Chapter 7: The $z$-Transform

- Continuous-time signals
  \[ x(t) = \mathcal{F} \leftrightarrow X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt \]

- Discrete-time signals
  \[ x[n] = \mathcal{DFT} \leftrightarrow X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \]

- FT does not exist for signals that are not absolutely integrable.

- DTFT does not exist for signals that are not absolutely summable.

- More general form: a transform as a function of an arbitrary point in the 2-dimensional plane: Laplace transform.

- More general form: a transform as a function of an arbitrary circle in the 2-dimensional plane: $z$-transform.
The \( z \)-Transform - definition

- **Continuous-time systems:** \( e^{st} \rightarrow H(s) \Rightarrow y(t) = e^{st}H(s) \)
  \( e^{st} \) is an eigenfunction of the LTI system \( h(t) \), and \( H(s) \) is the corresponding eigenvalue.

- **Discrete-time systems:** \( x[n] = z^n \rightarrow h[n] \Rightarrow y[n] \)

\[
y[n] = x[n] \ast h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]
\]

\[
= \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \left( \sum_{k=-\infty}^{\infty} h[k]z^{-k} \right) = z^n H(z)
\]

\( z^n \) is an eigenfunction of the LTI system \( h[n] \), and \( H(z) \) is the corresponding eigenvalue.
The $z$-Transform - definition (cont.)

The *transfer function*:

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}.$$  

Generally, let $z = re^{j\Omega}$. Then,

$$H(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} \left(h[n] z^{-n}\right) e^{-j\Omega n}.$$  

Thus, $H(z)$ is the DTFT of $h[n] r^{-n}$. The inverse DTFT of $H(re^{j\Omega})$ must be $h[n] r^{-n}$. 
The $z$-Transform - definition (cont.)

So we may write

$$h[n]r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(re^{j\Omega})e^{j\Omega n} d\Omega.$$

- $z = re^{j\Omega} \rightarrow dz = jre^{j\Omega} d\Omega$. $d\Omega = \frac{1}{j} z^{-1} dz$.

- As $\Omega$ goes from $-\pi$ to $\pi$, $z$ traverses a circle of radius $r$ in a counterclockwise direction. Thus, we may write

$$h[n] = \frac{1}{2\pi j} \oint H(z)z^{n-1} dz$$
The $z$-Transform - definition (cont.)

For an arbitrary signal $x[n]$, the $z$-transform and inverse $z$-transform are expressed as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z)z^{n-1}dz$$

We express this relationship between $x[n]$ and $X(z)$ as

$$x[n] \leftrightarrow X(z)$$
The $z$-Transform - convergence

- A necessary condition for convergence: $$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$
  (absolute summability)

- The range of $r$ for which this condition is satisfied is termed the **region of convergence (ROC)**.

- Complex number $z$ is represented as a location in a complex plane, termed the **$z$-plane**.

- If $x[n]$ is **absolutely summable**, then the DTFT of $x[n]$ is obtained as
  $$X(e^{j\Omega}) = X(z)|_{z=e^{j\Omega}}$$

- The contour $z = e^{j\Omega}$ is termed the **unit circle**.
The $z$-Transform - poles and zeros

The most commonly encountered form of the $z$-transform is a ratio of two polynomials in $z^{-1}$, as shown by the rational function

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \frac{\tilde{b} \prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

- $\tilde{b} = b_0 / a_0$.

- $c_k$: zeros of $X(z)$. Denoted with the “○” symbol in the $z$ plane.

- $d_k$: poles of $X(z)$. Denoted with the “×” symbol in the $z$ plane.
The $z$-Transform - Review of commonly used series

- Geometric series: Let $s_n = a + ar + ar^2 + \cdots + ar^n$, then

$$s_n = \frac{a(1 - r^{n+1})}{1 - r}$$

$$\lim_{n \to \infty} s_n = \frac{a}{1 - r}, \text{ if } |r| < 1$$

Proof:

$$s_n = a + ar + ar^2 + \cdots + ar^n$$

$$rs_n = ar + ar^2 + \cdots + ar^n + ar^{n+1}$$

$$s_n - rs_n = a(1 - r^{n+1})$$

$$s_n = \frac{a(1 - r^{n+1})}{1 - r}$$
The \( z \)-Transform - Review of commonly used series (cont.)

- Arithmetic progression: Let 
  \[ s_n = a + (a + r) + (a + 2r) + \cdots + (a + nr), \]
  then

  \[
  s_n = \frac{(n + 1)(a + (a + nr))}{2}
  \]

  \[
  \lim_{n \to \infty} s_n = \begin{cases} 
  \infty, & \text{if } r > 0 \\
  -\infty, & \text{if } r < 0
  \end{cases}
  \]

Proof:

\[
\begin{align*}
  s_n &= a + (a + r) + (a + 2r) + \cdots + (a + nr) \\
  s_n &= (a + nr) + (a + (n - 1)r) + \cdots + (a + r) + a \\
  2s_n &= (n + 1)(a + (a + nr)) \rightarrow s_n = (n + 1)(a + (a + nr))/2
\end{align*}
\]
The $\zeta$-Transform - Review of commonly used series (cont.)

- $\sum_{n=1}^{\infty} \frac{1}{2n} = 1$

- $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

- $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

- $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{24}$
The $z$-Transform - Convergence of commonly used series

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 0$:
  - Convergent, if $p > 1$
  - Divergent, if $p \leq 1$.

Example:

- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$: divergent
- $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$: convergent
The $z$-Transform - Convergence of commonly used series (cont.)

- **Ratio test:** Suppose \( \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = r. \)

  - \( r > 1 \): divergent
  - \( r < 1 \): convergent
  - \( r = 1 \): test gives no information

- **Comparison test:** Assume \( 0 \leq a_n \leq b_n, \ \forall n. \)

  - If \( \sum b_n \) is convergent \( \Rightarrow \) \( \sum a_n \) is convergent (For convenience, we use \( \sum b_n \) to represent an infinite series in the notes)

**Example:** Let \( a_n = \frac{2n}{3n^3 - 1}, \ b_n = \frac{1}{n^2}. \)
\[ \sum b_n \text{ is convergent. Thus, } \sum a_n \text{ is convergent because} \]
\[ n \geq 1 \rightarrow n^3 \geq 1 \rightarrow 3n^3 - 1 \geq 2n^3 \Rightarrow a_n \leq b_n. \]
The $z$-Transform - Convergence of commonly used series (cont.)

- Corollary of comparison test (limiting form): Suppose that $a_n > 0$, $b_n > 0$ and then $\lim_{n \to \infty} \frac{a_n}{b_n} = k > 0$.

$$\sum a_n \text{ convergent} \iff \sum b_n \text{ convergent}$$

Example: Let $a_n = \frac{n}{n^2 + 1}$, $b_n = \frac{1}{n}$.

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2}{n^2 + 1} = 1.$$  
Because $\sum b_n$ is divergent $\Rightarrow$ $\sum a_n$ is divergent.
The $z$-Transform - Convergence of commonly used series (cont.)

- Necessary condition for convergence of $\sum a_n$:

  $$\lim_{n \to \infty} a_n = 0$$

- It is not a sufficient condition. For example, $a_n = \frac{1}{n}$, $\sum a_n$ is divergent.
The $z$-Transform - Examples

Determine the $z$-transform of the following signals and depict the ROC and the locations of the poles and zeros of $X(z)$ in the $z$-plane:

- $x[n] = \alpha^n u[n]$ (causal signal)
- $x[n] = -\alpha^n u[-n - 1]$ (anticausal signal)

For signal $x[n] = \alpha^n u[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n.$$
The $z$-Transform - Examples (cont.)

This infinite series converges to

$$X(z) = \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}, \quad \text{for } |z| > |\alpha|.$$

For signal $x[n] = -\alpha^n u[-n-1]$:

$$X(z) = \sum_{n=-\infty}^{\infty} (-\alpha^n u[-n-1] z^{-n})$$

$$= - \sum_{n=\infty}^{1} \left(\frac{\alpha}{z}\right)^n$$

$$= - \sum_{k=\infty}^{-1} \left(\frac{\alpha}{z}\right)^k = 1 - \sum_{k=0}^{\infty} \left(\frac{\alpha}{z}\right)^k$$

$$= 1 - \frac{1}{1 - z\alpha^{-1}} = \frac{z}{z - \alpha}, \quad \text{for } |z| < |\alpha|.$$
Observations:

• As bilateral Laplace transform, the relationship between $x[n]$ and $X(z)$ is not unique.

• The ROC differentiates the two transforms.

• We must know the ROC to determine the correct inverse $z$-transform.
The $z$-Transform - Examples (cont.)

- Read Example 7.4, $p_{560}$.

- Problem 7.1(c), $p_{561}$: Determine the $z$-transform, the ROC, and the locations of poles and zeros of $X(z)$ for the following signal

$$x[n] = -\left(\frac{3}{4}\right)^n u[-n - 1] + \left(-\frac{1}{3}\right)^n u[n]$$

Using the results given in the previous two slides:

$$-\left(\frac{3}{4}\right)^n u[-n - 1] \quad \leftrightarrow \quad \frac{z}{z - 3/4}$$

$$\left(-\frac{1}{3}\right)^n u[n] \quad \leftrightarrow \quad \frac{z}{z + 1/3}.$$ 

Thus, $X(z) = \frac{z}{z - 3/4} + \frac{z}{z + 1/3} = \frac{z(2z - 5/12)}{(z - 3/4)(z + 1/3)}$
Properties of the ROC

- As the Laplace transform, the ROC cannot contain any poles.

- ROC for a finite-duration signal includes the entire $z$-plane, except possibly $z = 0$ or $z = \infty$ or both.

- **Left-sided sequence:** $x[n] = 0$ for $n \geq 0$ (notice the difference between the left-sided signal for Laplace transform).

- **Right-sided sequence:** $x[n] = 0$ for $n < 0$.

- **Two-sided sequence:** a signal that has infinite duration in both the positive and negative directions.
Properties of the ROC

- RSS: ROC is of the form $|z| > r_+$
- LSS: ROC is of the form $|z| < r_-$
- TSS: ROC if of the form $r_+ < |z| < r_-$

where the boundaries $r_+$ and $r_-$ are determined by the pole locations. See the figure next page.
Properties of the ROC (cont.)
Properties of the ROC - Examples

Example 7.5, identify the ROC associated with the $z$-transform for each of the following signals.

- $x[n] = (-1/2)^n u[-n] + 2(1/4)^n u[n]$
- $y[n] = (-1/2)^n u[n] + 2(1/4)^n u[n]$
- $w[n] = (-1/2)^n u[-n] + 2(1/4)^n u[-n]$
Properties of the ROC - Examples (cont.)

For $x[n]$, the $z$-transform is written as

$$X(z) = \sum_{n=-\infty}^{0} \left(-\frac{1}{2z}\right)^n + 2 \sum_{n=0}^{\infty} \left(\frac{1}{4z}\right)^n$$

$$= \sum_{k=0}^{\infty} (-2z)^k + 2 \sum_{n=0}^{\infty} \left(\frac{1}{4z}\right)^n$$

• The first sum converges for $|z| < \frac{1}{2}$.

• The second sum converges for $|z| > \frac{1}{4}$.

• Thus, the ROC is $\frac{1}{4} < z < \frac{1}{2}$. Summing the two geometric series:

$$X(z) = \frac{1}{1+2z} + \frac{2z}{z-1/4}.$$
Properties of the ROC - Examples (cont.)

Observations:

- The first term on the right side of $z[n]$ is a left-sided sequence. Its ROC is $|z| < r_-$, where $r_-$ is determined by its pole location.

- The second term on the right side of $z[n]$ is a right-sided sequence. Its ROC is $|z| > r_+$, where $r_+$ is determined by its pole location.
Properties of the ROC - Examples (cont.)

For $y[n]$, both terms are right-sided sequences. Thus, the ROC is $|z| > r_+$, where $r_+$ is determined by the pole locations.

$$Y(z) = \sum_{n=0}^{\infty} \left( \frac{-1}{2z} \right)^n + 2 \sum_{n=0}^{\infty} \left( \frac{1}{4z} \right)^n$$

The first series converges for $|z| > 1/2$ and the second series converges for $|z| > 1/4$. Thus, the ROC is $|z| > 1/2$, and we write $Y(z)$ as

$$Y(z) = \frac{z}{z + 1/2} + \frac{2z}{z - 1/4}.$$
Properties of the ROC - Examples (cont.)

For $w[n]$, both terms are left-sided sequences. Thus, the ROC is $|z| < r_-$, where $r_-$ is determined by the pole locations.

$$W(z) = \sum_{n=-\infty}^{0} \left(\frac{-1}{2z}\right)^n + 2 \sum_{n=-\infty}^{0} \left(\frac{1}{4z}\right)^n$$

$$= \sum_{k=0}^{\infty} (-2z)^k + 2 \sum_{k=0}^{\infty} (4z)^k$$
Properties of the ROC - Examples (cont.)

- The first series converges for $|z| < 1/2$.
- The second series converges for $|z| < 1/4$.
- Thus, the ROC is $|z| < 1/4$, and we write $W(z)$ as

$$W(z) = \frac{1}{1 + 2z} + \frac{2}{1 - 4z}.$$  

The pole locations of sequences $z[n], y[n], w[n]$ are shown in the figure next slide.
Properties of the ROC (cont.)
Properties of the $z$-transform

- **Linearity**
  - Let $x[n] \leftrightarrow X(z)$ (ROC $R_x$) and $y[n] \leftrightarrow Y(z)$. □
  - $ax[n] + by[n] \leftrightarrow aX(z) + bY(z)$, with ROC at least $R_x \cap R_y$
    - The ROC can be larger than the intersection if one or more terms in $x[n]$ or $y[n]$ cancel each other in the sum.
    - In the $z$-plane, this corresponds to a zero canceling a pole that defines one of the ROC boundaries.
Properties of the $z$-transform (cont.)

Example: Example 7.5, $p_{567}$. Suppose

\[ x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{3}{2}\right)^n u[-n-1] \zrightarrow X(z) = \frac{-z}{(z - 1/2)(z - 3/2)} \]

with ROC $1/2 < |z| < 3/2$, and

\[ y[n] = \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[n] \zrightarrow Y(z) = \frac{-\frac{1}{4}z}{(z - 1/4)(z - 1/2)} \]

with ROC $|z| > 1/2$. Evaluate the $z$-transform of $ax[n] + by[n]$, where $a$ and $b$ are constants.
Properties of the $z$-transform (cont.)

Using the linearity property, we have

$$ax[n] + by[n] \leftrightarrow z \rightarrow a \frac{-z}{(z - 1/2)(z - 3/2)} + b \frac{-\frac{1}{4}z}{(z - 1/4)(z - 1/2)}$$

Must be careful in determining ROC. In general, the ROC is the intersection of individual ROCs. For some special cases, however, the ROC could be larger. For instance, let $a = b = 1$. 
Properties of the $z$-transform (cont.)

Then,

$$aX(z) + bY(z) = \frac{-z}{(z - 1/2)(z - 3/2)} + \frac{-1/4z}{(z - 1/4)(z - 1/2)}$$

$$= \frac{-5/4z(z - 1/2)}{(z - 1/4)(z - 1/2)(z - 3/2)}$$

$$= \frac{-5/4z}{(z - 1/4)(z - 3/2)}$$

The ROC can be verified to be $1/4 < |z| < 3/2$ because the pole-zero cancellation ($z = 1/2$), and the $(1/2)^nu[n]$ no longer presents.
Properties of the $z$-transform (cont.)

- **Time reversal**
  - $x[-n] \overset{z}{\leftrightarrow} X\left(\frac{1}{z}\right)$ with ROC $\frac{1}{R_x}$.
  - If $R_x$ is of the form $a < |z| < b$, the ROC of the reflected signal is $1/b < |z| < 1/a$.

- **Time shift**
  - $x[n - n_0] \overset{z}{\leftrightarrow} z^{-n_0}X(z)$ with ROC $R_x$, except possibly $z = 0$ and $z = \infty$.
  - If $n_0 > 0$, $z^{-n_0}$ introduces a pole $z = 0$.
  - If $n_0 < 0$, $z^{-n_0}$ introduces a pole $z = \pm\infty$. 
Properties of the $z$-transform (cont.)

- Multiplication by an exponential sequence
  - $α^n x[n] \leftrightarrow X\left(\frac{z}{α}\right)$ with ROC $|α|R_x$.
  - $|α|R_x$ implies that the ROC boundaries are multiplied by $|α|$.
  - If $|α| = 1$, then the ROC is unchanged.

- Convolution
  - $x[n] * y[n] \leftrightarrow X(z)Y(z)$ with ROC at least $R_x \cap R_y$.
  - The ROC may be larger than the intersection of $R_x$ and $R_y$ if a pole-zero cancellation occurs in the product of $X(z)Y(z)$. 
Properties of the $z$-transform (cont.)

- Differentiation in the $z$-domain
  \[ nx[n] \leftrightarrow -z \frac{d}{dz} X(z), \text{ with ROC } R_x. \]

- Read Example 7.8, p570.

Example: Example 7.7, p570. Find the $z$-transform of

\[ x[n] = \left( n \left( \frac{-1}{2} \right)^n u[n] \right) \ast \left( \frac{1}{4} \right)^{-n} u[-n]. \]
Properties of the $z$-transform: example

- Basic signal of first term: $\left( \frac{-1}{2} \right)^n u[n] \leftrightarrow \frac{z}{z + 1/2}$, with ROC $|z| > 1/2$.

- Applying the $z$-domain differentiation property:

$$n \left( \frac{-1}{2} \right)^n u[n] \leftrightarrow -z \frac{d}{dz} \frac{z}{z + 1/2} = \frac{-1/2z}{(z + 1/2)^2}, \text{ with ROC } |z| > 1/2$$
Properties of the \( z \)-transform: example (cont.)

- Applying time reversal property: \( (\frac{1}{4})^n u[n] \leftrightarrow z \rightarrow \frac{z}{z-1/4} \), with ROC \(|z| > 1/4\). Thus, \( (\frac{1}{4})^{-n} u[-n] \leftrightarrow z \rightarrow \frac{1/z}{1/z-1/4} = \frac{-4}{z-4} \) with ROC \(|z| < 4\).

- Applying convolution property: \( x[n] \leftrightarrow z \rightarrow \frac{2z}{(z-4)(z+1/2)^2} \), with ROC \( \frac{1}{2} < |z| < 4\).