations (FNS)

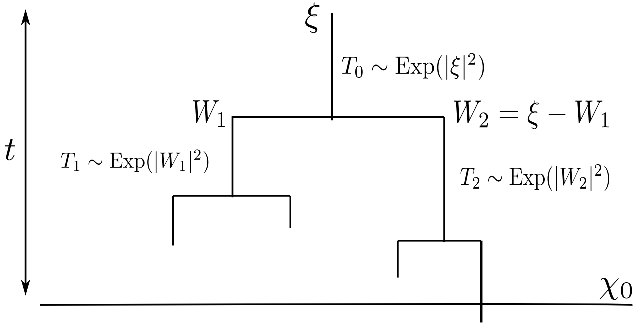


Normalization to (FNS): LJS 1997, Bhattacharya et al 2003.

$$\begin{aligned}\chi(\xi,t) &= e^{-t|\xi|^2}\chi_0(\xi) \\ &+ \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^d} \chi(\eta,t-s) \odot_\xi \chi(\xi-\eta,t-s) H(\eta|\xi)\, d\eta\, ds\end{aligned}$$

where $\chi = c_0\hat{u}/h$ and $H(\eta|\xi) = \frac{h(\eta)h(\xi-\eta)}{|\xi|h(\xi)}$.

h: *majorizing kernel*, i.e. $h * h = |\xi| h$.



Define a stochastic multiplicative functional recursively as

$$\mathbf{X}_{\text{FNS}}(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T_0 > t, \\ \mathbf{X}^{(1)}_{\text{FNS}}(W_1, t - T_0) \odot_\xi \mathbf{X}^{(2)}_{\text{FNS}}(\xi - W_1, t - T_0) & \text{if} \quad T_0 \le t. \end{cases}$$

Explosion time

The explosion time is

$$\zeta = \zeta(\xi) = \lim_{n\to\infty} \min_{|v|=n} \sum_{j=0}^{n} \frac{T_{v|j}}{|W_{v|j}|^2}.$$

The event $[\zeta = \infty]$ is called the non-explosion event. If $\zeta = \infty$ a.s then the cascade is called non-explosive.

Stochastic explosion

Branching process may never stop, potentially making the stochastic multiplicative functional not well-defined.

- 3D self-similar cascade $h_{\mathrm{dilog}}(\xi) = C|\xi|^{-2}$: stochastic explosion a.s. (Dascaliuc, Pham, Thomann, Waymire 2019)
- 3D Bessel cascade $h_{\mathrm{b}}(\xi) = C|\xi|^{-1}e^{-|\xi|}$: non-explosive a.s. (Orum, Pham 2019)

Branching Markov chain

Definition: A tree-indexed family $Y = \{Y_v\}_{v \in \mathcal{V}}$ of random variables on a countable or continuous state space $S \subset (0, \infty)$ is called a *(binary) branching Markov chain* if it satisfies the following:

- (A) For any path $s \in \mathcal{T}_\infty$, the sequence $Y_\emptyset,\ Y_{s|1},\ Y_{s|2}, \ldots$ is a time homogeneous Markov chain.
- (B) For any path $s \in \mathcal{T}_\infty$, the stationary transition probability $p(x, dy)$ does not depend on s.
- (C) For each $\sigma \in \mathcal{V}$, the two subtrees $\{Y_{\sigma \cdot 1 \cdot v}\}_{v \in \mathcal{V}}$ and $\{Y_{\sigma \cdot 2 \cdot v}\}_{v \in \mathcal{V}}$ are independent of each other given $Y_{\sigma \cdot 1}$ and $Y_{\sigma \cdot 2}$.

Le Jan - Sznitmann (LJS) cascade

Function $g : [0, \infty) \to [0, \infty)$ satisfying

- g is locally bounded,
- $g(x) > 0$ for all $x > 0$.

We will refer to this stochastic model as *Le Jan-Sznitmann (LJS) cascade*.
Definition: The explosion time of an LJS cascade (Y, T, g) is a $[0, \infty]$-valued random variable ζ defined by

$$\zeta = \lim_{n\to\infty} \min_{|s|=n} \sum_{j=0}^{n} \frac{T_{s|j}}{g(Y_{s|j})}.$$

The event of no-explosion is defined by $[\zeta = \infty]$. The cascade is said to be non-explosive if $\mathbb{P}(\zeta = \infty) = 1$, and explosive if $\mathbb{P}(\zeta = \infty) < 1$.

Main statement

Theorem: Let Y be a branching Markov chain. Suppose that

(D) There exist constants $c > 0$, $r > 2$ and a locally bounded function $\psi : [0, \infty) \to \mathbb{R}$ such that $I_n(a) \le \psi(a) r^{-n}$ for all $n \in \mathbb{N}$ and $a > 0$, where

$$I_n(a) = \mathbb{P}_a(Y_{s|1} > c, Y_{s|2} > c, \ldots, Y_{s|n} > c).$$

Here $\mathbb{P}_a$ denotes the probability measure given the initial state $Y_\emptyset = a > 0$ (constant).

Then for any $a > 0$, an LJS cascade (Y, T, g) with initial state $Y_\emptyset = a$ is non-explosive.

Proof

The proof is done in two stages. The first stage is to show that the branching Markov chain Y visits the region $(0, c]$ "infinitely often". We start an inspection process of whether $Y_v > c$ as follows...

$$
\begin{aligned}
\mathbb{P}_a(\mathcal{O}_n = 2) &= p_{n,2} := \mathbb{P}_a\big(Y_{v\cdot 1} > c,\ Y_{v\cdot 2} > c \mid Y_{v|1}, \ldots, Y_{v|n} > c\big), \\
\mathbb{P}_a(\mathcal{O}_n = 1) &= \underbrace{\mathbb{P}_a\big(Y_{v\cdot 1} > c, Y_{v\cdot 2} \le c \mid Y_{v|1}, \ldots, Y_{v|n} > c\big)}_{p_{1,n}} \\
&\quad + \underbrace{\mathbb{P}_a\big(Y_{v\cdot 1} \le c, Y_{v\cdot 2} > c \mid Y_{v|1}, \ldots, Y_{v|n} > c\big)}_{\tilde{p}_{1,n}}, \\
\mathbb{P}_a(\mathcal{O}_n = 0) &= p_{n,0} := \mathbb{P}_a\big(Y_{v\cdot 1} \le c,\ Y_{v\cdot 2} \le c \mid Y_{v|1}, \ldots, Y_{v|n} > c\big)
\end{aligned}
$$

Proof

Claim: the inspection process terminates a.s. after finitely many steps.

$$\begin{aligned}
\mu_n := \mathbb{E}\mathcal{O}_n &= 2p_{n,2} + p_{n,1} + \tilde{p}_{n,1} = (p_{n,2} + p_{n,1}) + (p_{n,2} + \tilde{p}_{n,1}) \\
&= \mathbb{P}(Y_{s\cdot 1} > c \,|\, Y_{s|1}, \ldots, Y_{s|n} > c) \\
&+ \mathbb{P}(Y_{s\cdot 2} > c \,|\, Y_{s|1}, \ldots, Y_{s|n} > c) \\
&= 2\mathbb{P}(Y_{s\cdot 1} > c \,|\, Y_{s|1}, \ldots, Y_{s|n} > c) \\
&= 2\frac{I_{n+1}}{I_n}.
\end{aligned}$$

$$\mathbb{E}Z_n = \mathbb{E}[\mathbb{E}[Z_n|Z_{n-1}]] = \mathbb{E}\left[\sum_{j=1}^{Z_{n-1}} \mathbb{E}\mathcal{O}_{n-1,j}\right] = \mathbb{E}[\mu_{n-1}Z_{n-1}] = \mu_{n-1}\mathbb{E}Z_{n-1}.$$

Proof

Construct a sequence of cutsets:

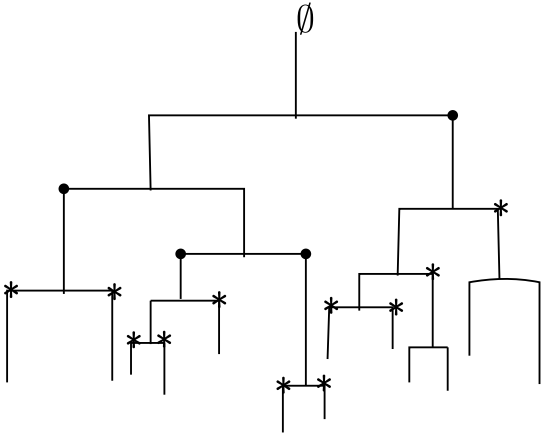


Figure 1: The first cutset $\mathcal{C}_1$ consists of the bold dots. The second cutset $\mathcal{C}_2$ consists of the stars.

Proof

Each cutset has bounded expectation:
For any $\sigma \in \{\emptyset\} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots$, we have

$$\mathbb{E}_a[\mathrm{card}\, \mathcal{C}^{(\sigma)}] \leq \mu$$

where $\mu = \mu(a) = \max\left\{1,\ M(a)\frac{2r}{r-2}\right\}$ and $M(a) = \sup_{x \leq \max\{a,c\}} \psi(x)$.

Proof

Reduce the problem:

$$\zeta_n(\omega, \tilde{\omega}) \geq \min_{|v|=n} \sum_{i=1}^{k} \frac{T_{s|I_i}(\tilde{\omega})}{g(Y_{v|I_i}(\omega))} \geq \frac{1}{C} \min_{|v|=n} \sum_{i=1}^{k} T_{v|I_i}(\tilde{\omega})$$

Proof

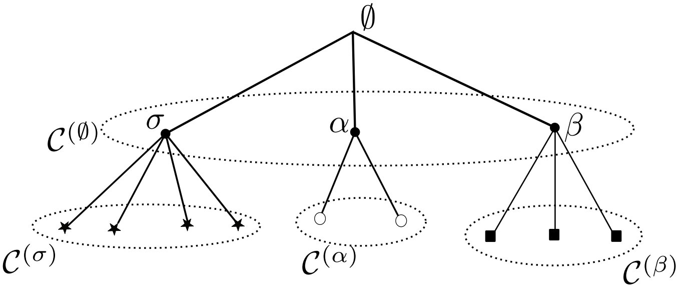


Figure 2: Tree $\mathcal{T}'(\omega)$.

Thank You!