Stability Analysis of Differential Equations with Stochastic Boundary Conditions

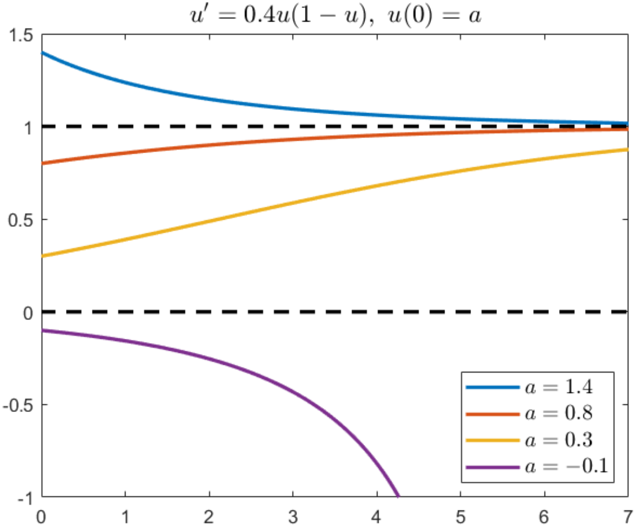
Tuan Pham

Brigham Young University-Hawaii

UH-Manoa Math Colloquium

November 20, 2025

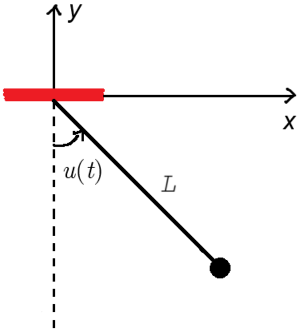
The logistic equation $u' = ru(1 - u/M)$



Equilibria: $u \equiv 1$ (asymptotically stable), $u \equiv 0$ (unstable)

The pendulum motion

$$u'' + cu' + \frac{g}{L}\sin u = 0,\ \ u(0) = a,\ \ u'(0) = b$$



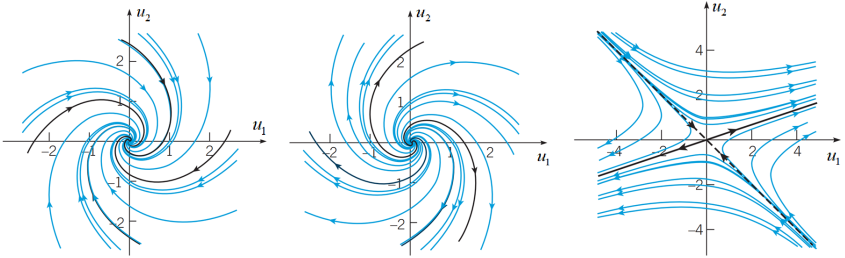
Equilibria: $u \equiv 0$ (asymptotically stable), $u \equiv \pi$ (unstable)

Simulation with different values of c

$$u' = Au, \quad u = [u_1 \ u_2 \ \ldots \ u_n]^T$$

- The equilibrium solution $u = 0$ is **asymptotically stable** if and only if $\mathfrak{Re}(\lambda) < 0$ for all complex eigenvalues λ of A.
- Associated eigenvalue problem: $\lambda u = Au$.

$$\text{asymptotically stable} \iff \text{exponentially stable (decay rate} = -\mathfrak{Re}(\lambda))$$



The heat equation - Linear stability analysis

$$u_t = \nu u_{xx}, \;\; u(0,t) = u(1,t) = 0$$

Steady state: $u_*'' = 0$, $u_*(0) = u_*(1) = 0$, which gives $u_* \equiv 0$.

Linear stability analysis: the associated eigenvalue problem

$$\lambda u = \nu u'', \;\; u(0) = u(1) = 0.$$

A nontrivial solution exists if and only if

$$\lambda = \lambda_n = -\nu n^2 \pi^2 \quad \text{for } n \in \mathbb{N}.$$

All eigenvalues are negative, so the steady-state solution is always *exponentially stable* (for any $\nu > 0$).

$$u_t = \nu u_{xx}, \;\; u(0,t) = u(1,t) = 0$$

Nonlinear stability analysis:

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2(x,t)dx = -\nu \int_0^1 u_x^2(x,t)dx \le -\lambda \int_0^1 u^2(x,t)dx$$

where

$$\lambda = \inf_{u \in H_0^1(0,1)} \frac{\nu \int_0^1 (u')^2 dx}{\int_0^1 u^2 dx} \ge 0.$$

The energy $E(t) = \int_0^1 u^2(x,t)dx$ satisfies $E(t) \le E(0)e^{-2\lambda t}$.

Steady-state solution $u_* \equiv 0$ is L^2-*exponentially stable* with decay rate λ.

The heat equation - Nonlinear stability analysis

$$\lambda = \inf_{u \in H_0^1(0,1)} \frac{\nu \int_0^1 (u')^2 dx}{\int_0^1 u^2 dx}. \quad \text{How to check if } \lambda > 0?$$

Rewrite: $\lambda = \inf_{\substack{u \in H_0^1(0,1) \\ \int_0^1 u^2 dx = 1}} I[u]$, where $I[u] = \nu \int_0^1 (u')^2 dx$.

If a minimizer v exists, we use *Lagrange multiplier*:

$$v = \operatorname*{argmin}_{u \in H_0^1(0,1)} J[u], \quad \text{where } J[u] = I[u] - \lambda \int_0^1 u^2 dx$$

v can be found using *Calculus of Variations*: for all $\phi \in H_0^1(0,1)$,

$$0 = \lim_{\epsilon \to 0} \frac{J[v + \epsilon\phi] - J[v]}{\epsilon} = -2 \int_0^1 (\nu v'' + \lambda v)\phi dx.$$

The heat equation - Nonlinear stability analysis

One obtains the *Euler-Lagrange equation* of the variational problem:

$$\nu v'' + \lambda v = 0, \quad v(0) = v(1) = 0.$$

A nontrivial solution exists if and only if $\lambda \in \{\nu n^2 \pi^2,\ n \in \mathbb{N}\}$.
The exponential decay rate is $\lambda = \nu \pi^2 > 0$.

Allen-Cahn equation - homogeneous boundary conditions

$$u_t = \nu u_{xx} + u - u^3,\ u(0,t) = u(1,t) = 0.$$

There are *infinitely many* steady-state solutions $u_* = u_*(x)$:

$$0 = \nu u_*'' + u_* - u_*^3,\ u_*(0) = u_*(1) = 0.$$

Linear stability analysis for $\mathbf{u_* \equiv 0}$: for small u,

$$u_t \approx \nu u_{xx} + u,\ u(0) = u(1) = 0.$$

Associated eigenvalue problem: $\lambda u = \nu u'' + u$.

• $\nu > \pi^{-2}$, all eigenvalues are negative. One has asymptotic stability for small perturbations.

• $\nu < \pi^{-2}$, there are positive and negative eigenvalues. 0 is a saddle point.

Simulation with $\nu = 0.01$ and different initial data

Burgers equation - Dirichlet boundary conditions

$$u_t = \nu u_{xx} - u u_x,\ \ u(0,t) = a,\ \ u(1,t) = b$$

Steady-state solution $u_* = u_*(x)$ is unique and can be computed explicitly:

$$0 = \nu u_*'' - u_* u_*',\ \ u_*(0) = a,\ \ u_*(1) = b.$$

Simulation with $b = 0$ and different values of a

The difference $v = u - u_*$ satisfies $v_t - \nu v_{xx} + v\, v_x + (u_* v)_x = 0$.
Linear stability analysis:

$$(-\lambda) w = -\nu w'' + (u_* w)',\quad w(0) = w(1) = 0.$$

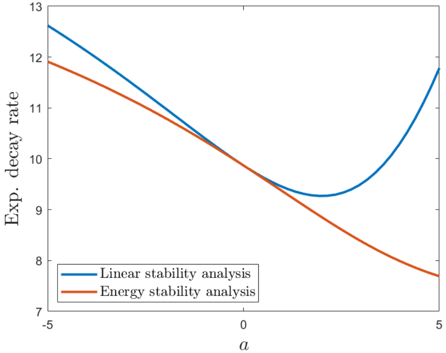
Nonlinear stability analysis (energy-based):

$$\lambda w = -\nu w'' + \frac{1}{2} u_*' w,\quad w(0) = w(1) = 0.$$

Burgers equation - Dirichlet boundary conditions

Comparison of exponential decay rates in linear and energy stability analysis.

$$u_t = u_{xx} - uu_x,\ \ u(0,t) = a,\ \ u(1,t) = 0$$



Nonlinear stability analysis based on *Cole-Hopf transformation*: steady-state solution is exponentially stable for Dirichlet boundary condition $u(0,t) = a$, $u(1,t) = b$ (Kourbatov, 1992).

Burgers equation - Time-dependent boundary conditions

$$u_t = \nu u_{xx} - uu_x, \;\; u(0,t) = a(t), \;\; u(1,t) = b(t)$$

Definition (stability of time-dependent solutions)

Time-dependent solution $u_*(x,t)$ is L^2-*asymptotically stable* if for any solutions $u(x,t)$, $u(\cdot,0) - u_*(\cdot,0) \in L^2$ implies $\|u(\cdot,t) - u_*(\cdot,t)\|_{L^2} \to 0$ as $t \to \infty$.

The difference $v = u - u_*$ satisfies $v_t - \nu v_{xx} + v\, v_x + (u_* v)_x = 0$.

Energy stability analysis:

$$E'(t) \leq -2\lambda(t)E(t), \quad \text{where } E(t) = \int_0^1 v^2(x,t)dx$$

$$\lambda(t) = \inf_{\substack{w \in H_0^1(0,1) \\ \int_0^1 w^2 dx = 1}} \int_0^1 \left(\nu (w')^2 + \frac{u_*}{2} w^2\right) dx.$$

Burgers equation - Time-dependent boundary conditions

$$u_t = \nu u_{xx} - uu_x, \;\; u(0,t) = a(t), \;\; u(1,t) = b(t)$$

Energy $E(t) = \int_0^1 v^2(x,t)dx$, where $v = u - u_*$, satisfies

$$E(t) \leq E(0) \exp\left(-2\int_0^t \lambda(s)ds\right)$$

Theorem (L^2-asymptotic stability criterion)

$$\bar{\lambda} \stackrel{\text{def}}{=\!=} \liminf_{t\to\infty} \frac{1}{t}\int_0^t \lambda(s)ds.$$

If $\bar{\lambda} > 0$ then the solution $u_*(x,t)$ is L^2-asymptotically stable.

Proof. For sufficiently large t,

$$E(t) \leq E(0) \exp\left(-2t\frac{1}{t}\int_0^t \lambda(s)ds\right) \leq E(0)\exp(-2t(\bar{\lambda} - \epsilon)).$$

Burgers equation - Time-dependent boundary conditions

$$u_t = \nu u_{xx} - uu_x,\ \ u(0,t) = a(t),\ \ u(1,t) = b(t)$$

How to compute $\bar{\lambda}$ for a given value of $\nu > 0$?

- Find one solution $u_*(x,t)$ by Cole-Hopf transformation $u_* = -2\nu\frac{\phi_x}{\phi}$ where ϕ satisfies $\phi_t - \nu\phi_{xx} = 0$ with Robin boundary conditions.
- For each t in a large time-interval $[0, T]$, find the smallest eigenvalue $\lambda(t)$ from Euler-Lagrange equation:

$$\lambda(t)w = -\nu w'' + \frac{1}{2}u_*(\cdot, t)w, \quad w(0) = w(1) = 0.$$

  This is a *Sturm-Liouville problem*.
- Take the average in time $\bar{\lambda} \approx \frac{1}{T}\int_0^T \lambda(t)dt$.

Observations: $\bar{\lambda}$ increases as ν increases. One can use the Bisection method to search for the *critical value* of ν where $\bar{\lambda} = 0$.

Burgers equation - Stochastic boundary conditions

$$u_t = \nu u_{xx} - u u_x, \;\; u(0,t) = a(t), \;\; u(1,t) = b(t)$$

Here, $\{a(t)\}_{t\geq 0}$ and $\{b(t)\}_{t\geq 0}$ are stochastic processes.

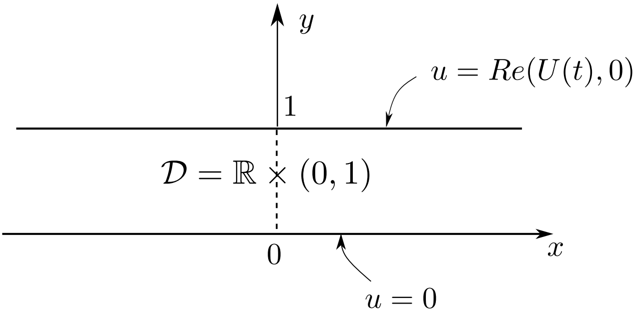
$$\begin{aligned} \lambda(t) w &= -\nu w'' + \frac{1}{2} u_*(\cdot, t) w, \quad w(0) = w(1) = 0, \\ \bar{\lambda} &= \liminf_{t\to\infty} \frac{1}{t} \int_0^t \lambda(s) ds. \end{aligned}$$

The computational cost is much cheaper if $\{a(t)\}$ and $\{b(t)\}$ each

- asymptotically stable in distribution: converges to an distribution π,
- mean-ergodic: (time average of $f(a(t))$ = state average of $f(a(t))$ in distribution π).

$$\begin{aligned} \lambda^S w &= -\nu w'' + \frac{1}{2} u_*^S w, \quad w(0) = w(1) = 0, \\ \bar{\lambda} &= \mathbb{E}[\lambda^S]. \end{aligned}$$

Plane Couette flow



$$\begin{cases} u_t - \Delta u + (u \cdot \nabla) u + \nabla p = 0, \ \operatorname{div} u = 0, \\ \qquad u(x, 1, t) = Re\,(U(t), 0), \\ \qquad u(x, 0, t) = 0\,, \end{cases}$$

$\{U(t)\}_{t \geq 0}$ is a *asymptotically stable* and *mean-ergodic* process with $\mathbb{E}[U(t)] = 1$ for all t.

Plane Couette flow - Literature

Some literature for deterministic case $U(t) \equiv 1, \;\; u_*(x, y, t) = Re\,(y, 0)$.

- *At given parameter values, will the flow return to its laminar state u_* no matter how it is perturbed?*
- Laminar state is stable for small perturbations for any $Re > 0$ (Romanov 1973).
- $Re_c \approx 177.2$ (Orr 1907).
- Using non-quadratic Lyapunov functional: $Re_c \approx 252.4$ (Fuentes, Goluskin, Chernyshenko '22).

Motivations:

- In practice, the velocity of the moving plane is stochastic. Does stochasticity improve or worsen the stability threshold?
- Most literature deals with periodic domains (in horizontal direction), which makes the problem mathematically simpler. Infinite domain is a more natural physical setting.

Plane Couette flow

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla) u + \nabla p = 0, \ \operatorname{div} u = 0, \\ u(x,1,t) = Re\,(U(t), 0), \quad u(x,0,t) = 0. \end{cases}$$

One solution is the *unsteady laminar flow* $u_*(x,t) = Re(\chi(y,t), 0)$ where

$$\chi_t - \chi_{yy} = 0, \quad \chi(1,t) = U(t), \quad \chi(0,t) = 0, \quad \chi(y,0) = yU(0).$$

Theorem (Foldes, Pham, Whitehead '25)

Suppose $\{U(t)\}_{t \geq 0}$ is an Ornstein-Uhlenbeck process, i.e. $dU = \alpha(1-U)dt + \sigma dW_t$ for $\alpha, \sigma > 0$. There exists a unique critical Reynolds number $Re_c = Re_c(\alpha, \sigma)$ in the energy stability analysis such that u_* is L^2-asymptotically stable whenever $Re < Re_c$. Also,

- Re_c is decreasing in σ.
- $\lim_{\sigma \to 0^+} Re_c \approx 177.2$ and $Re_c \sim \sigma^{-1}$ as $\sigma \to \infty$.

Plane Couette flow - Energy stability analysis

$$\mathcal{H} = \{f \in H_0^1(\mathcal{D}, \mathbb{R}) : \ \operatorname{div} f = 0\}.$$

Let $v = u - u_*$. Then

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{D}} |v(\cdot,t)|^2 dxdy \leq -\lambda(t)\int_{D} |v(\cdot,t)|^2 dxdy$$

where

$$\lambda(t) = \inf_{\substack{w\in\mathcal{H} \\ \|w\|_{L^2(\mathcal{D})}=1}} \int_{\mathcal{D}} (|\nabla w|^2 + Re\,\chi_y w_1 w_2) dxdy$$

Stability criterion

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \lambda(s)ds = \mathbb{E}[\lambda^S] > 0 \implies L^2\text{-asymptotic stability}$$

Plane Couette flow - Adjust minimization problem

Minimum value is, unfortunately, **not attainable** due to lack of compactness in infinite domain. Solution to Euler-Lagrange equation is not a minimizer.

Let ϕ be the *stream function*, i.e. $\phi_x = -w_2$, $\phi_y = w_1$, and $\psi = \psi(\xi, y)$ be the Fourier transform of $\phi(x, y)$ in the x-direction.

$$\lambda(t) = \inf_{\substack{\psi \in \hat{H}_0^2(D) \\ \|\nabla\psi\|_{L^2} = 1}} I(\psi, Re\chi_y) \geq \lambda_*(t) = \inf_{\xi \geq 0} \left(\inf_{\substack{f \in \hat{H}_0^2(\xi) \\ J_\xi(f) = 1}} I_\xi(f, Re\chi_y) \right)$$

where

$$\begin{aligned} I_\xi(f, g) &= \int_0^1 \left(\xi^4 |f|^2 + 2\xi^2 |f'|^2 + |f''|^2 - g\xi \mathfrak{Im}(f'\bar{f}) \right) dy, \\ J_\xi(f) &= \int_0^1 (\xi^2 |f|^2 + |f'|^2) dy. \end{aligned}$$

Plane Couette flow - Adjust minimization problem

Stability criterion (weakened)

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \lambda_*(s)ds = \mathbb{E}[\lambda_*^S] > 0 \implies L^2\text{-asymptotic stability}$$

Minimizer to each minimization problem exists. Euler-Lagrange equation:

$$f'''' - (2\xi^2 - \lambda_*^S)f'' + (\xi^4 - \lambda_*^S\xi^2)f + \frac{i}{2}Re\xi(2\chi_y^S f' + \chi_{yy}^S f) = 0.$$

Boundary conditions: $f(0) = f(1) = f'(0) = f'(1) = 0.$

Plane Couette flow - Algorithm

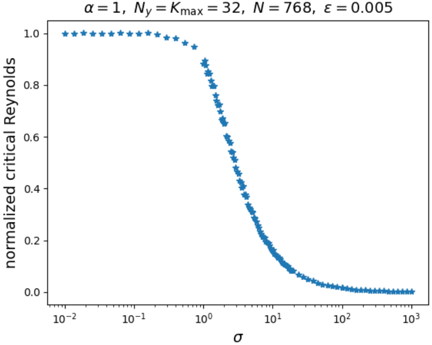
$$f'''' - (2\xi^2 - \lambda_*^S)f'' + (\xi^4 - \lambda_*^S\xi^2)f + \frac{i}{2}Re\xi(2\chi_y^S f' + \chi_{yy}^S f) = 0.$$

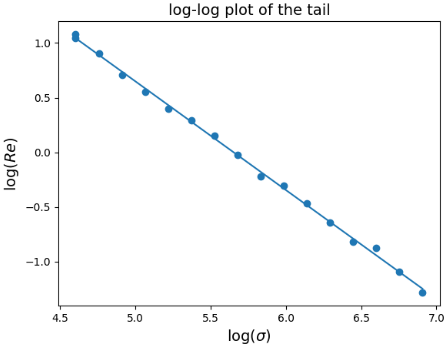
Boundary conditions: $f(0) = f(1) = f'(0) = f'(1) = 0.$

Algorithm to find *critical Reynolds number*, i.e. one that gives $\mathbb{E}[\lambda_*^S] = 0$:

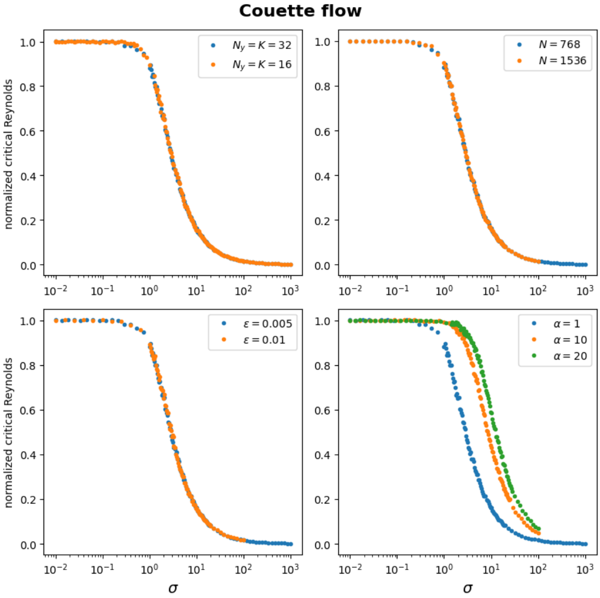
- For each ξ in an interval $[a, b]$, sample χ^S from the stationary distribution.
- Compute the smallest eigenvalue $\lambda(\xi)$ of the ODE.
- Minimize $\lambda(\xi)$ over $\xi \in [a, b]$. Call it λ_*.
- Compute the average of λ_* over all samples. This approximates $\mathbb{E}[\lambda_*]$.
- Use Bisection method to adjust Re so that $\mathbb{E}[\lambda_*] \approx 0$.

Numerical results

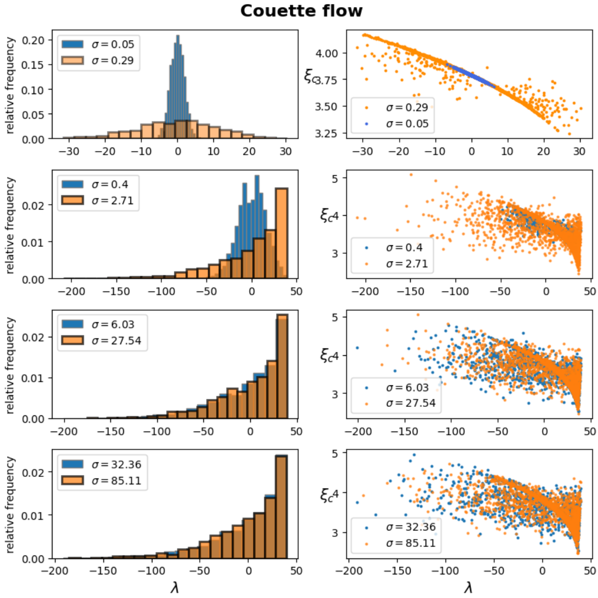




Numerical results



Numerical results



Thank You!