Complex Burgers singularities

Tuan N. Pham
Faculty of Math and Computing, Brigham Young University-Hawaii*

November 2, 2023

The purpose of this note is to give a correction to Equation (4.5) in the paper "Zeros of complex caloric functions and singularities of complex viscous Burgers equation" by Poláčik and Šverák (J Reine Angew Math 616:205–217, 2008).

Proposition 1. *Let $v = v(x,t)$ be a complex-valued smooth function on a neighborhood $\Omega \subset \mathbb{R}^2$ of $(0,0)$ such that $v_t - v_{xx} = 0$ and that $(0,0)$ is the only zero of v in Ω. Suppose that $a = v_x(0,0)$ and $b = v_t(0,0)$ are linearly independent over $\mathbb{R}$. Then the function $u = -2\frac{v_x}{v}$ satisfies*

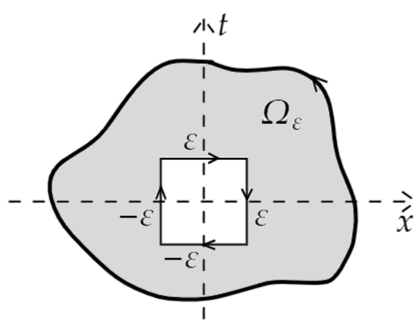
$$u_t + uu_x - u_{xx} = 4\pi i \sigma \delta - \frac{4}{c}\mathrm{Log}\left(\frac{1-c}{1+c}\right)\delta_x$$

where σ is the sign of the imaginary part of $c = b/a$, and δ denotes the Dirac distribution at the point $(0,0)$.

Proof of Proposition 1. Let ϕ be a smooth function compactly supported on Ω and let

$$\begin{aligned} J &= \text{p.v.} \iint\limits_{\Omega} (-u\phi_t - u^2\phi_x/2 - u\phi_{xx})dA \\ &= \lim_{\varepsilon\to 0} \iint\limits_{\Omega_\varepsilon} (-u\phi_t - u^2\phi_x/2 - u\phi_{xx})dA \end{aligned}$$

where $\Omega_\varepsilon = \Omega \backslash [-\varepsilon, \varepsilon]^2$. By Divergence Theorem,



$$\begin{aligned} J_\varepsilon &:= \iint\limits_{\Omega_\varepsilon} (-u\phi_t - u^2\phi_x/2 - u\phi_{xx})dA \\ &= \iint\limits_{\Omega_\varepsilon} (u_t + uu_x - u_{xx})\phi\, dA - \int_{\partial\Omega_\varepsilon} \left\{ u\phi\, n_2 + \left(\frac{u^2}{2}\phi + u\phi_x - u_x\phi\right) n_1 \right\} ds. \end{aligned} \tag{1}$$

*Email address: tpham@byuh.edu

where $n = (n_1, n_2)$ is the normal vector on $\partial\Omega_\varepsilon$ pointing outward. Note that double integral on RHS(1) vanishes because u satisfies the Burgers equation pointwise in $\Omega\backslash\{(0,0)\}$. We have

$$\frac{1}{2}u^2\phi + u\phi_x - u_x\phi \quad = \quad 2\frac{v_x^2}{v^2}\phi - 2\frac{v_x}{v}\phi_x + 2\frac{v_{xx}v - v_x^2}{v^2}\phi = -2\frac{v_x\phi_x}{v} + 2\frac{v_t\phi}{v}.$$

One can rewrite (1) as

$$J_\varepsilon = \underbrace{\int_{\partial\Omega_\varepsilon}\left(2\frac{v_x\phi}{v}n_2 - 2\frac{v_t\phi}{v}n_1\right)ds}_{I_1} + \underbrace{\int_{\partial\Omega_\varepsilon} 2\frac{v_x\phi_x}{v}n_1 ds}_{I_2}.$$

Write $\partial\Omega_\varepsilon = C_\varepsilon^- \cup C_\varepsilon^+$, where C_ε^- is the inner boundary (clockwise oriented square), and C_ε^+ is the outer boundary. Note that the integrands of I_1 and I_2 vanish on C_ε^+. One can rewrite I_1 as

$$I_1 = \int_{C_\varepsilon^-}\frac{-2\phi}{v}(v_x, v_t)\cdot(-n_2, n_1)ds = \int_{C_\varepsilon^-}\frac{-2\phi}{v}(v_x, v_t)\cdot(dx, dt) = \int_{\Gamma_\varepsilon}\frac{-2\phi}{v}dv$$

where $\Gamma_\varepsilon = v(C_\varepsilon^-)$. Viewing $v(x,t) = \xi + i\eta \equiv (\xi, \eta)$ as a transformation on the plane, we can compute its Jacobian as follows:

$$\frac{\partial(\xi,\eta)}{\partial(x,t)} = \begin{vmatrix}\xi_x & \xi_t \\ \eta_x & \eta_t\end{vmatrix} = \xi_x\eta_t - \xi_t\eta_x = \mathrm{Im}(\bar{v}_x v_t).$$

At $(x,t) = (0,0)$, the sign of the Jacobian is σ, which is not zero. By the Inverse Function Theorem, there exists a neighborhood of $(0,0)$ on which v is a one-to-one mapping and on which the sign of $\mathrm{Im}(\bar{v}_x v_t)$ does not change and is equal to σ. Thus, for sufficiently small $\varepsilon > 0$, Γ_ε is a simple curve with positive orientation if $\sigma = -1$, and negative orientation if $\sigma = 1$. Let $\psi(z) = -2\phi(x,t)$ where $z = \xi + i\eta = v(x,t)$. By Cauchy-Pompeiu integral formula,

$$-\sigma I_1 = -\sigma\int\limits_{\Gamma_\varepsilon}\frac{\psi}{z}dz = 2\pi i\psi(0) + 2i\iint\limits_{D_\varepsilon}\frac{\psi_{\bar{z}}}{z}dA \tag{2}$$

Here D_ε is the region enclosed by Γ_ε, and $\psi_{\bar{z}} = \frac{1}{2}(\psi_\xi + i\psi_\eta)$, which is a continuous function in (ξ,η). Since $\frac{\psi_{\bar{z}}}{z}$ is Lebesgue integrable in a neighborhood of $(0,0)$, the double integral in (2) tends to 0 as $\varepsilon \to 0$. Therefore,

$$\lim_{\varepsilon\to 0} I_1 = -2\pi i\sigma\psi(0) = 4\pi i\sigma\phi(0,0).$$

To evaluate I_2, denote $f(x,t) = -2v_x\phi_x$. Then

$$I_2 = \int_{-\varepsilon}^{\varepsilon}\left(\frac{f(\varepsilon,s)}{v(\varepsilon,s)} - \frac{f(-\varepsilon,s)}{v(-\varepsilon,s)}\right)ds$$

One can rewrite I_2 as

$$I_2 = \underbrace{\int_{-\varepsilon}^{\varepsilon} f(\varepsilon,s)\left(\frac{1}{v(\varepsilon,s)} - \frac{1}{v(-\varepsilon,s)}\right)ds}_{\{1\}} + \underbrace{\int_{-\varepsilon}^{\varepsilon}\frac{1}{v(-\varepsilon,s)}\left(f(\varepsilon,s) - f(-\varepsilon,s)\right)ds}_{\{2\}}.$$

Note that $v(\varepsilon, s) = a\varepsilon + bs + O(\varepsilon^2)$ for all $s \in [-\varepsilon, \varepsilon]$. Therefore,

$$\begin{aligned}
\{1\} &= \int_{-\varepsilon}^{\varepsilon} f(\varepsilon, s)\left(\frac{1}{a\varepsilon + bs + O(\varepsilon^2)} - \frac{1}{-a\varepsilon + bs + O(\varepsilon^2)}\right) ds \\
&= \int_{-\varepsilon}^{\varepsilon} f(\varepsilon, s)\frac{-2a\varepsilon + O(\varepsilon^2)}{-a^2\varepsilon^2 + b^2 s^2 + O(\varepsilon^3)} ds \\
&\overset{s=\varepsilon\tau}{=\!=} \int_{-1}^{1} f(\varepsilon, \varepsilon\tau)\frac{-2a + O(\varepsilon)}{-a^2 + b^2\tau^2 + O(\varepsilon)} d\tau.
\end{aligned}$$

By Lebesgue's Dominated Convergence Theorem,

$$\lim_{\varepsilon\to 0}\{1\} = \int_{-1}^{1} f(0,0)\frac{-2a}{-a^2 + b^2\tau^2} d\tau \overset{z=b\tau/a}{=\!=} -\frac{f(0,0)}{b}\int_C \frac{2}{z^2-1} dz$$

where C is the line segment from $-b/a$ to b/a on the complex plane. Note that C lies inside the region $\mathbb{C}\backslash\{z \in \mathbb{R} : |z| \geq 1\}$. In this region, the function $\frac{2}{z^2-1}$ has an antiderivative $\mathrm{Log}\left(\frac{1-z}{1+z}\right)$, where Log denotes the principal branch of the complex logarithm. Hence,

$$\lim_{\varepsilon\to 0}\{1\} = -\frac{f(0,0)}{b}\,\mathrm{Log}\left(\frac{1-z}{1+z}\right)\bigg|_{-b/a}^{b/a} = \phi_x(0,0)\frac{4}{c}\,\mathrm{Log}\left(\frac{1-c}{1+c}\right)$$

where $c = b/a$. To estimate $\{2\}$, we use the change of variables $s = \varepsilon\tau$ and rewrite $\{2\}$ as

$$\begin{aligned}
\{2\} &= \int_{-1}^{1} \frac{1}{v(-\varepsilon, \varepsilon\tau)}\left(f(\varepsilon,\varepsilon\tau) - f(-\varepsilon,\varepsilon\tau)\right)\varepsilon d\tau \\
&= \int_{-1}^{1} \frac{1}{-a\varepsilon + b\varepsilon\tau + O(\varepsilon^2)}\left(f(\varepsilon,\varepsilon\tau) - f(-\varepsilon,\varepsilon\tau)\right)\varepsilon d\tau \\
&= \int_{-1}^{1} \frac{1}{-a + b\tau + O(\varepsilon)} O(\varepsilon) d\tau.
\end{aligned}$$

By Lebesgue's Dominated Convergence Theorem, $\lim\limits_{\varepsilon\to 0}\{2\} = 0$. Thus,

$$\lim_{\varepsilon\to 0} I_2 = \lim_{\varepsilon\to 0}\{1\} + \lim_{\varepsilon\to 0}\{2\} = \phi_x(0,0)\frac{4}{c}\,\mathrm{Log}\left(\frac{1-c}{1+c}\right).$$

Therefore,

$$J = \lim_{\varepsilon\to 0} J_\varepsilon = \lim_{\varepsilon\to 0} I_1 + \lim_{\varepsilon\to 0} I_2 = 4\pi i\sigma\phi(0,0) + \phi_x(0,0)\frac{4}{c}\,\mathrm{Log}\left(\frac{1-c}{1+c}\right). \tag{3}$$

□

Numerical confirmation. A program written the Wolfram Mathematica language is attached to this document. It confirms numerically that $J_\varepsilon \approx \mathrm{RHS}(3)$. In the program, we use:

$$\begin{aligned}
\Omega &= [-1,1]^2 \\
v(x,t) &= x + ct + c\frac{x^2}{2} \\
u(x,t) &= -2\frac{v_x}{v} \\
\phi(x,t) &= \lambda\rho(\lambda(x-\alpha), \lambda t) \\
\rho(x,t) &= \begin{cases} (x^2-1)^3(t^2-1)^2 & \text{if} \quad |x|,\ |t| \leq 1 \\ 0 & \quad\text{otherwise} \end{cases}
\end{aligned}$$

```
In[1]:= c = (I - 1) / 5;
λ = 2;
α = 0.1;
ϵ = 0.00005;
Ωϵ = ImplicitRegion[((-1 ≤ x ≤ -ϵ || ϵ ≤ x ≤ 1) && -1 ≤ t ≤ 1) ||
    (-ϵ ≤ x ≤ ϵ && (-1 ≤ t ≤ -ϵ || ϵ ≤ t ≤ 1)), {x, t}];
ρ[x_, t_] := Piecewise[{{(x^2 - 1)^3 * (t^2 - 1)^2, -1 ≤ x ≤ 1 && -1 ≤ t ≤ 1}}, 0];
ρx[x_, t_] := Piecewise[{{6 x (x^2 - 1)^2 * (t^2 - 1)^2, -1 ≤ x ≤ 1 && -1 ≤ t ≤ 1}}, 0];
ρxx[x_, t_] := Piecewise[{{6 (-1 + t^2)^2 (-1 + x^2) (-1 + 5 x^2), -1 ≤ x ≤ 1 && -1 ≤ t ≤ 1}}, 0];
ρt[x_, t_] := Piecewise[{{4 t (t^2 - 1) * (x^2 - 1)^3, -1 ≤ x ≤ 1 && -1 ≤ t ≤ 1}}, 0];
u[x_, t_] := -2 (1 + c x) / (x + c t + c * x^2 / 2);
ux[x_, t_] := (8 + 8 c x + c^2 (-8 t + 4 x^2)) / (2 x + c (2 t + x^2))^2;
uxx[x_, t_] := -(8 (1 + c x) (4 + 2 c x + c^2 (-6 t + x^2))) / (2 x + c (2 t + x^2))^3;
ut[x_, t_] := (8 c (1 + c x)) / (2 x + c (2 t + x^2))^2;
ϕ[x_, t_] := ρ[λ (x - α), λ * t];
ϕt[x_, t_] := λ ρt[λ (x - α), λ * t];
ϕx[x_, t_] := λ ρx[λ (x - α), λ * t];
ϕxx[x_, t_] := λ^2 ρxx[λ (x - α), λ * t];
(*The double integral Jϵ*)
NIntegrate[-u[x, t] * (ϕt[x, t] + u[x, t] * ϕx[x, t] / 2 + ϕxx[x, t]), {x, t} ∈ Ωϵ]
(*The number 4πiσϕ(0,0)+ϕx(0,0) 4/c Log((1-c)/(1+c)) *)
N[Sign[Im[c]] * 4 * Pi * I * ϕ[0, 0] + 4 / c * ϕx[0, 0] * Log[(1 - c) / (1 + c)]]

NIntegrate: Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small.

Out[18]=
17.6721 - 11.5885 i

Out[19]=
17.6722 - 11.5885 i
```