Activities as Time Series of Human-Body Snapshots Supplemental Material

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In the supplemental material, we present the proof that our four alternative algorithms for solving the LP, given by (5)–(6) and (9)–(10) in the paper, converge to a globally optimal solution. This is formally stated in Theorem 1.

Theorem 1. *Let $f(\boldsymbol{v})$ be a strictly convex (or concave) function of $\boldsymbol{v} \in \mathbb{R}^K$ and $g(\boldsymbol{w}) = f(\boldsymbol{v})$, where $\boldsymbol{w} = [w_1, \ldots, w_k, \ldots, w_K]^T = [v_1^2, \ldots, v_k^2, \ldots, v_K^2]^T$. If the point $\boldsymbol{w}^*$ is found through gradient descent (or gradient ascent), $\frac{\partial g}{\partial \boldsymbol{w}}|_{\boldsymbol{w}=\boldsymbol{w}^*} = \mathbf{0}$, with an initial value $w_k^{(0)} \neq 0$, for $k = 1, \ldots, K$, then $\boldsymbol{w}^*$ is a global minimizer (or maximizer) of $g(\boldsymbol{w})$.*

The key idea of the proof is to show that our formulations in (4) and (8) in the paper are convex and concave, respectively. If this holds, then the gradient descent in (5)-(6), and the gradient ascent in (9)-(10) in the paper will each provide an optimal solution. To this end, we will use the following Theorems 2–4, and Lemma 1 from Chapter 3 of the book " Convex Optimization and Euclidean Distance Geometry" By Jon Dattorro, 2005.

Theorem 2. *Given functions $g : \mathbb{R}^K \rightarrow \mathbb{R}$ and $h : \mathbb{R}^N \rightarrow \mathbb{R}^K$, their composition $f = g(h) : \mathbb{R}^N \rightarrow \mathbb{R}$ is convex if*

- *g is convex non-decreasing monotonic, and h is convex; or*
- *g is convex non-increasing monotonic, and h is concave;*

and composite function f is concave if

- *g is concave non-decreasing monotonic, and h is concave;*
- *g is concave non-increasing monotonic, and h is convex;*

where ∞ $(-\infty)$ is assigned to convex (concave) g when evaluated outside its domain. Convexity (concavity) of any g is preserved when h is affine.

Lemma 1. *The following properties follow from Theorem 2:*

1. *f is convex $\Leftrightarrow -f$ is concave,*
2. *minimization of $f \Leftrightarrow$ maximization of $-f$,*
3. *f is positive and convex $\Leftrightarrow 1/f$ is positive and concave,*
4. *minimization of $f \Leftrightarrow$ maximization of $1/f$,*
5. *f is convex (concave) $\Rightarrow \exp(f)$ and f^2 are convex (concave)*
6. *f is positive and convex (concave) $\Rightarrow \log(f)$ and $\sqrt{f}$ are convex (concave)*

Theorem 3. *A norm on $\mathbb{R}^N$ is a convex function $f : \mathbb{R}^N \to \mathbb{R}$, satisfying for $x, y \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$ the following:*

1. $f(x) \geq 0$ $(f(x) = 0 \Leftrightarrow x = 0)$ *(non-negativity)*
2. $f(x+y) \leq f(x) + f(y)$ *(triangle inequality)*
3. $f(\alpha x) = |\alpha| f(x)$ *(non-negative homogeneity)*

Theorem 4. *Non-negatively weighted sum of (strictly) convex (concave) functions remains (strictly) convex (concave)*

We now prove that our formulations in (4) and (7) in the paper are convex and concave, respectively. Define an auxiliary variable, $\boldsymbol{z}$, as

$$\boldsymbol{z} = \begin{cases} \boldsymbol{\delta}_m - \boldsymbol{\delta}_h & \text{, if } \boldsymbol{\delta}_m - \boldsymbol{\delta}_h > 0 \\ 0 & \text{, otherwise .} \end{cases}$$

The original LP formulation, given by (2) in the paper, reads

$$\underset{\boldsymbol{w}}{\operatorname{argmax}}\ \boldsymbol{z}^{\mathrm{T}}\boldsymbol{w}, \quad \text{s.t.} \quad \boldsymbol{w} \geq 0,\ ||\boldsymbol{w}||_n \leq \gamma\ ,$$

where $\boldsymbol{w} \geq 0$ means that all elements of vector $\boldsymbol{w}$ are nonnegative. This is a concave optimization problem, where $\|\cdot\|_n$ is the ℓ_n norm, $n \in \{1, 2\}$. Replacing w_k with v_k^2, $k = 1, 2, \ldots$, as described in the paper, gives the following formulation:

$$\underset{\boldsymbol{v}}{\operatorname{argmax}}\ \sum v_k^2 z_k, \quad \text{s.t.} \quad \|\boldsymbol{v}\|_{2n}^2 \leq \gamma,\ n \in \{1, 2\}\ .$$

Using the diagonal matrix $\boldsymbol{Z} = \text{diag}(\boldsymbol{z})$, and $\beta = \sqrt{\gamma}$, the above formulation can be re-written as

$$\underset{\boldsymbol{v}}{\operatorname{argmax}}\ \boldsymbol{v}^{\mathrm{T}}\boldsymbol{Z}\boldsymbol{v}, \quad \text{s.t.} \quad \|\boldsymbol{v}\|_{2n} \leq \beta,\ n \in \{1, 2\}\ .$$

This is a standard concave QCQP problem, since $z_k \in \mathbb{R}^+$, $k = 1, 2, \ldots$. It follows that the substitution $w_k = v_k^2$, $k = 1, 2, \ldots$, does not change the concavity of the original LP formulation, given by (2) in the paper. Based on Lemma 1 and Theorem 4, it also follows that the logistic regression formulations

$$\underset{\boldsymbol{v}}{\operatorname{argmin}}\ \log[1 + \exp(-\boldsymbol{v}^{\mathrm{T}}\boldsymbol{Z}\boldsymbol{v})] + \lambda\|\boldsymbol{v}\|_{2n}^2, \quad n \in \{1, 2\}\ .$$

are convex.

Using the same auxiliary variable, $\boldsymbol{z}$, the alternative LP formulation, given by (7) in the paper, becomes:

$$\underset{\boldsymbol{w}}{\operatorname{argmax}}\ \boldsymbol{z}^{\mathrm{T}}\boldsymbol{w}, \quad \text{s.t.} \quad \boldsymbol{w} \geq 0,\ \|\boldsymbol{w}\|_n = \gamma$$

which can be reformulated as:

$$\underset{\boldsymbol{v}}{\operatorname{argmax}}\ \boldsymbol{v}^{\mathrm{T}}\boldsymbol{Z}\boldsymbol{v}, \quad \text{s.t.} \quad \|\boldsymbol{v}\|_{2n} = \beta$$

Let's now consider the function $h : \mathbb{R}^K \to \mathbb{R}^K$ defined by $h(\boldsymbol{v}) = \boldsymbol{v}/\|\boldsymbol{v}\|_{2n}$. From Theorem 3 and Lemma 1, h is a concave function. From Theorem 2 the following formulation

$$\underset{\boldsymbol{v}}{\operatorname{argmax}} \frac{\boldsymbol{v}^{\mathrm{T}} \boldsymbol{Z} \boldsymbol{v}}{\|\boldsymbol{v}\|_{2n}^2}$$

is concave. This completes the proof.