Lecture 5: Introduction to Entropy Coding

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Codes

- Definitions:
  - Alphabet: is a collection of symbols.
  - Letters (symbols): is an element of an alphabet.
  - Coding: the assignment of binary sequences to elements of an alphabet.
  - Code: A set of binary sequences.
  - Codewords: Individual members of the set of binary sequences.

Examples of Binary Codes

- English alphabets:
  - 26 uppercase and 26 lowercase letters and punctuation marks.

  - ASCII code for the letter “a” is 1000011
  - ASCII code for the letter “A” is 1000001
  - ASCII code for the letter “,” is 0011010

Note: all the letters (symbols) in this case use the same number of bits (7). These are called fixed length codes.

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The average number of bits per symbol (letter) is called the rate of the code.

Code Rate

- Average length of the code is important in compression.

- Suppose our source alphabet consists of four letters a_1, a_2, a_3, and a_4 with probabilities $P(a_1) = 0.5$ $P(a_2) = 0.25$, and $P(a_3) = P(a_4) = 0.125$.

- The average length of the code is given by

$$l = \sum_{i=1}^{4} P(a_i) n(a_i)$$

- $n(a_i)$ is the number of bits in the codeword for letter a_i

Uniquely Decodable Codes

Letters	Probabilitity	Code 1	Code 2	Code 3	Code 4
a_1	0.5	0	0	0	0
a_2	0.25	0	1	10	01
a_3	0.125	1	00	110	011
a_4	0.125	10	11	111	0111
Average Length		1.125	1.25	1.75	1.875

Code 1: not unique a_1 and a_2 have the same codeword

Code 2: not uniquely decodable: 100 could mean a_2a_3 or $a_2a_1a_1$

Codes 3 and 4: uniquely decodable: What are the rules?

Code 3 is called instantaneous code since the decoder knows the codeword the moment a code is complete.

How do we know a uniquely decodable code?

- Consider two codewords: 011 and 011101
  - Prefix: 011
  - Dangling suffix: 101
- Algorithm:
  1. Construct a list of all the codewords.
  2. Examine all pairs of codewords to see if any codeword is a prefix of another codeword. If there exists such a pair, add the dangling suffice to the list unless there is one already.
  3. Continue this procedure using the larger list until:
     1. Either a dangling suffix is a codeword -> not uniquely decodable.
     2. There are no more unique dangling suffixes -> uniquely decodable.

Examples of Unique Decodability

- Consider {0,01,11}
  - Dangling suffix is 1 from 0 and 01
  - New list: {0,01,11,1}
  - Dangling suffix is 1 (from 0 and 01, and also 1 and 11), and is already included in previous iteration.
  - Since the dangling suffix is not a codeword, {0,01, 11} is uniquely decodable.

Examples of Unique Decodability

- Consider {0,01,10}
  - Dangling suffix is 1 from 0 and 01
  - New list: {0,01,10,1}
  - The new dangling suffix is 0 (from 10 and 1).
  - Since the dangling suffix 0 is a codeword, {0,01, 10} is not uniquely decodable.

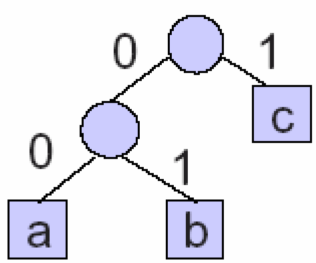
Prefix Codes

- Prefix codes: A code in which no codeword is a prefix to another codeword.

- A prefix code can be defined by a binary tree

  Example:

tree



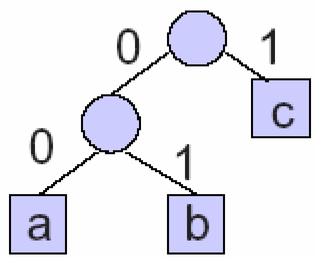
input	output
a	00
b	01
c	1

code

c c a b c c b c c c

1 1 00 01 1 1 01 1 1 1

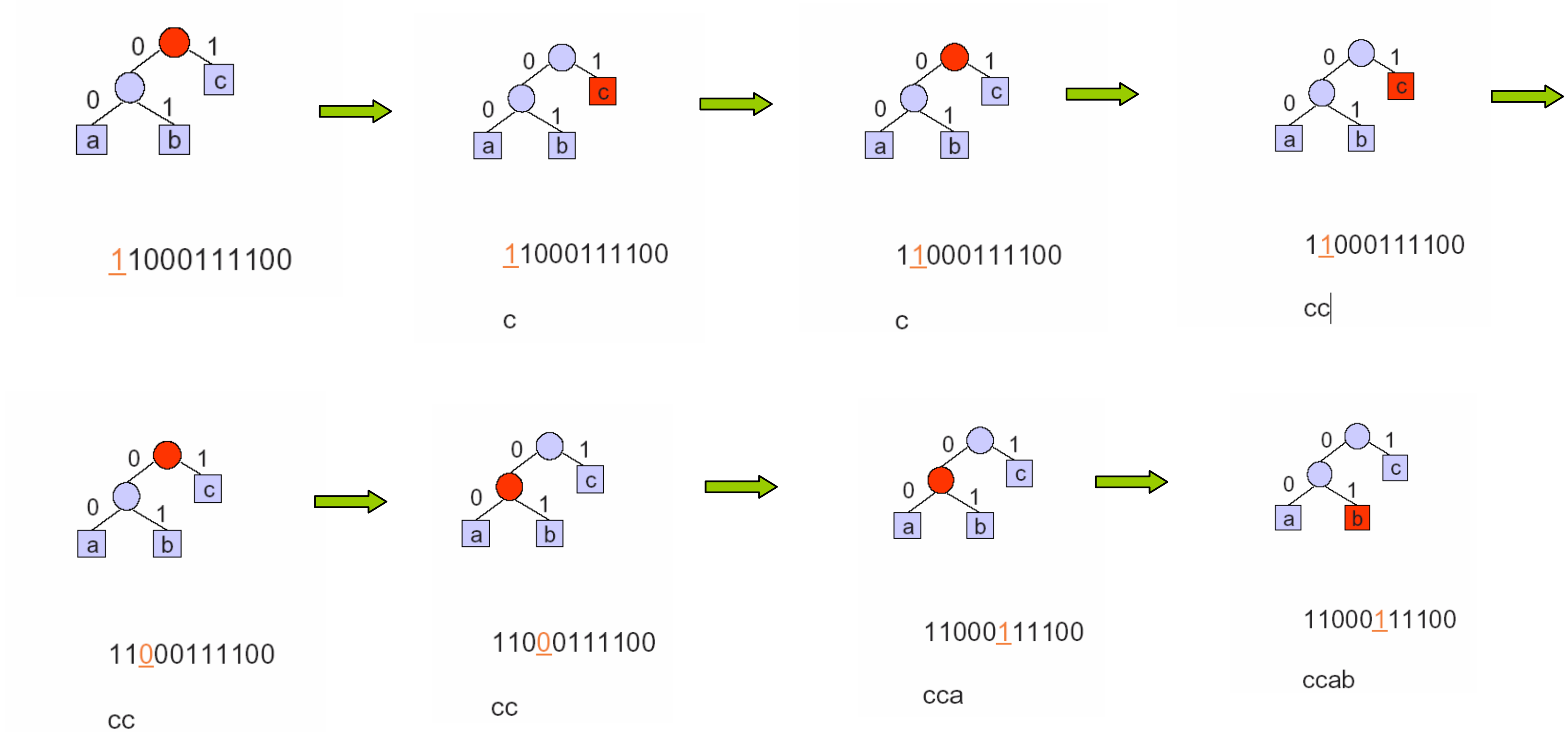
Decoding a Prefix Codeword




```
repeat
start at root of tree
   repeat
      if read bit = 1 then go right
      else go left
   until node is a leaf
   report leaf
until end of the code
```

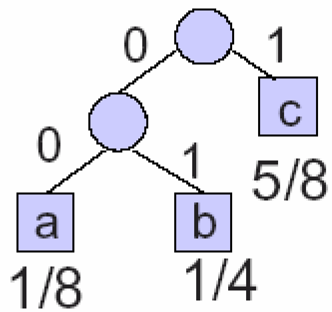
11000111100

Decoding a Prefix Codeword



How good is the code?

Suppose a, b, and c occur with probabilities 1/8, 1/4, and 5/8, respectively.



bit rate = (1/8)2 + (1/4)2 + (5/8)1 = 11/8 = 1.375 bps
Entropy = 1.3 bps
Standard code = 2 bps

(bps = bits per symbol)

Are we losing any efficiency by using prefix code?

- The answer is NO!

- Theorem 1: Let C be a code with N code words with lengths l_1, l_2, ... l_N . If C is uniquely decodable, then

$$K(C) = \sum_{i=1}^{N} 2^{-l_i} \leq 1$$

- Theorem 2: Given a set of integers l_1, l_2, ... l_N that satisfy the inequality

$$\sum_{i=1}^{N} 2^{-l_i} \leq 1$$

we can always find a prefix code with codeword lengths l_1, l_2, ... l_N .

Proof of Theorem 1

$$K(C)=\sum_{i=1}^{N}2^{-l_i}\le 1$$

$$\left[\sum_{i=1}^{N}2^{-l_i}\right]^n=\left(\sum_{i=1}^{N}2^{-l_{i1}}\right)\left(\sum_{i=1}^{N}2^{-l_{i2}}\right)\cdots\left(\sum_{i=1}^{N}2^{-l_{i3}}\right)=\sum_{i1=1}^{N}\sum_{i2=1}^{N}\cdots\sum_{in=1}^{N}2^{-(l_{i1}+l_{i2}+\ldots+l_{in})}$$

The exponent $k=(l_{i1}+l_{i2}+\ldots+l_{in})$ is simply the length of n codewords

Smallest value of k is n and largest value is

So,

$$[K(C)]^n=\sum_{k=n}^{nl}A_k 2^{-k}$$

A_k is the number of combinations of n codewords that have a combined length of k

$A_k\le 2^k$ Since for a uniquely decodable code, each sequence can represent one and only one sequence of codewords. This implies

Growth linearly!!!!

$$[K(C)]^n=\sum_{k=n}^{nl}A_k 2^{-k}\le\sum_{k=n}^{nl}2^k 2^{-k}=nl-n+1$$

Thus, $K(C)\le 1$

Proof of Theorem 2: If $\sum_{i=1}^{N} 2^{-l_i} \le 1$ we can always find a prefix codes with the length $l_1, l_2 \ldots l_N$

Assume: $l_1 \le l_2 \le \ldots \le l_N$

Define: $w_1 = 0, w_j = \sum_{i=1}^{j-1} 2^{l_j - l_i} \qquad j > 1$

Fact 1: binary representation of w_j would take up $ceil[\log_2(w_j + 1)]$

Fact 2: The number of bits in the binary representation of w_j is less than l_j

$$\log_2(w_j + 1) = \log_2\left(\sum_{i=1}^{j-1} 2^{l_j - l_i} + 1\right) = \log_2\left(2^{l_j}\left[\sum_{i=1}^{j-1} 2^{-l_i} + 2^{-l_j}\right]\right)$$

$$= l_j + \log_2\left(\sum_{i=1}^{j} 2^{-l_i}\right) \le l_j$$

Proof of Theorem 2: If $\sum_{i=1}^{N} 2^{-l_i} \le 1$ we can always find a prefix codes with the length $l_1, l_2 \ldots l_N$

Now using the binary representation of w_j , we define the codeword as:

If $ceil(\log_2(w_j+1)) = l_j$, then the jth codeword c_j is the binary representation of w_j

If $ceil(\log_2(w_j+1)) \le l_j$, then the jth codeword c_j is the binary representation of w_j *with* $l_j - ceil(\log_2(w_j+1))$ zeros

This is clearly a decodable code (w_j are all different since $\sum_{i=1}^{j-1} 2^{l_j - l_i}$ is an increased function, each w_j also has length l_j)

Proof of Theorem 2: If $\sum_{i=1}^{N} 2^{-l_i} \le 1$ we can always find a prefix codes with the length $l_1, l_2 \ldots l_N$

Suppose the claim is not true, then for some $j < k$, c_j is the prefix of c_k

This means l_j most significant bits fo w_k form the binary represention of w_j

$$w_j = \left\lfloor \frac{w_k}{2^{l_k - l_j}} \right\rfloor$$, However $$w_k = \sum_{i=1}^{k-1} 2^{l_k - l_j}$$

Therefore,

$$\frac{w_k}{2^{l_k - l_j}} = \sum_{i=1}^{k-1} 2^{l_j - l_i} = w_j + \sum_{i=j}^{k-1} 2^{l_j - l_i} = w_j + 1 + \sum_{i=j+1}^{k-1} 2^{l_j - l_i} \ge w_j + 1$$

That is the smallest value for $\frac{w_k}{2^{l_k - l_j}}$ is $w_j + 1$

Hence, contradicts!