ECE 468: Digital Image Processing

Lecture 8

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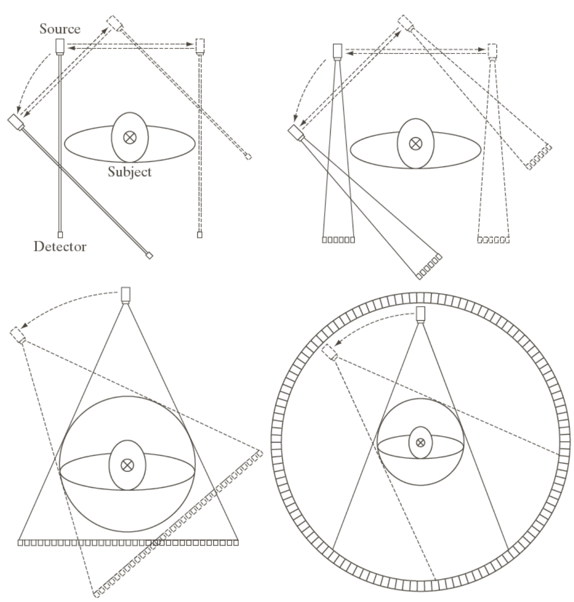
Image Reconstruction from Projections

X-ray computed tomography:

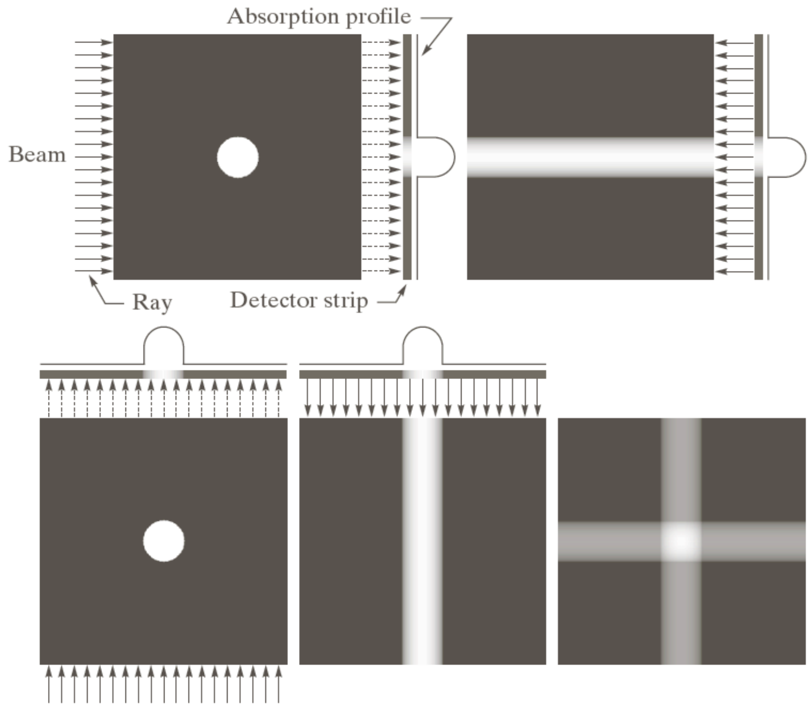
X-raying an object from different directions ⇒ 3D object representation



a b
c d

FIGURE 5.35 Four generations of CT scanners. The dotted arrow lines indicate incremental linear motion. The dotted arrow arcs indicate incremental rotation. The cross-mark on the subject's head indicates linear motion perpendicular to the plane of the paper. The double arrows in (a) and (b) indicate that the source/detector unit is translated and then brought back into its original position.

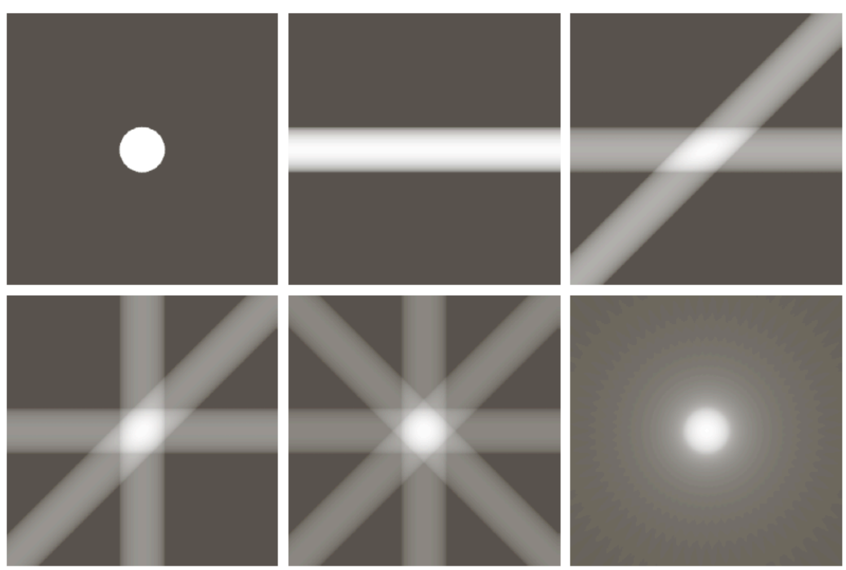
Example: Backprojecting a 1D signal



a b
c d e

FIGURE 5.32 (a) Flat region showing a simple object, an input parallel beam, and a detector strip. (b) Result of back-projecting the sensed strip data (i.e., the 1-D absorption profile). (c) The beam and detectors rotated by 90°. (d) Back-projection. (e) The sum of (b) and (d). The intensity where the back-projections intersect is twice the intensity of the individual back-projections.

...As We Increase the Number of Backprojections



a b c
d e f

FIGURE 5.33 (a) Same as Fig. 5.32(a). (b)–(e) Reconstruction using 1, 2, 3, and 4 backprojections 45° apart. (f) Reconstruction with 32 backprojections 5.625° apart (note the blurring).

halo effect

Example: Backprojecting a 1D signal

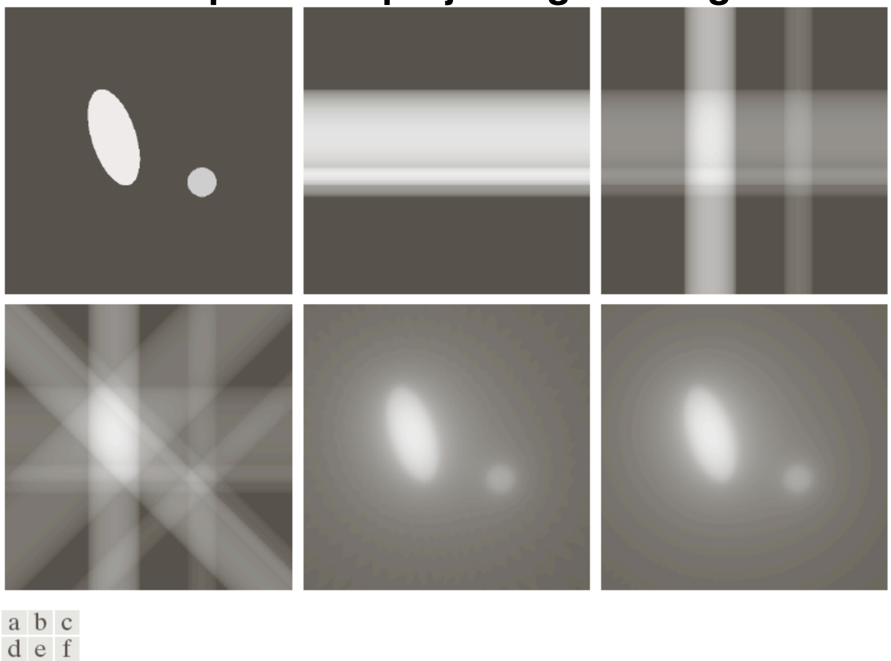


FIGURE 5.34 (a) A region with two objects. (b)–(d) Reconstruction using 1, 2, and 4 backprojections 45° apart. (e) Reconstruction with 32 backprojections 5.625° apart. (f) Reconstruction with 64 backprojections 2.8125° apart.

Projection

$$y = ax + b$$

$$x \cos\theta + y \sin\theta = \rho$$

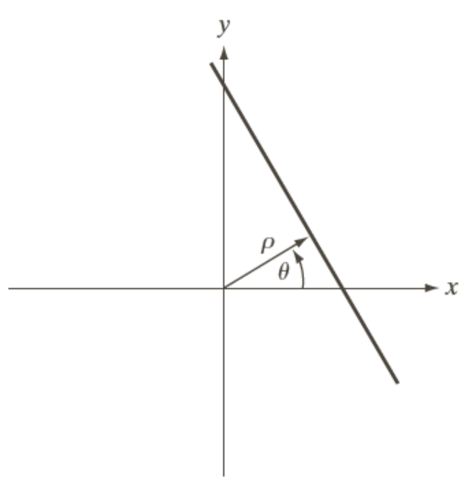
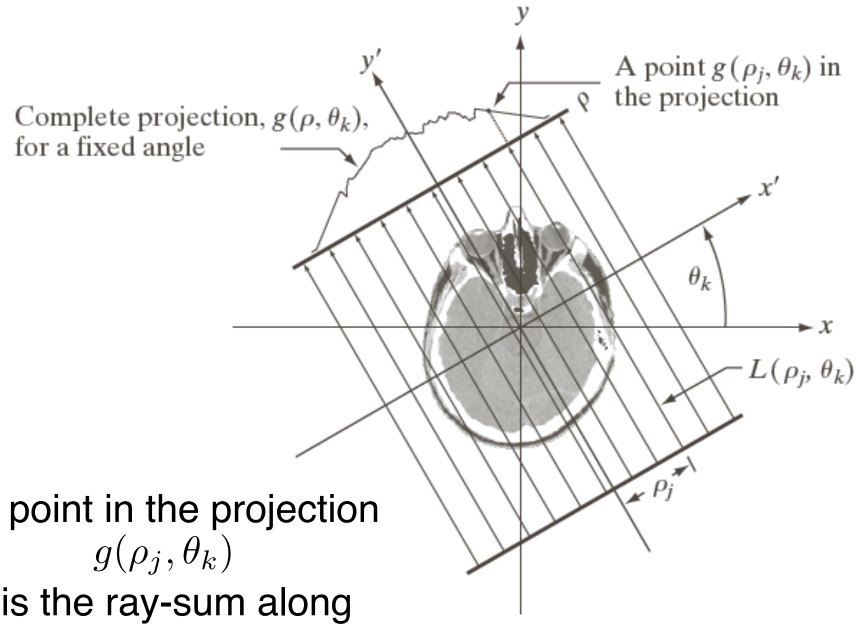


FIGURE 5.36 Normal representation of a straight line.

Radon Transform



A point in the projection

$g(\rho_j, \theta_k)$

is the ray-sum along

$$x \cos\theta_k + y \sin\theta_k = \rho_j$$

Radon Transform

$$g(\rho, \theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\delta(x\cos\theta + y\sin\theta - \rho)dxdy$$

continuous space coordinates

$$g(\rho, \theta) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x, y)\delta(x\cos\theta + y\sin\theta - \rho)$$

discrete space coordinates

key tool for reconstruction from projections

Example: Radon Transform

$$f(x,y) = \begin{cases} A & , \quad x^2+y^2 \le r^2 \\ 0 & , \quad \text{o.w} \end{cases}$$

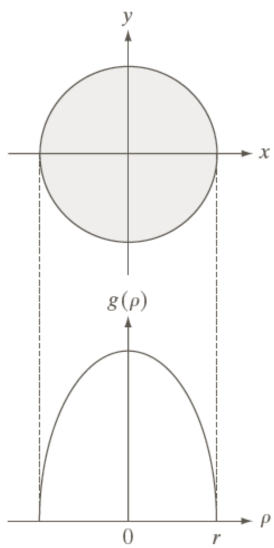
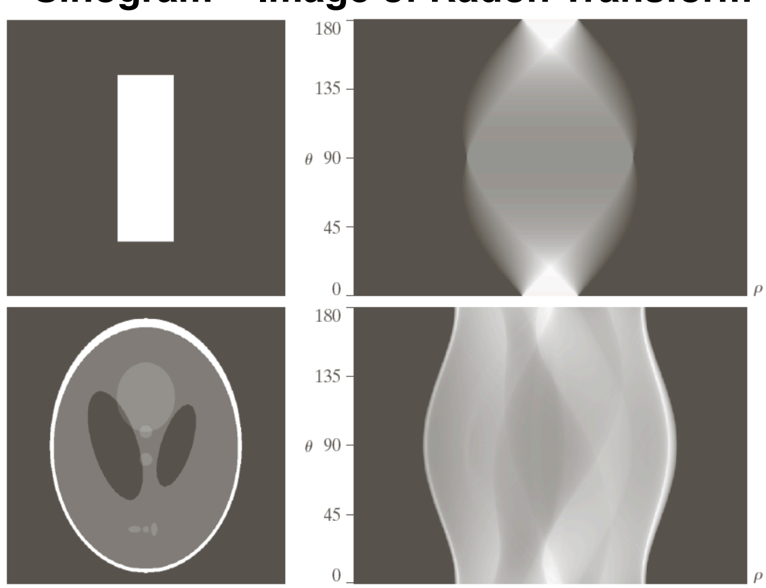


FIGURE 5.38 A disk and a plot of its Radon transform, derived analytically. Here we were able to plot the transform because it depends only on one variable. When g depends on both ρ and θ, the Radon transform becomes an image whose axes are ρ and θ, and the intensity of a pixel is proportional to the value of g at the location of that pixel.

Sinogram = Image of Radon Transform



a b
c d

FIGURE 5.39 Two images and their sinograms (Radon transforms). Each row of a sinogram is a projection along the corresponding angle on the vertical axis. Image (c) is called the *Shepp-Logan phantom*. In its original form, the contrast of the phantom is quite low. It is shown enhanced here to facilitate viewing.

Properties of Objects from Sinogram

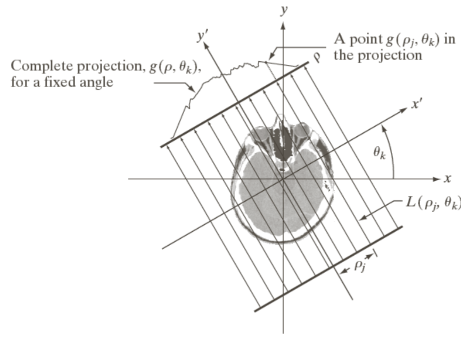
- Sinogram symmetric = Object symmetric
- Sinogram symmetric about image center = Object symmetric and parallel to x and y axes
- Sinogram smooth = Object has uniform intensity

Computed Tomography (CT)

- Key objective:
- Obtain a 3D representation of a volume from its projections

- How:
- Backproject all projections and sum them all in one image
- By stacking all images we obtain the 3D volume

Backprojection from the Radon Transform

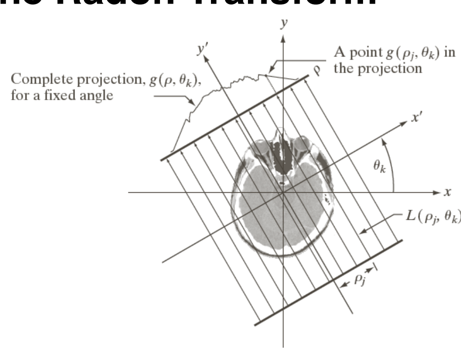


- Given point $g(\rho_j, \theta_k)$
- Backprojection = copy the value of $g(\rho_j, \theta_k)$ on the entire line

$$\forall \rho \quad \Rightarrow \quad f_{\theta_k}(x, y) = g(x\cos\theta_k + y\sin\theta_k, \theta_k)$$

$$\Rightarrow \quad f(x, y) = \int_0^\pi f_\theta(x, y) d\theta$$

Backprojection from the Radon Transform



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Laminogram Obtained from Sinogram

- Backprojection for a specific angle

$$f_{\theta_k}(x,y) = g(x\cos\theta_k + y\sin\theta_k, \theta_k)$$

- Summation over all theta

$$f(x,y) = \sum_{\theta=0}^{\pi} f_\theta(x,y)$$

Laminogram Obtained from Sinogram

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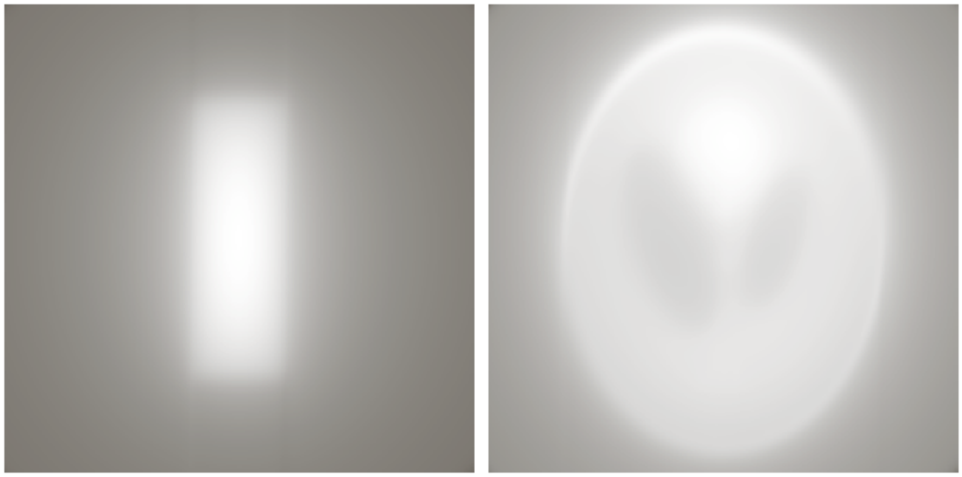
Example Laminograms



Significant improvements can be obtained
by reformulating backprojections!

Relating

1D Fourier Transform of the projection

with

2D Fourier Transform of the image

from which the projection was obtained.

1D Fourier Transform of the Projection

$$G(\omega, \theta) = \int_{-\infty}^{\infty} g(\rho, \theta) \mathrm{e}^{-j2\pi\omega\rho} d\rho$$

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by definition

$$G(\omega, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos\theta + y \sin\theta - \rho) \mathrm{e}^{-j2\pi\omega\rho} dx\ dy\ d\rho$$

1D Fourier Transform of the Projection

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by definition

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$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{e}^{-j2\pi\omega(x\cos\theta + y\sin\theta)}dx\ dy$$

1D Fourier Transform of the Projection

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$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{e}^{-j2\pi\omega(x\cos\theta + y\sin\theta)}dx\ dy$$

$$= F(\omega\cos\theta, \omega\sin\theta)$$

1D Fourier Transform of the Projection

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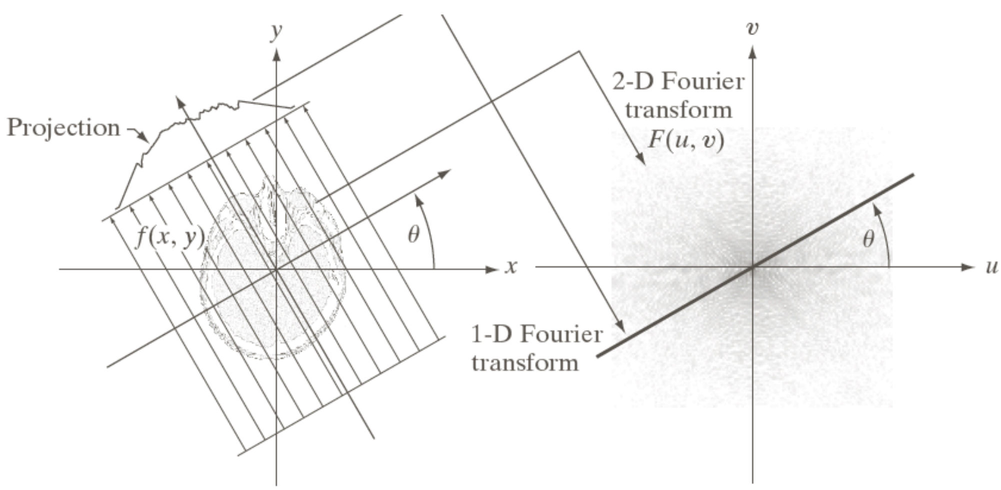
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{e}^{-j2\pi\omega(x\cos\theta+y\sin\theta)}dx\ dy$$

$$= F(\omega\cos\theta, \omega\sin\theta)$$ Fourier Slice Theorem

Fourier Slice Theorem



1D FT = a slice of 2D FT

Reconstruction Using Filtered Backprojections

by definition

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\mathrm{e}^{j2\pi(ux+vy)}du\ dv$$

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$$u = \omega\cos\theta,\ \ v = \omega\sin\theta,\ \ \Rightarrow\ \ dudv = \omega d\omega d\theta$$

$$f(x,y) = \int_{0}^{2\pi}\int_{0}^{\infty} F(\omega\cos\theta,\omega\sin\theta)\mathrm{e}^{j2\pi\omega(x\cos\theta+y\sin\theta)}\omega\ d\omega\ d\theta$$

Reconstruction Using Filtered Backprojections

by definition

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$$f(x,y) = \int_{0}^{2\pi}\int_{0}^{\infty} F(\omega\cos\theta, \omega\sin\theta)\mathrm{e}^{j2\pi\omega(x\cos\theta + y\sin\theta)}\omega\ d\omega\ d\theta$$

by Fourier Slice Theorem

$$f(x,y) = \int_{0}^{2\pi}\int_{0}^{\infty} G(\omega,\theta)\mathrm{e}^{j2\pi\omega(x\cos\theta + y\sin\theta)}\omega\ d\omega\ d\theta$$

Reconstruction Using Filtered Backprojections

$$G(\omega, \theta + 180^\circ) = G(-\omega, \theta)$$

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Reconstruction Using Filtered Backprojections

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$$f(x, y) = \int_0^\pi \int_0^\infty |\omega| G(\omega, \theta) \mathrm{e}^{j2\pi\omega(x\cos\theta + y\sin\theta)} \; d\omega \; d\theta$$

$$f(x, y) = \int_0^\pi \left[\int_0^\infty |\omega| G(\omega, \theta) \mathrm{e}^{j2\pi\omega\rho} \; d\omega \right]_{\rho = x\cos\theta + y\sin\theta} d\theta$$

Reconstruction Using Filtered Backprojections

$$G(\omega, \theta + 180°) = G(-\omega, \theta)$$

$$f(x, y) = \int_0^{\pi} \int_0^{\infty} |\omega| G(\omega, \theta) \mathrm{e}^{j2\pi\omega(x\cos\theta + y\sin\theta)} \; d\omega \; d\theta$$

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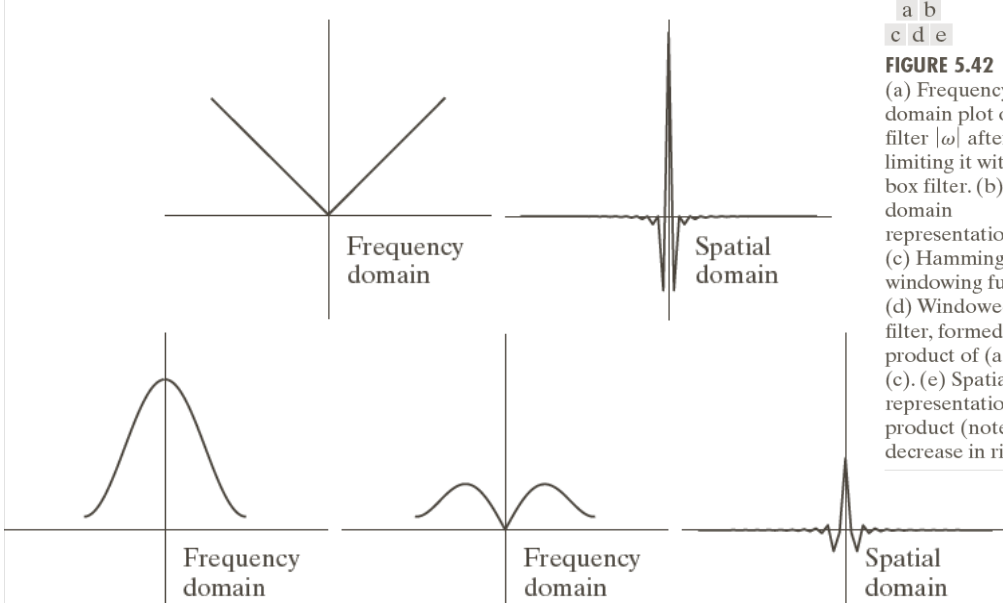
1D filtering

Box + Ramp Filter



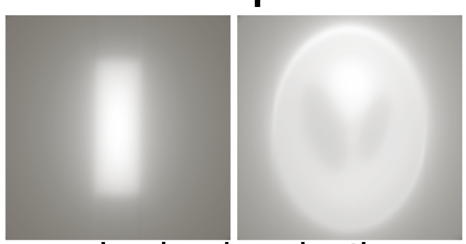
a b
c d e

FIGURE 5.42 (a) Frequency domain plot of the filter $|\omega|$ after band-limiting it with a box filter. (b) Spatial domain representation. (c) Hamming windowing function. (d) Windowed ramp filter, formed as the product of (a) and (c). (e) Spatial representation of the product (note the decrease in ringing).

Algorithm for Filtered Backprojection

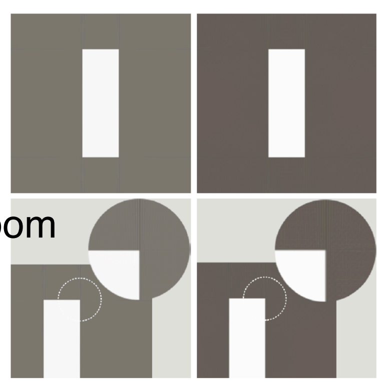
1. Given projections $g(\rho,\theta)$ obtained at each fixed angle θ
2. Compute $G(\omega,\theta)$ = 1D Fourier Transform of each projection $g(\rho,\theta)$
3. Multiply $G(\omega,\theta)$ by the filter function $|\omega|$ modified by Hamming window
4. Compute the inverse of the results from 3.
5. Integrate (sum) over θ all results from 4.

Examples



naive backprojection

zoom



ramp filter | windowed ramp filter

ramp filter | windowed ramp filter