ECE 468: Digital Image Processing

Lecture 11

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Problem with Fourier Transform

- Available frequency content
- But not where that content is in the space domain of 2D signals
- (or in the time domain for 1D signals)

$$e^{j2\pi\omega x}$$

Problem with Fourier Transform

- Available frequency content
- But not where that content is in the space domain of 2D signals
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$$e^{j2\pi\omega x}$$

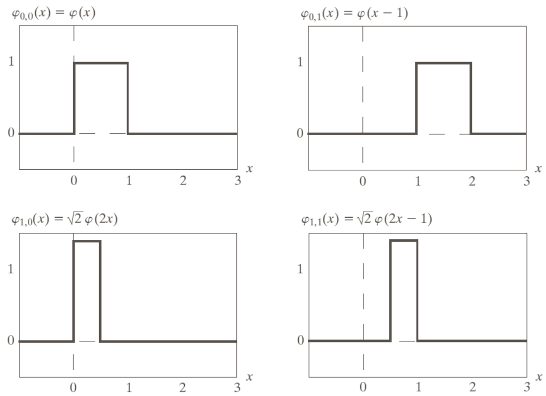
just scaling of the generating function w/o translation

Scaling Functions

Given: $\varphi(x)$

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$$

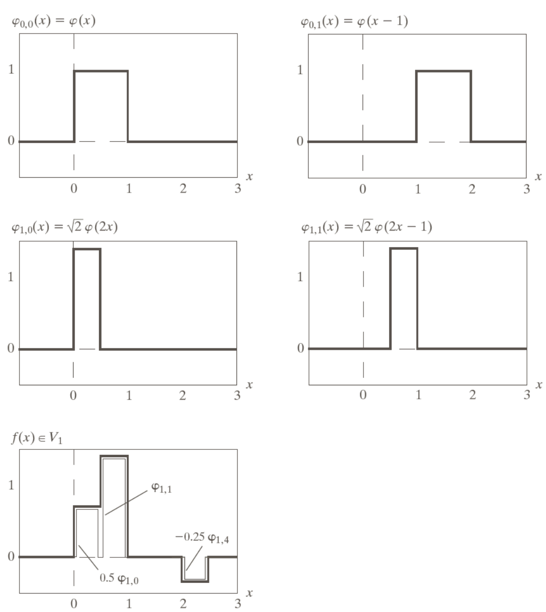
Example: Haar Scaling Functions



a b
c d
e f

FIGURE 7.11 Some Haar scaling functions.

Example: Haar Scaling Functions

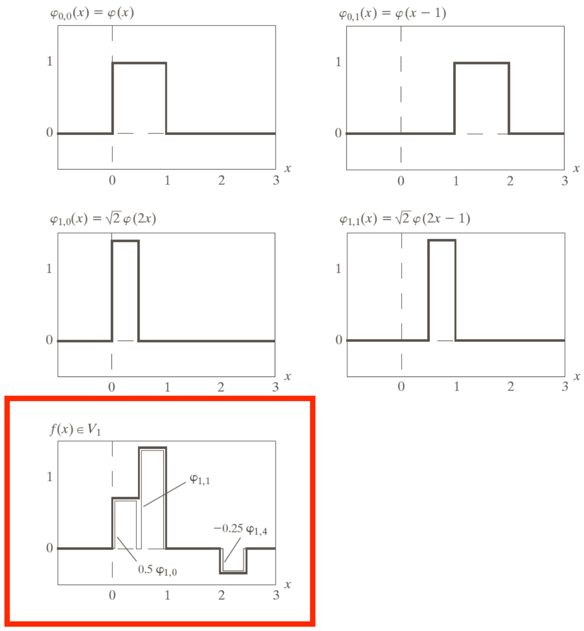


a b
c d
e f

FIGURE 7.11 Some Haar scaling functions.

$$f(x) = 0.5\varphi_{1,0}(x) + \varphi_{1,1}(x) - 0.25\varphi_{1,4}(x)$$

Example: Haar Scaling Functions



Function in V₁

a b
c d
e f

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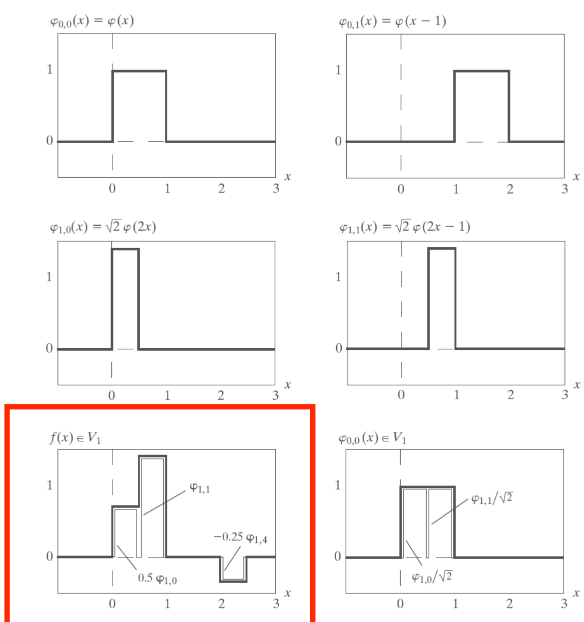

Function in V₁

a b
c d
e f

FIGURE 7.11 Some Haar scaling functions.

$$f(x) = 0.5\varphi_{1,0}(x) + \varphi_{1,1}(x) - 0.25\varphi_{1,4}(x)$$

$$\varphi_{0,k}(x) = \frac{1}{\sqrt{2}}\varphi_{1,2k}(x) + \frac{1}{\sqrt{2}}\varphi_{1,2k+1}(x)$$

Nested Function Spaces

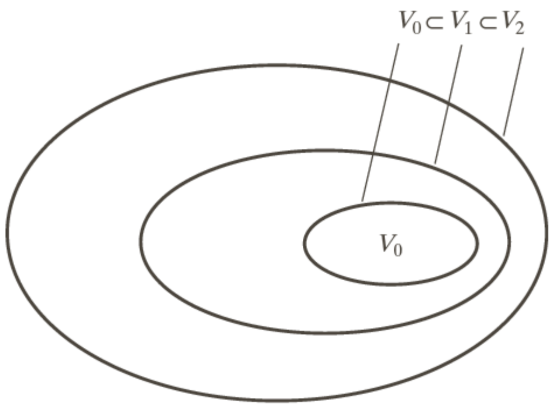

FIGURE 7.12 The nested function spaces spanned by a scaling function.

Any Function

If $f(x) \in V_j$

$$f(x) = \sum_k \alpha_k \varphi_{j,k}(x)$$

$$f(x) = \sum_k \beta_k \varphi_{j+1,k}(x)$$

$$f(x) = \sum_k \gamma_k \varphi_{j+2,k}(x)$$

$$\vdots$$

Wavelet Function Spaces

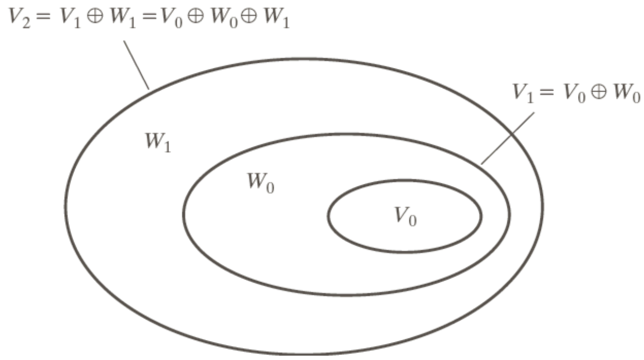

FIGURE 7.13 The relationship between scaling and wavelet function spaces.

Wavelet Function

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$$

$$\langle \psi_{j,k}(x), \varphi_{j,l}(x) \rangle = 0$$

$$V_{j+1} = V_j \oplus W_J$$

Any Function

If $f(x) \in V_j$

Any Function

If $f(x) \in V_j$

$$f(x) = \underbrace{\underbrace{\sum_k \alpha_k \varphi_{j-1,k}(x)}_{V_{j-1}} + \sum_k \beta_k \psi_{j-1,k}(x)}_{V_j}$$

Any Function

If $f(x) \in V_j$

$$f(x) = \underbrace{\underbrace{\sum_k \alpha_k \varphi_{j-1,k}(x)}_{V_{j-1}} + \sum_k \beta_k \psi_{j-1,k}(x)}_{V_j}$$

$$f(x) = \underbrace{\underbrace{\underbrace{\sum_k \alpha_k \varphi_{j-2,k}(x)}_{V_{j-2}} + \sum_k \beta_k \psi_{j-2,k}(x)}_{V_{j-1}} + \sum_k \beta_k \psi_{j-1,k}(x)}_{V_j}$$

Any Function

If $f(x) \in V_j$

$$f(x) = \underbrace{\underbrace{\sum_k \alpha_k \varphi_{j-1,k}(x)}_{V_{j-1}} + \sum_k \beta_k \psi_{j-1,k}(x)}_{V_j}$$

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$$\vdots$$

$$f(x) = \sum_k \alpha_k \varphi_{j_0,k}(x) + \sum_{l=j_0}^{j-1} \sum_k \beta_k \psi_{l,k}(x)$$

Relations Between the Scaling and Wavelet Functions

$$\begin{aligned}\varphi_{j,k}(x) &= \sum_n \alpha_n \varphi_{j+1,n} \ , \\ &= \sum_n h_\varphi(n) 2^{\frac{j+1}{2}} \varphi(2^{j+1}x - n)\end{aligned}$$

$$\Rightarrow \quad \varphi(x) = \sum_n h_\varphi(n)\sqrt{2}\varphi(2x-n)$$

Relations Between the Scaling and Wavelet Functions

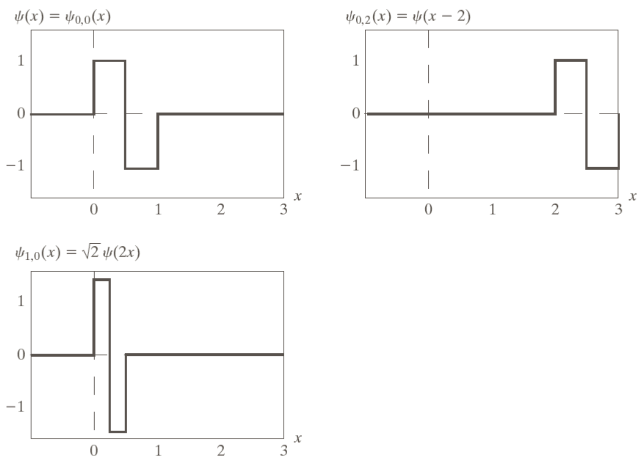
$$\begin{aligned}\varphi_{j,k}(x) &= \sum_n \alpha_n \varphi_{j+1,n} \ , \\ &= \sum_n h_\varphi(n) 2^{\frac{j+1}{2}} \varphi(2^{j+1}x - n)\end{aligned}$$

$$\Rightarrow \quad \varphi(x) = \sum_n h_\varphi(n)\sqrt{2}\varphi(2x-n)$$

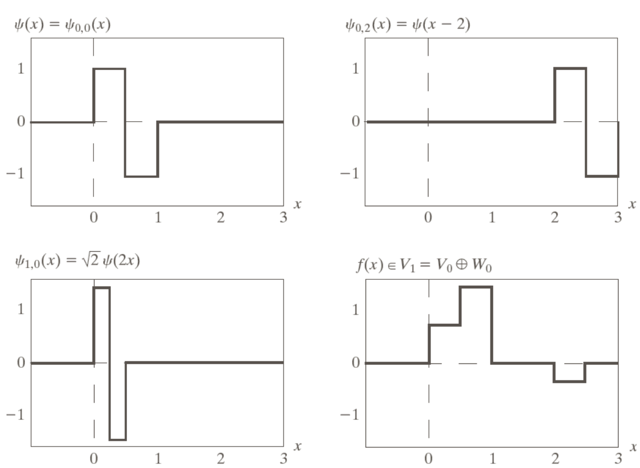
$$\Rightarrow \quad \psi(x) = \sum_n h_\psi(n)\sqrt{2}\varphi(2x-n)$$

$$h_\psi(n) = (-1)^n h_\varphi(1-n)$$

Haar Wavelet Functions



Haar Wavelet Functions



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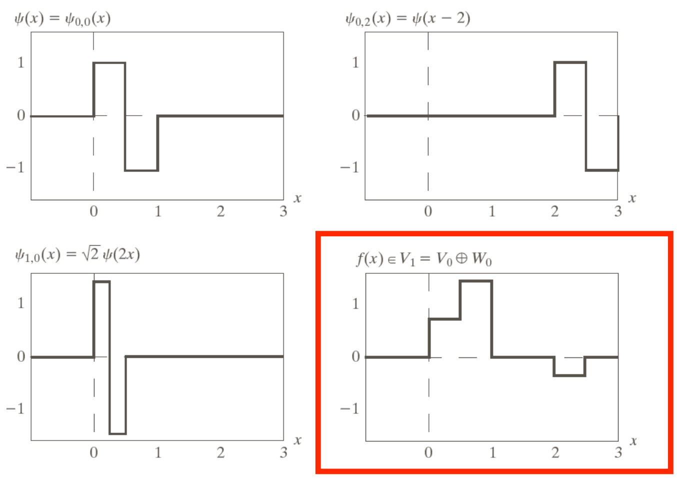
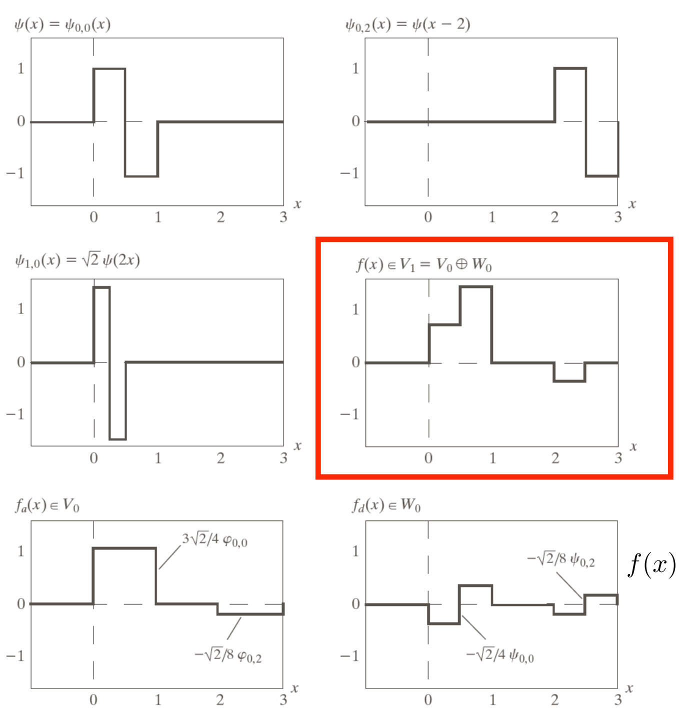
Haar Wavelet Functions



Function in V_1

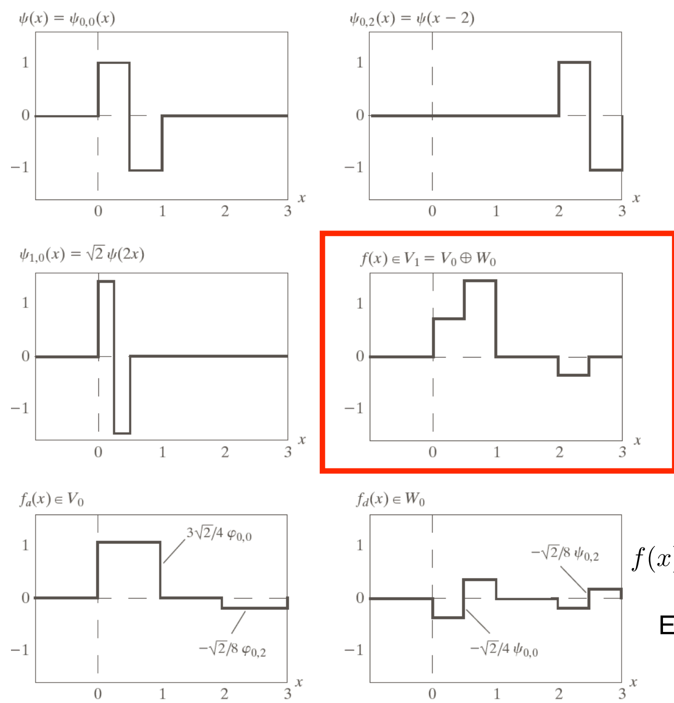
Haar Wavelet Functions



Function in V_1

$$f(x) = f_a(x) + f_d(x)$$

Haar Wavelet Functions



Function in V_1

$$f(x) = f_a(x) + f_d(x)$$

Expressed via V_0

Wavelet Series Expansion

$$f(x) = \sum_k c_{j_0}(k)\varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_k d_j(k)\psi_{j,k}(x)$$

$$c_{j_0}(k) \;\; = \langle \varphi_{j_0,k}(x), f(x) \rangle$$

$$d_j(k) \;\; = \langle \psi_{j,k}(x), f(x) \rangle$$

Example: Haar Wavelet Series Expansion