Minimal blowup data for potential Navier-Stokes singularities in the half-space

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Cauchy problem of NSE

For $\Omega = \mathbb{R}^3$ or $\mathbb{R}^3_+$, consider

$$(\mathrm{NSE})_\Omega : \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f & x \in \Omega, \quad t > 0, \\ \mathrm{div}\ u = 0 & x \in \Omega, \quad t > 0, \\ u(x,t) = 0 & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0 & x \in \Omega. \end{cases}$$

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- Scaling symmetry :

$$\begin{aligned} u(x,t) &\to \lambda u(\lambda x, \lambda^2 t) \\ p(x,t) &\to \lambda^2 p(\lambda x, \lambda^2 t) \\ f(x,t) &\to \lambda^3 f(\lambda x, \lambda^2 t) \\ u_0(x) &\to \lambda u_0(\lambda x) \end{aligned}$$

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- Critical spaces : $u_0 \in L^3$, $f \in L^{5/3}_{t,x}$, $u \in L^5_{t,x}, \ldots$

Mild Solutions

- **Helmholtz decomposition:** $g = v + \nabla\phi$

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- Local in time, unique, regular.
- Characterization of finite-time blowup: $\lim_{T \to T_*} \|u\|_{L^5(\Omega \times (0,T))} = \infty$.
- Globally well-posed if (u_0, f) is sufficiently small in critical spaces.

Weak Solutions

Suitable weak solution:

$$\text{Leray '34, Scheffer '77, C–K–N '82, Lemarié-Rieusset '02} \quad \begin{cases} \text{weak form,} \\ \text{local energy inequality,} \\ u(t) \to u_0 \text{ in } L^2_{\text{loc}} \text{ as } t \downarrow 0. \end{cases}$$

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Sw-solution:

Seregin–Sverak (2017) $\begin{cases} u = v + w \\ v \text{ satisfies linear Stokes eq. with data } (u_0, f) \\ w \text{ satisfies } \begin{cases} \partial_t w - \Delta w + \nabla \pi = -u \cdot \nabla u \text{ weakly} \\ \text{energy inequality} \end{cases} \end{cases}$

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ε-regularity criterion (C–K–N '82, Lin '98, Seregin 2002)

There are two positive constants ε and C such that

$$\frac{1}{r^2} \iint_{Q_r(x_0,t_0)} \left(|u|^3 + |p|^{\frac{3}{2}}\right) dxdt \le \varepsilon \quad \Longrightarrow \quad \sup_{Q_{r/2}(x_0,t_0)} |u(x,t)| \le \frac{C}{r}.$$

Minimal blowup data

$$\rho^{\Omega}_{\mathsf{max}} = \sup\left\{\rho : T_{\mathsf{max}}(u_0, f) = \infty \ \text{ if } \ \|(u_0, f)\|_{X\times Y} < \rho\right\}.$$

Question

If $\rho^{\Omega}_{\max}$ is finite, does there exist a data $(u_0, f) \in X \times Y$ with $\|(u_0, f)\| = \rho^{\Omega}_{\max}$, such that the solution u of $(\mathrm{NSE})_{\Omega}$ blows up in finite time ?

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Affirmation for $\Omega = \mathbb{R}^3$, $f \equiv 0$ and $u_0 \in X$

- $X = \dot{H}^{1/2}$: Rusin–Sverak 2011.
- $X = L^3$: Jia–Sverak 2013, Gallagher–Koch–Planchon 2013.
- $X = \dot{B}^{-1+3/p}_{p,q}$ $(3 < p, q < \infty)$: G–K–P 2016.

Main Results

Assume $u_0 = 0$.

$$Y_q = \left\{ f : \quad t^{q^*} f \in L^q_{t,x} \right\}, \quad \frac{5}{2} < q < 3, \quad q^* = \frac{3}{2} - \frac{5}{2q}$$

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Theorem 1

For $\Omega = \mathbb{R}^3$ and $Y = Y_q$, minimal blowup data exists, provided that a blowup data exists.

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Theorem 2

(a) $\rho^+_{\max} \leq \rho_{\max}$,

(b) If $\rho^+_{\max} < \rho_{\max}$ then there exists a minimal blowup data for $\Omega = \mathbb{R}^3_+$.

Theorem 1: Sketch of proof

- Step 1 : Blowup happens only if there occurs a singular point.

$$\|u\|_{Q_r(x_0, T_{\max})} = \infty \quad \forall r > 0.$$

This is an application of ε-regularity criterion !

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- Step 2 : Set up minimizing sequence.

$$\|f^k\| \downarrow \rho_{\max}$$

  (u^k, p^k) is mild solution with data f^k, singular at (x^k, t^k).

Theorem 1: Sketch of proof

- Step 3 : Normalize (x^k, t^k) to $(0,1)$ by translation/scaling symmetry.

$$u^k(x,t) \quad \to \quad \lambda_k u^k\left(\frac{x - x^k}{\lambda_k}, \frac{t}{\lambda_k^2}\right), \qquad \lambda_k = \sqrt{t^k}$$

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- Step 4 : Compactness Theorem (Seregin–Sverak '17, Lin '98).

$$u^k \to u \;\text{ in }\; L^3_{\mathrm{loc}}$$

$$p^k \rightharpoonup p \;\text{ in }\; L^{3/2}_{\mathrm{loc}}$$

$$f^k \rightharpoonup f \;\text{ in }\; Y_q$$

(u,p) is sw-solution with data f.

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- Step 5 : $z_0 = (0,1)$ is a singularity of u ?

$$\frac{1}{r^2}\iint\limits_{Q_r(z_0)} \left(|u^k|^3 + |p^k|^{\frac{3}{2}}\right) dxdt > \varepsilon \quad \forall r > 0,\ k = 1, 2, \dots$$

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Interior pressure decomposition:

$$p^k = R_i R_j \left(u_i^k u_j^k \right) = \underbrace{R_i R_j \left(u_i^k u_j^k \chi \right)}_{p_1^k \to p_1} + \underbrace{R_i R_j \left(u_i^k u_j^k (1 - \chi) \right)}_{p_2^k \text{ harmonic in } B_1}$$

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- Step 6 : u must blow up !

$$\rho_{\max} \leq \|f\| \leq \liminf_{k \to \infty} \|f^k\| = \rho_{\max}$$

Theorem 2 (b): Sketch of proof

Normalize (x^k, t^k) to $((0,0,d_k),1)$ by translation and scaling.

$$u^k(x,t) \quad \to \quad \lambda_k u^k\left(\frac{x-\overline{x}^k}{\lambda_k}, \frac{t}{\lambda_k^2}\right), \qquad \lambda_k = \sqrt{t^k}$$

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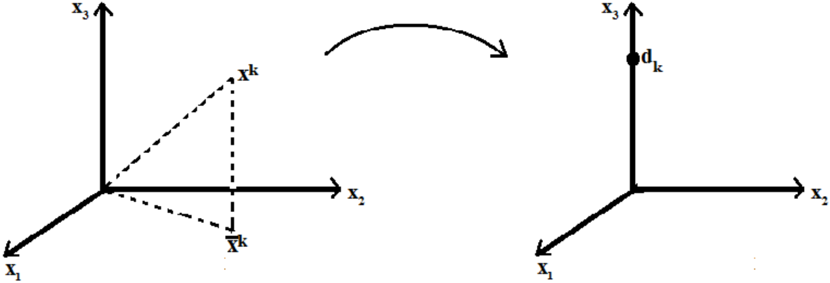

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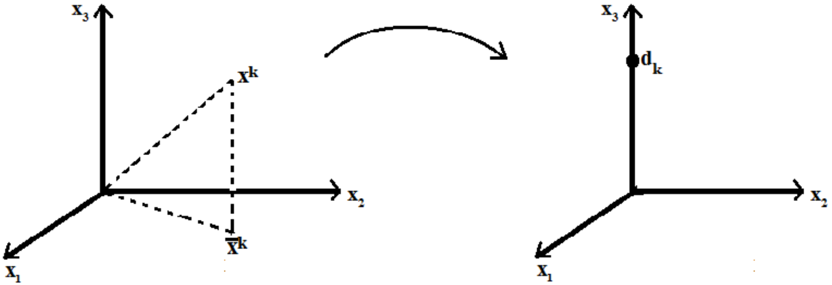

$d_k \to d$ **with** $\mathbf{d > 0}$**, or** $\mathbf{d = 0}$**, or** $\mathbf{d = \infty}$**.**

Theorem 2 (b): the case $d = 0$

Boundary pressure decomposition (Seregin 2002):

$$(u, p) = (v, q) + \underbrace{(w, \pi)}_{(w_1,\pi_1)+(w_2,\pi_2)}$$

$$\begin{cases} \partial_t w_1 - \Delta w_1 + \nabla \pi_1 = u \cdot \nabla u \\ \operatorname{div} w_1 = 0 \\ w_1 = 0 \quad \text{on} \quad \partial_p Q_1^+ \end{cases} \qquad \begin{cases} \partial_t w_2 - \Delta w_2 + \nabla \pi_2 = 0 \\ \operatorname{div} w_2 = 0 \\ w_2 = 0 \quad \text{on} \quad \Gamma \end{cases}$$

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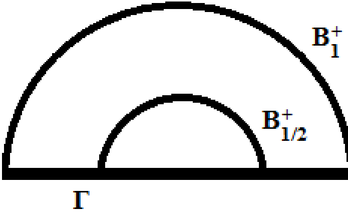

$$\mathbf{Q_r^+ = B_r^+ \times (1 - r^2, 1)}$$

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$$\frac{1}{r^2}\int_{Q_r^+} |\pi^k - \pi|^{\frac{3}{2}} dxdt \to 0 \quad \text{as} \quad k \to \infty, \quad r \to 0$$

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By maximal regularity,

$$\left\| \pi_1^k - \pi_1 \right\|_{L^{\frac{3}{2}}_{t,x}(Q_1^+)} \lesssim \left\| \nabla \pi_1^k - \nabla \pi_1 \right\|_{L_t^{\frac{3}{2}} L_x^{\frac{45}{44}}} \lesssim \left\| u^k \nabla u^k - u \nabla u \right\|_{L_t^{\frac{3}{2}} L_x^{\frac{45}{44}}} \to 0$$

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By Seregin's lemma,

$$\left\|\pi_2^k\right\|_{L_t^{3/2}L_x^{10}(Q_{1/2}^+)} \lesssim \left\|\left(w_2^k,\ \nabla w_2^k,\ \pi_2^k\right)\right\|_{L_t^{3/2}L_x^{9/8}(Q_1^+)} \lesssim \|f^k\| \le M$$

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Claim $\rho^{+}_{\text{max}} \geq \rho_{\text{max}}$!

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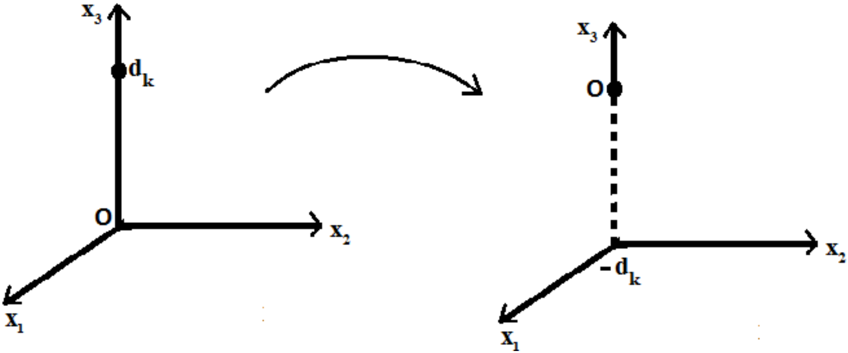

Theorem 2 (b): the case $d = \infty$

- Shift $(0, 0, d_k)$ to the origin (0,0,0).

$$\tilde{u}^k(x,t) = \begin{cases} u^k(x', x_3 + d_k, t), & x_3 > -d_k \\ 0 & x_3 \leq -d_k \end{cases}$$

Similarly, $p^k \rightsquigarrow \tilde{p}^k$, $f^k \rightsquigarrow \tilde{f}^k$.

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- Compactness Theorem :

$$\tilde{f}^k \rightharpoonup \tilde{f} \ \text{ in } \ Y_q, \quad \tilde{u}^k \to \tilde{u} \ \text{ in } \ L^3_{\text{loc}}, \quad \tilde{p}^k \rightharpoonup \tilde{p} \ \text{ in } \ L^{3/2}_{\text{loc}}$$

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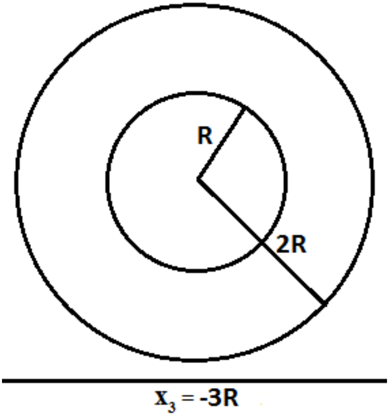
$(\tilde{u}, \tilde{p})$ is sw-solution with data $\tilde{f}$.

- Norm estimate :

$$\rho_{\max} \le \|\tilde{f}\| \le \liminf_{k\to\infty} \|\tilde{f}^k\| = \liminf_{k\to\infty} \|f^k\| = \rho^+_{\max}$$

Theorem 2 (a): $\rho^+_{\max} \leq \rho_{\max}$



Minimal blowup data f **(in whole space)** gives blowup solution u.

Theorem 2 (a): $\rho^+_{\max} \leq \rho_{\max}$

Theorem (Bogovskii 1979)

$D \subset \mathbb{R}^n$ ($n \geq 2$) bounded, $1 < p < \infty$. There exists $C = C(n, p, D) > 0$ such that : for each $g \in L^p_0(D)$, there exists $\phi : \Omega \to \mathbb{R}$ satisfying

$$\mathrm{div}\ \phi = g, \quad \phi|_{\partial D} = 0,$$

$$\|\nabla \phi\|_{L^p} \leq C\|g\|_{L^p}$$

Moreover, ϕ is compactly supported in D if g is compactly supported in D.

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Bogovskii localization: $\tilde{u} = u\chi_R + \phi_R$,

div $\phi = -u \cdot \nabla \chi_R$, supp $\phi \subset S_{R,2R}$,

$$\|\nabla \phi_R\|_{L^p} \le C(p) \|\text{div } \phi_R\|_{L^p} \le \frac{C(p)}{R} \|u\|_{L^p}$$

Theorem 2 (a): $\rho^+_{\max} \le \rho_{\max}$

- $\partial_t \tilde{u} - \Delta \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + \nabla \tilde{p} = \tilde{f}$ where

$$
\begin{aligned}
\tilde{f} &= f\chi - \partial_t \phi + p\nabla\chi + \nabla u \nabla \chi + u\Delta\chi + u\nabla u \chi(1-\chi) - u\nabla\phi\chi \\
&+ \phi\nabla u\chi + uu\chi\nabla\chi - \phi u \nabla\chi - \phi\nabla\phi - \Delta\phi
\end{aligned}
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\end{aligned}$$

- $\tilde{f} \to f$ in Y_q

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- $\partial_t \tilde{u} - \Delta \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + \nabla \tilde{p} = \tilde{f}$ where

$$\begin{aligned} \tilde{f} &= f\chi - \partial_t \phi + p\nabla\chi + \nabla u \nabla \chi + u\Delta\chi + u\nabla u \chi(1-\chi) - u\nabla\phi\chi \\ &+ \phi\nabla u \chi + uu\chi\nabla\chi - \phi u \nabla\chi - \phi\nabla\phi - \Delta\phi \end{aligned}$$

- $\tilde{f} \to f$ in Y_q
- $\tilde{u}$ blows up

$$\rho^+_{\max} \leq \|\tilde{f}\|_{Y_q} \to \rho_{\max} \quad \text{as} \quad R \to \infty$$

Thank You!