CS450/550 — Homework 2 — Solutions

1. Consider a 2-D window defined by

$$\begin{aligned} left &= 2 \\ right &= 4 \\ top &= 0 \\ bottom &= -5 \end{aligned}$$

Suppose this is to be mapped onto an X-window viewport having dimensions of 200 pixels by 200 pixels. (You may assume that the `viewport_t` normalized device coordinates are $(left = 0, right = 1, bottom = 0, top = 1)$).

(a) Give the device coordinates for each of the following points (given in world coordinates):

(2,0)	(0,0)
(2,-5)	(0,200)
(3,-2.5)	(100,100)
(3.5,-2.5)	(150,100)

(b) Give the world coordinates for each of the following pixels (given in X-window device coordinates):

(50,50)	(2.5,-1.25)
(200,200)	(4,-5)
(150,100)	(3.5,-2.5)
(160,160)	(3.6,-4.0)

2. Problem 16 p. 105. Assume that the sum s in Bresenham's circle generator is exactly 0. Show that, in this case, pixel A is closer to the ideal circular arc than pixel B, with the distance measured along the radials from the center of the circle to those pixels. Hint: Let a be the distance that pixel A is outside the circle and b be the distance that pixel B is inside the circle. Compute $f(A)$ and $f(B)$ in terms of the circle radius r and a and b. Set $f(A) + f(B) = 0$ and simplify. Then show that a must be less than b.

For point A, $x^2 + y^2 = (r + a)^2 = r^2 + 2ra + a^2$. Hence, $f(A) = 2ra + a^2$.
For point B, $x^2 + (y - 1)^2 = (r - b)^2 = r^2 - 2rb + b^2$. Hence, $f(B) = -2rb + b^2$.

$$f(A) + f(B) = 2ra - 2rb + a^2 + b^2 = 0$$

or

$$2r(a-b)+(a^2+b^2)=0$$

Now the second term is positive, and r is positive, so the only way to get equality with zero is if $a-b<0$, or in other words $a<b$.

For Students Enrolled in CS550:

3. Problem 17 p. 105. Assume that the sum s in Bresenham's circle generator is exactly 0. This does not imply that the ideal circular arc intersects the line between the two pixels at the midpoint. Show that the arc intersects at a point that is always a little closer to pixel A. Hint: $s=0$. That is

$$x^2+y^2-r^2+x^2+(y-1)^2-r^2=0$$

which can be compressed to

$$x^2+y^2-y+\frac{1}{2}-r^2=0 \quad (a)$$

Also, let $d \geq 0$ be the distance above $y-1$ where the circle intersects the vertical line at x. There,

$$x^2+(y-1+d)^2-r^2=0.$$

This can be expanded to

$$x^2+y^2+d^2+1+2dy-2d-2y-r^2=0 \quad (b)$$

Subtract (b) from (a) to get an equation in d and y alone. Isolate d from that relation and show that the resulting expression in y is greater than 1/2 for every positive y value.

Subtraction yields

$$-y+\frac{1}{2}-d^2-1-2dy+2d+2y=0$$

Negate both sides:

$$y-\frac{1}{2}+d^2+1+2dy-2d-2y=0$$

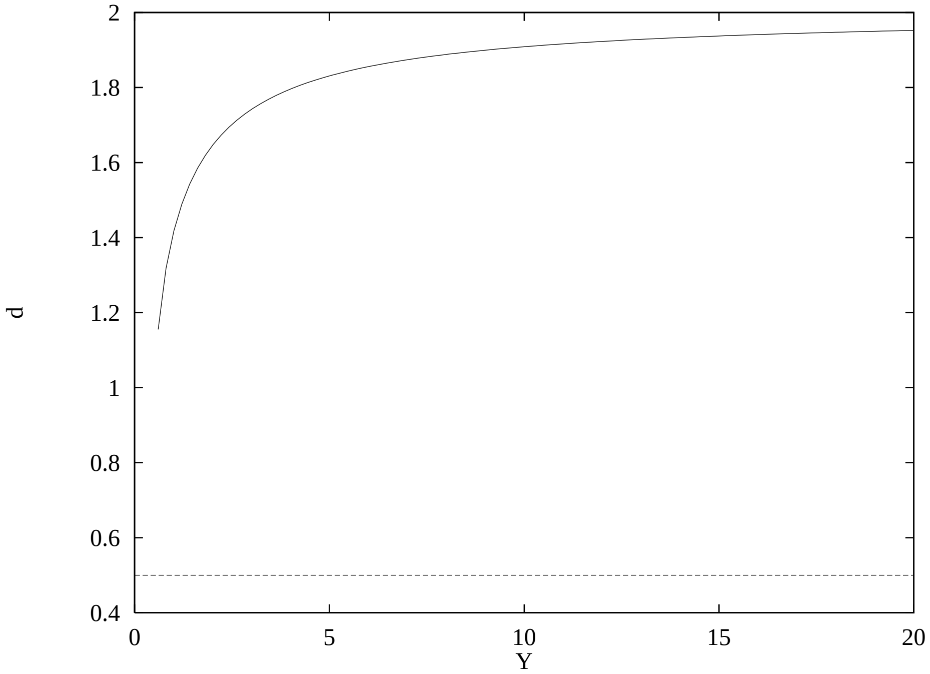
Gather terms for a quadratic in d:

$$d^2+d(2y-2)+(-y+\frac{1}{2})=0$$

Plug into the quadratic formula, and choose the $+$ sign to make d positive.

$$d=(1-y)+\sqrt{y^2+2y-1}$$

Here is a plot of this curve for positive values of y:



4. Prove that the line drawn by the Bresenham line drawing algorithm is symmetric about the middle of the line. In other words, if the starting pixel is at location $(0,0)$ and the ending location is at (x_e, y_e) and Bresenham sets pixel (p_x, p_y), then it also sets pixel $(x_e - p_x, y_e - p_y)$. You may assume that the line begins exactly at the origin and ends exactly a point (x_e, y_e) in device coordinates. How can this fact be used to reduce by half the number of decisions that the algorithm must make?

Let (p_x, y') be the point where the true line intersects the vertical column of pixels at $x = p_x$.

Claim 1: The true line intersects the vertical column of pixels $x = x_e - p_x$ at the point $y = y_e - y'$.

Proof of claim: To get from $(0,0)$ to (p_x, y'), we moved an amount p_x in the x direction and y' in the y direction (this is the slope of the line). Let the parametric equation of the line be

$$(x, y) = (0,0) + s(p_x, y')$$

We reach (p_x, y') when the parameter $s = 1$. There exists a value for the parameter $s = s_e$ such that $(x, y) = (x_e, y_e)$. If we therefore use the parameter value $s = s_e - 1$ in the parametric equation, we get

$$\begin{aligned}
(x, y) &= (0,0) + (s_e - 1)(p_x, y') \\
&= (0,0) + s_e(p_x, y') - (p_x, y') \\
&= (0,0) + (x_e, y_e) - (p_x, y') \\
&= (x_e - p_x, y_e - y')
\end{aligned}$$

QED.

Proof of main result. Bresenham chooses pixel (p_x, p_y), because it is closer than any other pixel. That is, because $|p_y - y'| < 0.5$. By the same argument, Bresenham will choose pixel $(x_e - p_x, y_e - p_y)$ if $|(y_e - p_y) - (y_e - y')| < 0.5$. This is true, because

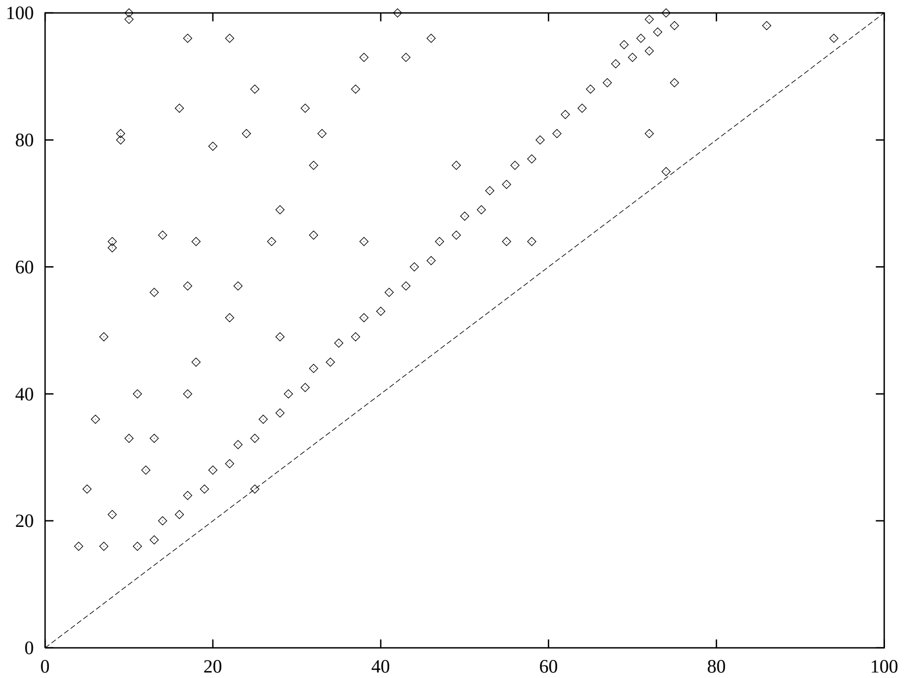
$$\begin{aligned}
|(y_e - p_y) - (y_e - y')| &= |y_e - p_y - y_e + y'| \\
&= |y' - p_y| \\
&= |p_y - y'| < 0.5
\end{aligned}$$

QED.

5. Consider the Bresenham circle algorithm. In the case where $f(A) + f(B) > 0$, the algorithm always chooses point B. Show an example where this makes an error. When will this error be most obvious? For small circles? Large ones? Never?

An error can occur when $f(A) > 0$ and $f(B) < 0$. If we refer to the solution to Problem 2, the case we are concerned about is basically the same. We have already seen that it is possible for $f(A) + f(B) = 0$ and yet $a < b$. If we change $f(A)$ by some small additional amount $-\epsilon$, then $f(A) + f(B) < 0$ and yet a will still be less than b.

To visualize when these cases occur, it is easier to compute cases where $a = b$. Here is a plot of these for $0 < x, y < 100$.



We can see that they occur fairly often and with a density that is quite uniform regardless of the radius of the circle.