On smallness condition of initial data for Le Jan–Sznitman cascade of the Navier-Stokes equations

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October 14, 2019

NSE, mild solutions

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\text{(NSE)}: \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^d \times (0, \infty), \\ \text{div } u = 0 & \text{in} \quad \mathbb{R}^d \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^d. \end{cases}
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Integro-differential equation:

$$u(x,t) = e^{\Delta t} u_0 - \int_0^t e^{\Delta s} \mathbf{P} \text{div}[u(t-s) \otimes u(t-s)] ds.$$

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Mild solutions – obtained by Picard's iteration:

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NSE, mild solutions

- ✓ Global existence and uniqueness in $L_t^\infty L_x^2$ for $d = 2$: Leray (1933).
- ✓ Local existence and uniqueness in subcritical spaces: Leray ('34), Kato ('84),...
- ✓ Global existence in critical spaces for small initial data: Kato ('84), Koch-Tataru (2001),...
- ? Global existence for arbitrarily large initial data.

NSE, weak solutions

Weak formulation = diff. eq. in distribution sense + energy inequality.

- Energy solutions: Leray '34, Hopf '51

$$\int_{\mathbb{R}^d} \frac{|u(x,t)|^2}{2} dx + \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 dxds \leq \int_{\mathbb{R}^d} \frac{|u_0(x)|^2}{2} dx$$

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- Local energy solutions: Scheffer '77, CKN '82, L-R 2002,...

$$\int_0^\infty \int_{\mathbb{R}^d} |\nabla u|^2 \phi dxdt \leq \int_0^\infty \int_{\mathbb{R}^d} \left[\frac{|u|^2}{2} (\partial_t \phi + \Delta \phi) + \left(\frac{|u|^2}{2} + p \right) u \nabla \phi \right] dxdt$$

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✓ Global existence

? Uniqueness, smoothness

NSE, weak solutions

Partial regularity:

Let $u_0 \in L^2$. How big is the set of singular points $S \subset \mathbb{R}^d \times (0, \infty)$?

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Partial regularity:
Let $u_0 \in L^2$. How big is the set of singular points $S \subset \mathbb{R}^d \times (0, \infty)$?

$$H^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$$

- $d = 2$: $S = \emptyset$ (Leray '33).
- $d = 3$: $\mathcal{H}^1_{\mathrm{par}}(S) = 0$ (CKN '82).
- $d = 4$: $\mathcal{H}^2_{\mathrm{par}}(S) = 0$ (Dong-Gu 2014, Wang-Wu '14).
- $d = 5$ (stationary): $S = \emptyset$ (Struwe 1995).
- $d = 6$ (stationary): $\mathcal{H}^2(S) = 0$ (Dong-Strain 2012).

Fourier transformed Navier-Stokes (FNS)

$$\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u}_0(\xi) + c_0 \int_0^t e^{-|\xi|^2 s}|\xi| \int_{\mathbb{R}^d} \hat{u}(\eta, t-s) \odot_\xi \hat{u}(\xi-\eta, t-s) d\eta ds$$

where $a \odot_\xi b = -i(e_\xi \cdot b)(\pi_{\xi^\perp} a)$.

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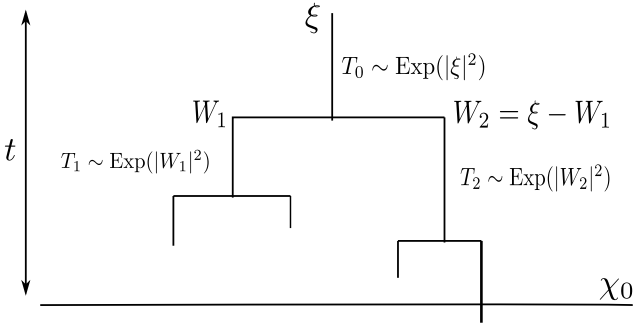
Normalization to (FNS): LJS '97, Bhattacharya et al (2003)

$$\begin{aligned}\chi(\xi,t) &= e^{-t|\xi|^2}\chi_0(\xi) \\ &+ \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^d} \chi(\eta, t-s) \odot_\xi \chi(\xi-\eta, t-s) H(\eta|\xi)\, d\eta\, ds\end{aligned}$$

where $\chi = c_0 \hat{u}/h$ and $H(\eta|\xi) = \frac{h(\eta)h(\xi-\eta)}{|\xi| h(\xi)}$.

h: *majorizing kernel*, i.e. $h * h = |\xi| h$.

Cascade structure of FNS



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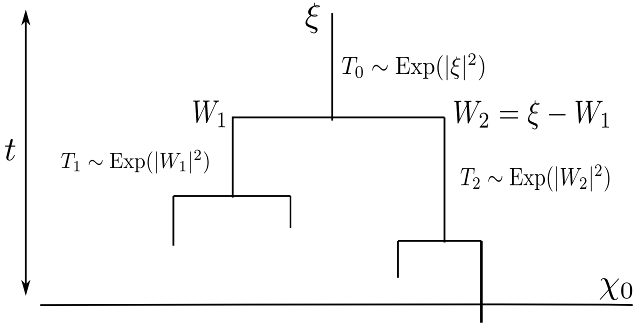
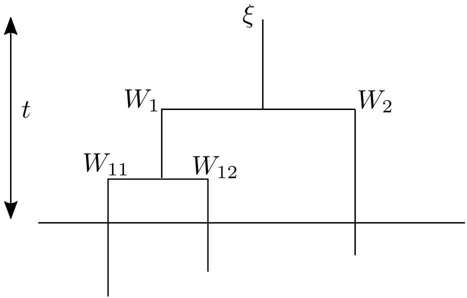


Define a stochastic multiplicative functional recursively as

$$\mathbf{X}_{\text{FNS}}(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T_0 > t, \\ \mathbf{X}^{(1)}_{\text{FNS}}(W_1, t - T_0) \odot_\xi \mathbf{X}^{(2)}_{\text{FNS}}(\xi - W_1, t - T_0) & \text{if} \quad T_0 \leq t. \end{cases}$$

Closed form of $\mathbf{X}_{\text{FNS}}$

Consider the following event:



On this event,

$$\mathbf{X}_{\mathsf{FNS}}(\xi, t) = (\chi_0(W_{11}) \odot_{W_1} \chi_0(W_{12})) \odot_{\xi} \chi_0(W_2).$$

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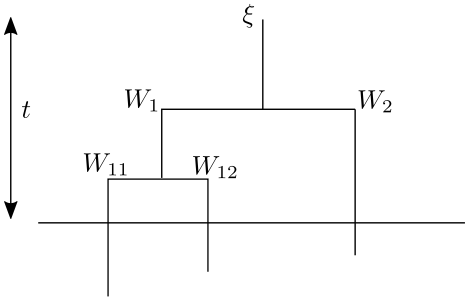

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Three ingredients: clocks, branching process, product.
Cascade structure = clocks + branching process.

FNS: mild solutions, cascade solutions

$$\begin{aligned}\chi(\xi, t) &= e^{-t|\xi|^2}\chi_0(\xi) \\ &+ \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^d} \chi(\eta, t-s) \odot_\xi \chi(\xi-\eta, t-s) H(\eta|\xi) d\eta ds\end{aligned}$$

- Mild solution:

$$\begin{aligned}\gamma_0 &\equiv 0 \\ \gamma_n &= e^{-t|\xi|^2}\chi_0 + \bar{B}(\gamma_{n-1}, \gamma_{n-1}) \\ \chi &= \lim \gamma_n\end{aligned}$$

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- Cascade solution ($\sim$ LJS 1997):

$$\chi(\xi,t)=\mathbb{E}_{\xi,t}\mathbf{X}_{\text{FNS}}$$

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- Cascade solution ($\sim$ LJS 1997):

$$\chi(\xi,t) = \mathbb{E}_{\xi,t}\mathbf{X}_{\mathrm{FNS}}$$

Two issues: (1) *stochastic explosion* and (2) *existence of expectation*.

Explosion

Branching process may never stop, potentially making $\mathbf{X}_{FNS}$ not well-defined.

- Property of cascade structure, not of product.
- Depending only on the majorizing kernel h and the clocks.

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- 3D self-similar cascade $h_{\mathrm{dilog}}(\xi) = C|\xi|^{-2}$: stochastic explosion a.s. (Dascaliuc, Pham, Thomann, Waymire 2019)
- 3D Bessel cascade $h_{\mathrm{b}}(\xi) = C|\xi|^{-1}e^{-|\xi|}$: no-explosion a.s. (Orum, Pham 2019)

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We bypass the explosion problem by defining instead

$$\chi(\xi, t) = \mathbb{E}_{\xi,t}[\mathbf{X}_{\mathrm{FNS}} \mathbb{1}_{S>t}],$$

where S is the shortest path.

Existence of expectation

It may happen that $\mathbb{E}_{\xi,t}[|\mathbf{X}_{\mathrm{FNS}}|\mathbb{1}_{S>t}] = \infty$.

$$\mathbf{X}_{\mathrm{FNS}}(\xi, t)\mathbb{1}_{S>t} = \bigodot_{s\in\mathcal{V}_0(\xi,t)} \chi_0(W_s) \quad (\textit{finite product})$$

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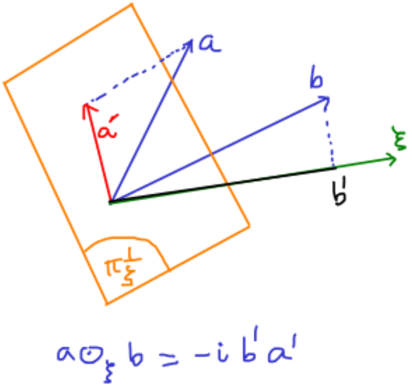
This issue depends on both cascade structure and the product.



Existence of expectation

LJS '97, Bhattacharya et al 2003: $|\chi_0| \leq 1$ leads to

1. Global existence
2. Uniqueness in the class $\{\chi : \ |\chi| \leq 1 \ a.e. \ (\xi, t)\}$
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Question: can smallness of χ_0 in a global sense guarantee existence of expectation?

$$\|u_0\|_{\dot{H}^{d/2-1}} = C_d \left\{ \int_{\mathbb{R}^d} |\xi|^{d-2} h^2(\xi) |\chi_0(\xi)|^2 d\xi \right\}^{1/2}.$$

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An iteration method was used by LJS (1997) to show uniqueness; by Bhattacharya et al (2003) to show cascade-mild agreement; by Dascaliuc et al (2018) to show nonuniqueness for α-Riccati equation.

Iteration process

Chain from initial condition to solution – Introduce a *ground state*.

$$\mathbf{X}_{\mathrm{FNS},0}(\xi, t) \equiv 0,$$
$$\mathbf{X}_{\mathrm{FNS},n}(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T_0 > t, \\ \mathbf{X}^{(1)}_{\mathrm{FNS},n-1}(W_1, \ldots) \odot_\xi \mathbf{X}^{(2)}_{\mathrm{FNS},n-1}(\xi - W_1, \ldots) & \text{if} \quad T_0 \leq t. \end{cases}$$

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Ignore the product:

$$\mathbf{X}_0(\xi, t) \equiv 0,$$
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Domination principle: $|\mathbf{X}_{\mathrm{FNS},n}| \leq \mathbf{X}_n$.

Majorizing NSE equation

$\mathbf{X}_n$ corresponds to the following scalar equation:

$$(\text{mNSE}): \quad \begin{cases} \partial_t u - \Delta u = \sqrt{-\Delta}(u^2) & \text{in} \quad \mathbb{R}^d \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^d. \end{cases}$$

called *majorizing NSE*.

It is called "cheap NSE" by Montgomery-Smith (2001).

Iteration process

Note that $\mathbf{X}_{\text{FNS},n}(\xi, t) \to \mathbf{X}_{\text{FNS}}(\xi, t)\mathbb{1}_{S>t}$ a.s.
Put $\phi_n(\xi, t) = \mathbb{E}_{\xi,t}\mathbf{X}_n$. By Fatou's lemma and domination principle,

$$\begin{aligned}\phi(\xi, t) := \mathbb{E}_{\xi,t}[|\mathbf{X}_{\text{FNS}}|\mathbb{1}_{S>t}] &\leq \liminf \ \mathbb{E}_{\xi,t}|\mathbf{X}_{\text{FNS},n}| \\ &\leq \liminf \ \mathbb{E}_{\xi,t}\mathbf{X}_n \\ &= \liminf \phi_n(\xi, t).\end{aligned}$$

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Admissible functional

A map $N_T : \mathcal{M}_T \to [0, \infty]$ is said to be an *admissible functional* if it has the following properties:

1. If $N_T[f] < \infty$ then $|f(\xi, t)| < \infty$ for a.e. $(\xi, t) \in \mathbb{R}^d \times (0, T)$.
2. If $f, f_n \in \mathcal{M}_T$ and $f \leq \liminf f_n$ a.e. then $N_T[f] \leq \liminf N_T[f_n]$.

$\mathcal{M}_T$: space of all Borel measurable functions from $\mathbb{R}^d \times (0, T)$ to $[0, \infty]$.

Admissible functionals

Example of admissible functionals:

$$N_T[f] = \|f\rho\|_{L_t^r L_\xi^q} = \left\| \|f(\cdot,t)\rho(\cdot,t)\|_{L_\xi^q(\mathbb{R}^d)} \right\|_{L_t^r(0,T)}$$

where $0 < r, q \leq \infty$ and $\rho : \mathbb{R}^d \times (0, T) \to [0, \infty]$ is a measurable function which vanishes only on a set of measure zero.

Key estimates

Recall:

$$\phi_n(\xi, t) = \mathbb{E}_{\xi,t}\mathbf{X}_n,$$
$$\phi(\xi, t) = \mathbb{E}_{\xi,t}[|\mathbf{X}_{\mathrm{FNS}}|\mathbb{1}_{S>t}].$$

Key estimates

Recall:

$$\begin{aligned} \phi_n(\xi, t) &= \mathbb{E}_{\xi,t}\mathbf{X}_n, \\ \phi(\xi, t) &= \mathbb{E}_{\xi,t}[|\mathbf{X}_{\mathrm{FNS}}|\mathbb{1}_{S>t}]. \end{aligned}$$

If $N_T[\phi_n] \leq M < \infty$ for all n then

By (2), $N_T[\phi] \leq \liminf N_T[\phi_n] \leq M$.

By (1), $\phi(\xi, t) < \infty$ a.e. $(\xi, t) \in \mathbb{R}^d \times (0, T)$.

What can we choose for N_T ?

$\phi_n(\xi, t)$

$$
\begin{aligned}
&= \mathbb{E}_{\xi,t}[\mathbf{X}_n \mathbb{1}_{T_0>t}] + \mathbb{E}_{\xi,t}[\mathbf{X}_n \mathbb{1}_{T_0\le t}] \\
&= e^{-t|\xi|^2}|\chi_0| \\
&+ \int_0^t |\xi|^2 e^{-s|\xi|^2} \int_{\mathbb{R}^d} \phi_{n-1}(\eta, t-s)\phi_{n-1}(\xi-\eta, t-s)H(\eta|\xi)d\eta ds.
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Therefore,

$$\phi_n = F_1[|\chi_0|] + F_2[\phi_{n-1}, \phi_{n-1}].$$

This is a *Picard iteration*.

Kato's settings

Problem:

What can we choose for E and $\mathcal{E}_T$ such that if $|\chi_0|$ is sufficiently small in E then ϕ_n is bounded in $\mathcal{E}_T$?

Kato's settings

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> *What can we choose for E and $\mathcal{E}_T$ such that if $|\chi_0|$ is sufficiently small in E then ϕ_n is bounded in $\mathcal{E}_T$?*

We call $(E, \mathcal{E}_T)$ a *Kato's setting* if

- F_1 is bounded linear from E to $\mathcal{E}_T$,
- F_2 is bounded bilinear from $\mathcal{E}_T \times \mathcal{E}_T$ to $\mathcal{E}_T$.

Lemarie-Rieusset calls E an *adapted value space*, $\mathcal{E}_T$ an *admissible path space*.

$$\|\phi_n\|_{\mathcal{E}_T} \leq \kappa \| |\chi_0| \|_E + \gamma \|\phi_{n-1}\|^2_{\mathcal{E}_T}.$$

Smallness of χ_0 in integral sense

Theorem (P. - Thomann 2019)

Let $(E, \mathcal{E}_T)$ be a Kato's setting such that $\| \cdot \|_{\mathcal{E}_T}$ is an admissible functional. If $|\chi_0|$ is sufficiently small in E then $\phi(\xi, t) = \mathbb{E}_{\xi,t}[|\mathbf{X}_{\mathrm{FNS}}| \mathbb{1}_{S>t}]$ is finite for a.e. $(\xi, t) \in \mathbb{R}^d \times (0, T)$.

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Choices of E include

1. From smallness of u_0 in $\dot{H}^{d/2-1}$:

$$\|\chi_0\|_E = \left\{ \int_{\mathbb{R}^d} |\xi|^{d-2} h^2(\xi) |\chi_0(\xi)|^2 d\xi \right\}^{1/2}.$$

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2. From smallness of u_0 in Lin-Lei's space (2011):

$$\|\chi_0\|_E = \int_{\mathbb{R}^d} |\xi|^{-1} h(\xi) |\chi_0(\xi)| d\xi.$$

Thank You!