ECE 468: Digital Image Processing

Lecture 12

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Scaling Functions

Given: $\varphi(x)$

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$$

Wavelet Functions

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$$

$$\langle \psi_{j,k}(x), \varphi_{j,l}(x) \rangle = 0$$

$$V_{j+1} = V_j \oplus W_J$$

Wavelet Function Spaces

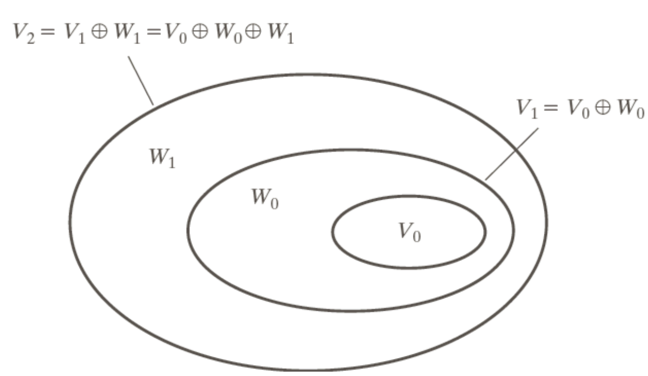

FIGURE 7.13 The relationship between scaling and wavelet function spaces.

Relations Between the Scaling and Wavelet Functions

$$\varphi(x) = \sum_n h_\varphi(n)\sqrt{2}\varphi(2x - n)$$

Relations Between the Scaling and Wavelet Functions

$$\varphi(x) = \sum_n h_\varphi(n)\sqrt{2}\varphi(2x - n)$$

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$$m = 2k + n$$

$$= \sum_m h_\varphi(m - 2k)\sqrt{2}\varphi(2^{j+1} x - m)$$

Relations Between the Scaling and Wavelet Functions

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Relations Between the Scaling and Wavelet Functions

$$\psi(x) = \sum_n h_\psi(n)\sqrt{2}\varphi(2x-n)$$

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$$m = 2k + n$$

$$= \sum_m h_\psi(m-2k)\sqrt{2}\varphi(2^{j+1}x - m)$$

Discrete Wavelet Transform

$$f(n) = \frac{1}{\sqrt{M}}\sum_k W_\varphi(j_0,k)\varphi_{j_0,k}(n) + \frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_k W_\psi(j,k)\psi_{j,k}(n)$$

$$W_\varphi(j_0,k) = \frac{1}{\sqrt{M}}\sum_n f(n)\varphi_{j_0,k}(n)$$

$$W_\psi(j,k) = \frac{1}{\sqrt{M}}\sum_n f(n)\psi_{j,k}(n),\ \ j \geq j_0$$

$M = 2^j$ $\qquad$ $j = 0, 1, 2, \ldots, J-1$

$n = 0, 1, 2, \ldots, M-1$ $\qquad$ $k = 0, 1, 2, \ldots, 2^j - 1$

Fast Wavelet Transform

$$W_{\varphi}(j,k) = \frac{1}{\sqrt{M}} \sum_{n} f(n) \varphi_{j,k}(n)$$

Fast Wavelet Transform

$$\begin{aligned} W_{\varphi}(j,k) &= \frac{1}{\sqrt{M}} \sum_{n} f(n) \varphi_{j,k}(n) \\ &= \frac{1}{\sqrt{M}} \sum_{n} f(n) 2^{\frac{j}{2}} \varphi(2^j n - k) \end{aligned}$$

Fast Wavelet Transform

$$W_\varphi(j,k) = \frac{1}{\sqrt{M}} \sum_n f(n)\varphi_{j,k}(n)$$

$$= \frac{1}{\sqrt{M}} \sum_n f(n) 2^{\frac{j}{2}} \varphi(2^j n - k)$$

$$= \frac{1}{\sqrt{M}} \sum_n f(n) 2^{\frac{j}{2}} \sum_m h_\varphi(m-2k)\sqrt{2}\varphi(2^{j+1}n - m)$$

Fast Wavelet Transform

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$$= \sum_m h_\varphi(m-2k) \frac{1}{\sqrt{M}} \sum_n f(n) 2^{\frac{j+1}{2}} \varphi(2^{j+1}n - m)$$

Fast Wavelet Transform

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$$= \sum_m h_\varphi(m-2k) \frac{1}{\sqrt{M}} \sum_n f(n) 2^{\frac{j+1}{2}} \varphi(2^{j+1}n - m)$$

$$= \sum_m h_\varphi(m-2k) W_\varphi(j+1, m)$$

Fast Wavelet Transform

$$W_\psi(j,k) = \frac{1}{\sqrt{M}} \sum_n f(n)\psi_{j,k}(n)$$

Fast Wavelet Transform

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Fast Wavelet Transform

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Fast Wavelet Transform

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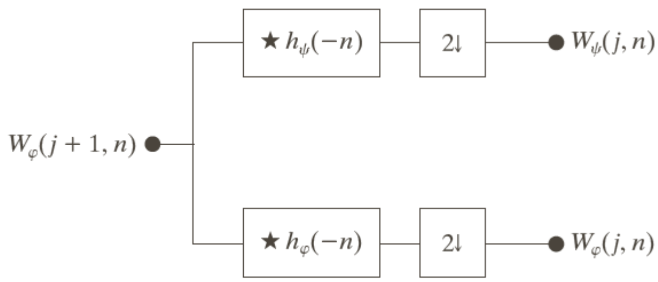
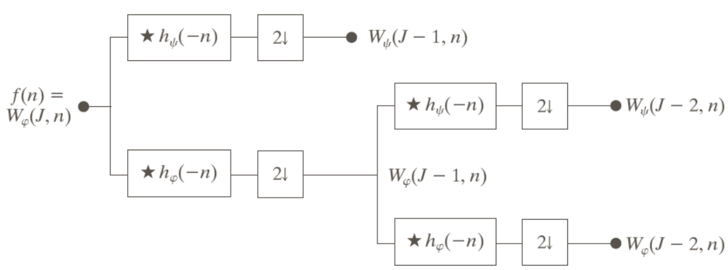
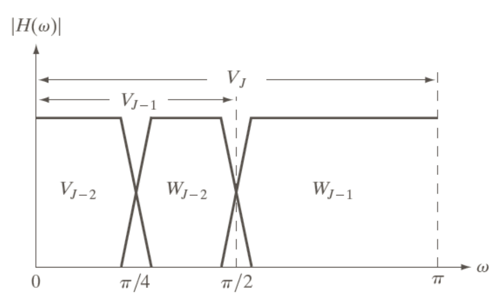
$$= \sum_m h_\psi(m-2k) W_\varphi(j+1, m)$$

Discrete Wavelet Transform



Discrete Wavelet Transform



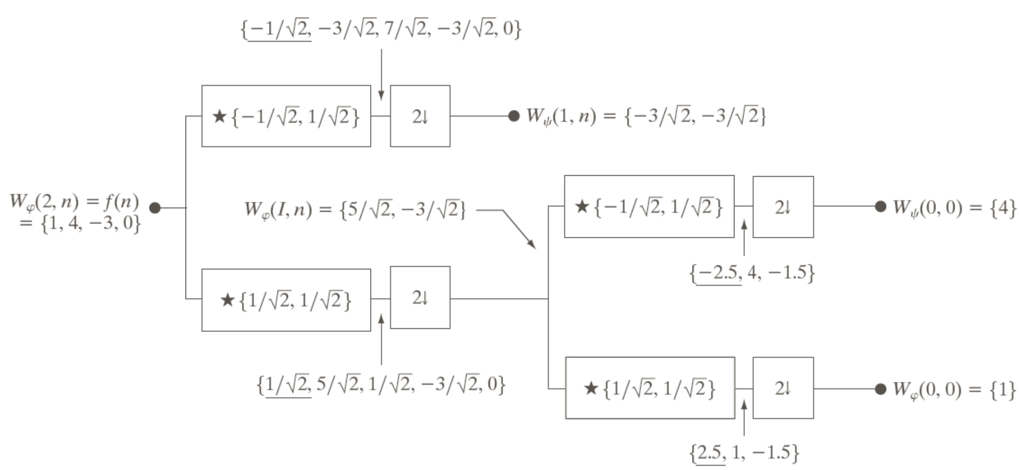


Example: Discrete Wavelet Transform

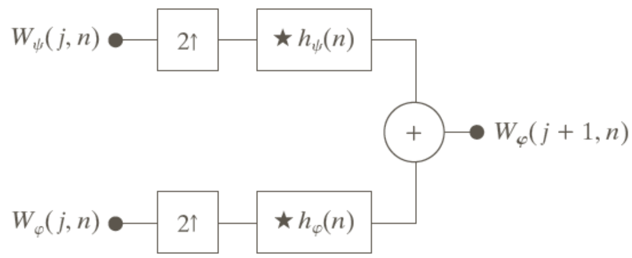
n	$h_\varphi(n)$
0	$1/\sqrt{2}$
1	$1/\sqrt{2}$

TABLE 7.2 Orthonormal Haar filter coefficients for $h_\varphi(n)$.

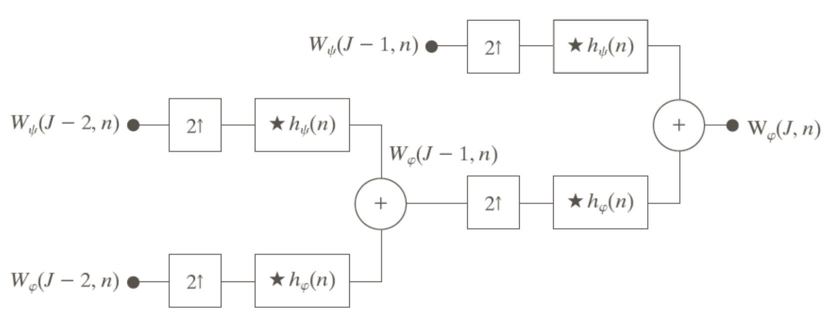
Example: Discrete Wavelet Transform



Inverse Discrete Wavelet Transform



Inverse Discrete Wavelet Transform



Example: Inverse Discrete Wavelet Transform

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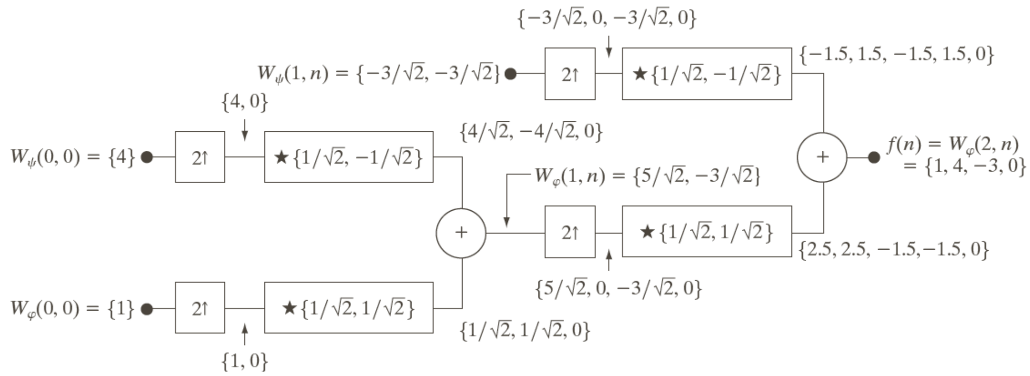
1	$1/\sqrt{2}$



2D Scaling and Wavelet Functions

$$\varphi(x, y) = \varphi(x)\varphi(y)$$

2D Scaling and Wavelet Functions

$$\varphi(x, y) = \varphi(x)\varphi(y)$$

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$$\psi^V(x, y) = \varphi(x)\psi(y)$$

$$\psi^D(x, y) = \psi(x)\psi(y)$$

2D Scaling and Wavelet Functions

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$$\varphi_{j,m,n}(x, y) = 2^{\frac{j}{2}}\varphi(2^j x - m, 2^j y - n)$$

$$\psi^i_{j,m,n}(x, y) = 2^{\frac{j}{2}}\psi^i(2^j x - m, 2^j y - n), \quad i = H, V, D$$

2D Discrete Wavelet Transform

$$\varphi(x,y) = \varphi(x)\varphi(y)$$

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$$W_\varphi(j,m,n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\varphi_{j,m,n}(x,y)$$
$$W_\psi(j,m,n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\psi_{j,m,n}(x,y)$$

2D Discrete Wavelet Transform

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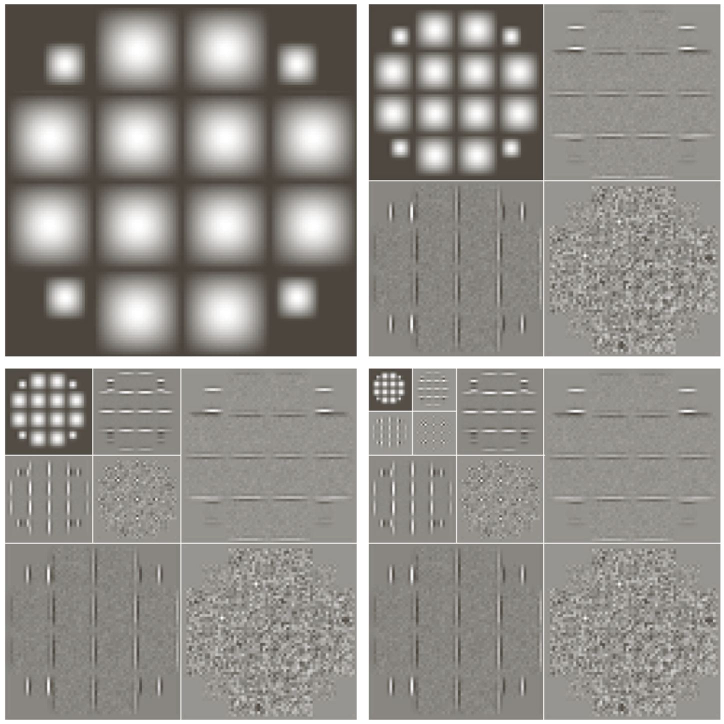
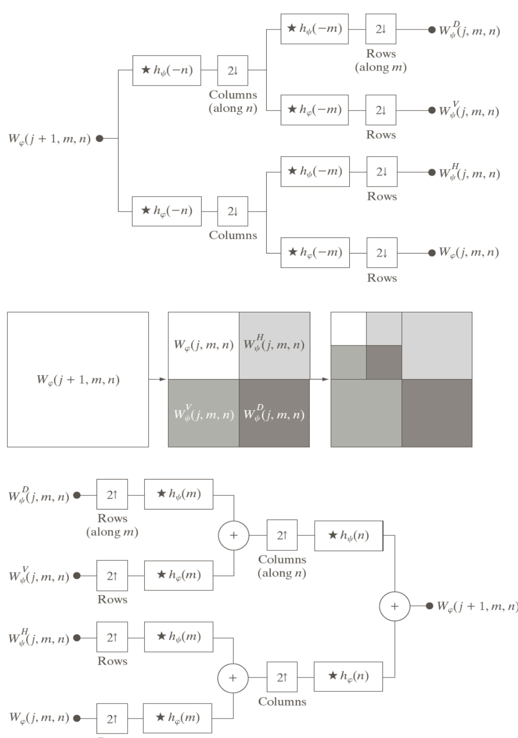
$$W_\psi(j,m,n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\psi_{j,m,n}(x,y)$$

$$f(x,y) = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} W_\varphi(j_0,m,n)\varphi_{j_0,m,n}(x,y) + \frac{1}{\sqrt{MN}} \sum_{i=H,V,D} \sum_{j=j_0}^{\infty} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} W^i_\psi(j,m,n)\psi^i_{j,m,n}(x,y)$$

2D DWT



2D DWT


$W_\varphi(j+1, m, n)$
$\star h_\psi(-n)$
$2\downarrow$
Columns (along n)
$\star h_\psi(-m)$
$2\downarrow$
Rows (along m)
$W_\psi^D(j, m, n)$
$\star h_\varphi(-m)$
$2\downarrow$
Rows
$W_\psi^V(j, m, n)$
$\star h_\varphi(-n)$
$2\downarrow$
Columns
$\star h_\psi(-m)$
$2\downarrow$
Rows
$W_\psi^H(j, m, n)$
$\star h_\varphi(-m)$
$2\downarrow$
Rows
$W_\varphi(j, m, n)$
$W_\varphi(j+1, m, n)$
$W_\varphi(j, m, n)$
$W_\psi^H(j, m, n)$
$W_\psi^V(j, m, n)$
$W_\psi^D(j, m, n)$
$W_\psi^D(j, m, n)$
$2\uparrow$
Rows (along m)
$\star h_\psi(m)$
$W_\psi^V(j, m, n)$
$2\uparrow$
Rows
$\star h_\varphi(m)$
$+$
$2\uparrow$
Columns (along n)
$\star h_\psi(n)$
$W_\psi^H(j, m, n)$
$2\uparrow$
Rows
$\star h_\psi(m)$
$W_\varphi(j, m, n)$
$2\uparrow$
Rows
$\star h_\varphi(m)$
$+$
$2\uparrow$
Columns
$\star h_\varphi(n)$
$+$
$W_\varphi(j+1, m, n)$

Uncertainty Principle

Energy spread of a function and its Fourier transform

CANNOT BE

simultaneously arbitrarily small.

Example: Uncertainty Principle

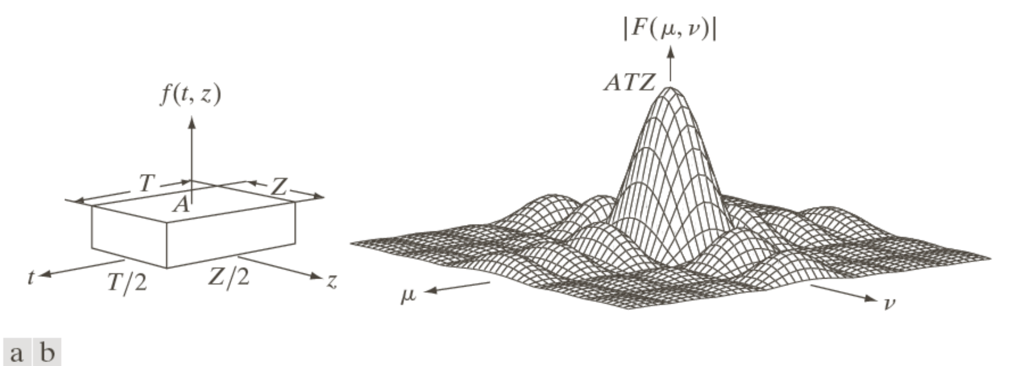


FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

Example: Uncertainty Principle

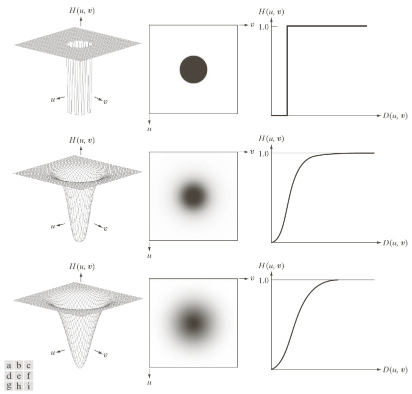


FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

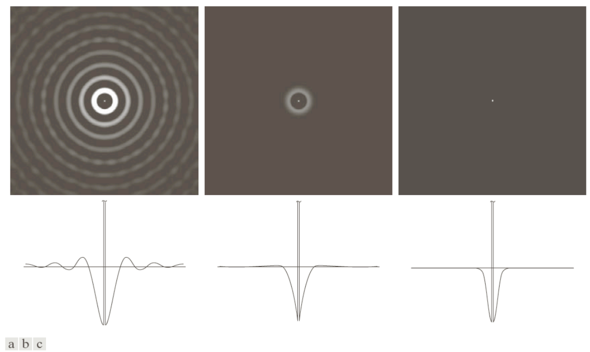


FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

1D Case: Gabor -- Ville -- Wigner

- Decompose a signal
- over elementary waveforms
- that have a minimal spread in the time-frequency plane

1D Case: Time-Frequency Plane

A Wavelet Tour of Signal Processing
Stéphane Mallat, Academic Press 1999 (2nd edition)

u,v = offsets in time
ξ, γ = offsets in frequency

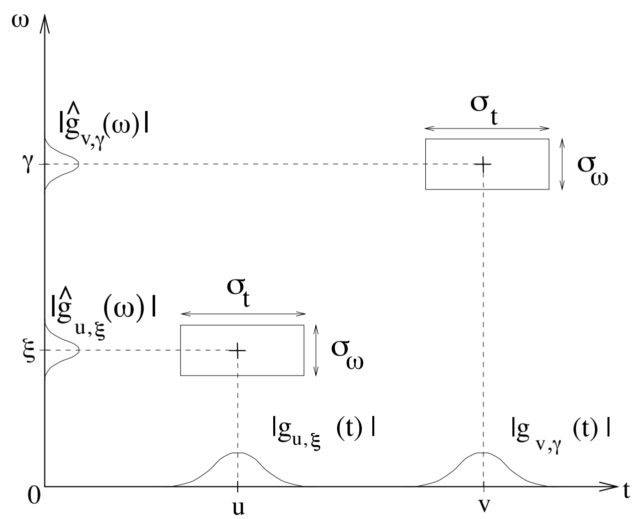


Figure 1.1: Time-frequency boxes ("Heisenberg rectangles") representing the energy spread of two Gabor atoms

Gabor Functions

Constructed by:

- Translating the window function g in
  - Time by u
  - Frequency by ξ
- Scaling the window function g

$$g_{k,u,\xi}(t) = 2^{\frac{k}{2}} \; g(2^k t - u) \; \mathrm{e}^{j\xi t}$$

Example