Lecture 3: Information Theory Continues

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Review:
Shannon's Information Theory

The

Claude Shannon: ~~A~~ Mathematical Theory of Communication

Bell System Technical Journal, 1948

- Shannon's measure of information is the number of bits to represent the amount of uncertainty (randomness) in a data source, and is defined as entropy

$$H = -\sum_{i=1}^{n} p_i \log(p_i)$$

Where there are n symbols 1, 2, … n, each with probability of occurrence of p_i

Entropy: Three properties

1. It can be shown that $0 \cdot H \cdot log\ N$.

2. Maximum entropy ($H = log\ N$) is reached when all symbols are equiprobable, i.e., $p_i = 1/N$.

3. The difference $log\ N - H$ is called the *redundancy* of the source.

Joint Information

- X and Y are random variables.
- X and Y can have n and m possibilities, respectively. Then, the joint information is defined as:

$$H(X,Y) = -\sum_{i=1}^{n}\sum_{j=1}^{m} r(x_i, y_j)\log(r(x_i, y_j))$$

- r(x,y) is the joint probability of x and y.
- Why this definition?

Conditional Information

- X and Y are random variables.
- X and Y can have n and m possibilities, respectively. Then, the conditional information is defined as:

$$H(Y \mid X) = -\sum_{i=1}^{n}\sum_{j=1}^{m} r(x_i, y_j)\log(q(y_j \mid x_i))$$

- q(y|x) is the conditional probability.
- Why this definition?

Conditional Information

Properties of Conditional Information:

1. $H(Y \mid X) \geq 0$
2. $H(Y \mid X) \leq H(Y)$ with equality if X and Y are independent.
3. $H(X,Y) = H(X) + H(Y \mid X) = H(Y) + H(X \mid Y)$

Mutual Information

- X and Y are random variables.
- X and Y can have n and m possibilities, respectively. Then, the mutual information is defined as:

$$I(X,Y) = H(Y) - H(Y \mid X) = \sum_{i=1}^{n} \sum_{j=1}^{m} r(x_i, y_j) \log[\frac{r(x_i, y_j)}{q(y_j)p(x_i)}]$$

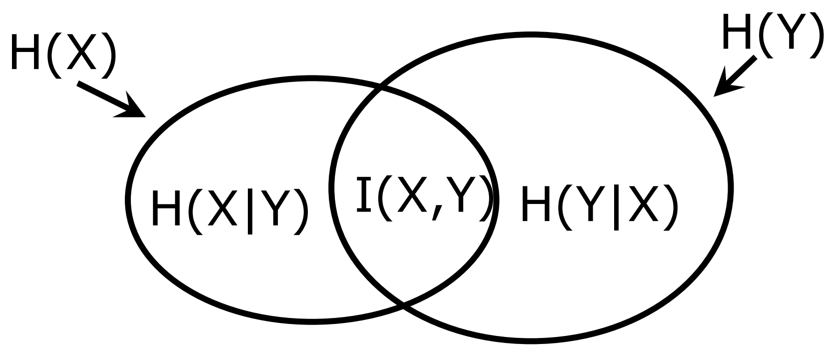
- Why this definition?

Mutual Information

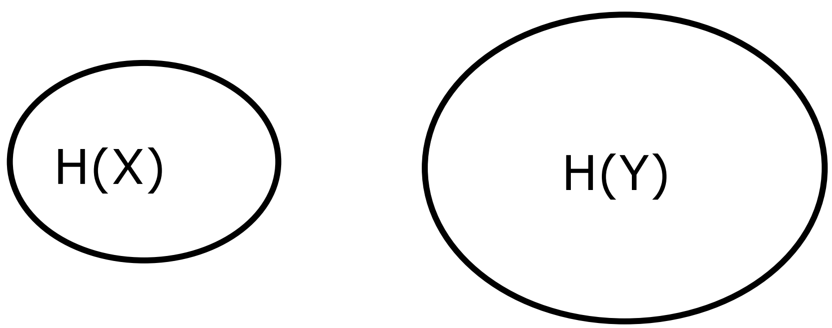
- Properties of Mutual Information:

$$I(X,Y) = I(Y,X)$$

Relationship among entropy, conditional, and mutual information



X and Y are dependent variables.

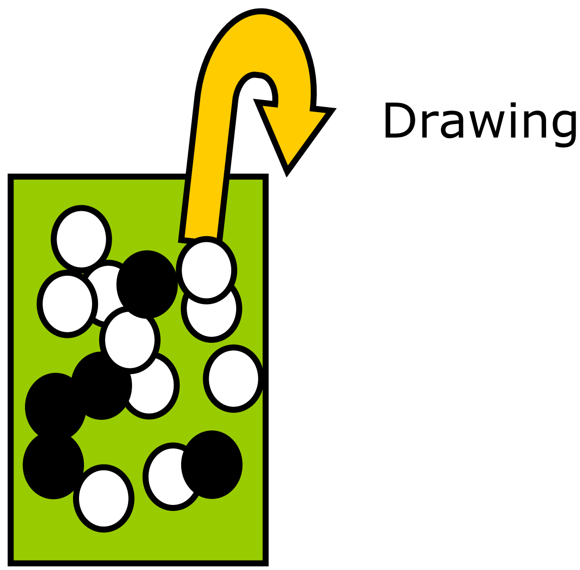


X and Y are independent variables.

Example:

- A vase contains 5 black balls and 10 white balls. Experiment x involves the random drawing of a ball, without being replaced in the vase. Experiment Y involves random drawing of the second ball.

- 5 black balls
- 10 white balls

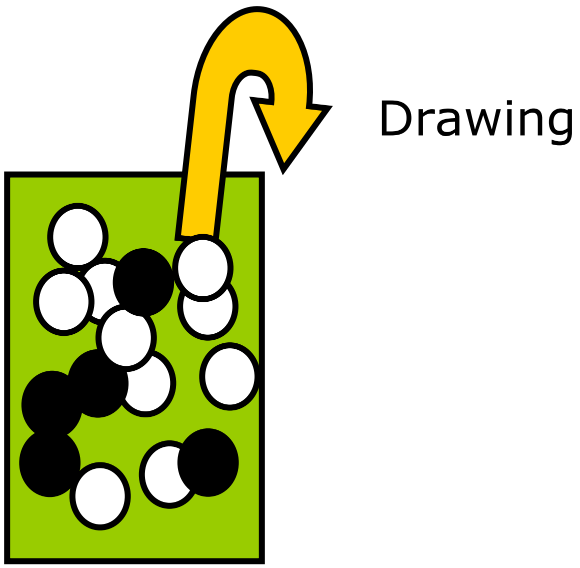


Example: Entropy

- How much uncertainty (information) does experiment X contain?

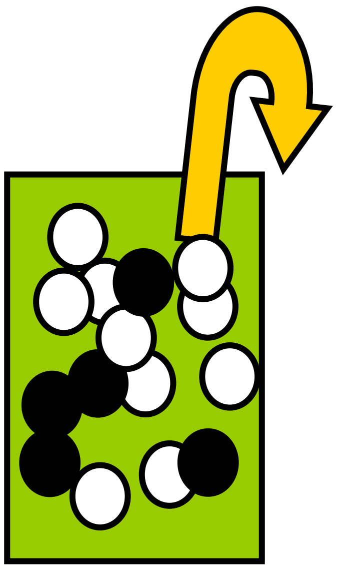
P(black_X) = 1/3, P(white_X) = 2/3)

$H(X) = -(1/3)\log(1/3) - (2/3)\log(2/3) = 0.92$ bit



Example:

- How much uncertainty (information) in experiment Y given that the ball in experiment X is white?



Drawing

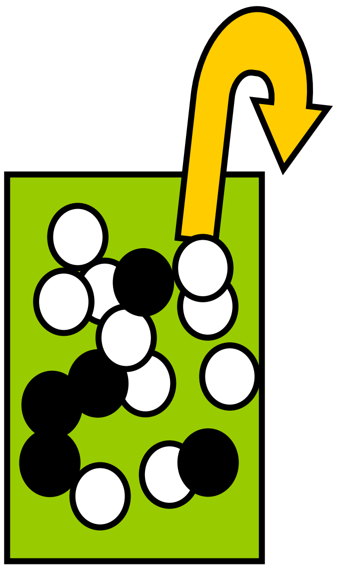
P(black_Y|white_X) = 5/14

P(white_Y|white_X) = 9/14

H(Y|white_X) = -(5/14)log(5/14) – (9/14)log(9/14) = .94 bit

Example:

- How much uncertainty (information) in experiment Y given that the ball in experiment X is black?



P(black_Y|black_X) = 4/14 = 2/7

P(white_Y|black_X) = 10/14 = 5/7

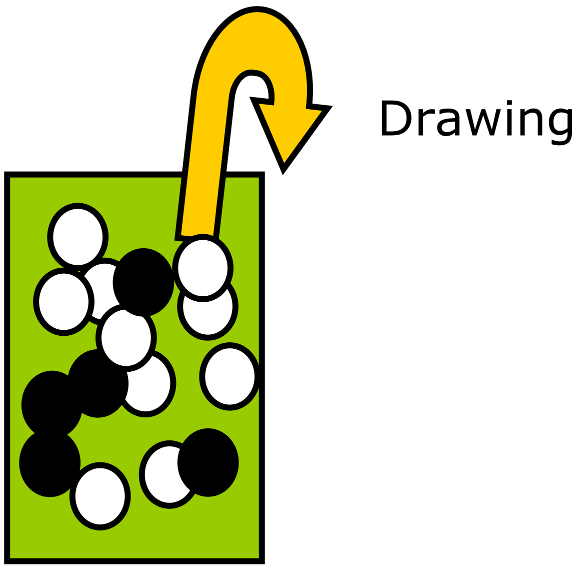
H(Y|black_X) = -(2/7)log(2/7) – (5/7)log(5/7) = 0.86 bit

Example:

- How much uncertainty does experiment Y contain?

H(Y) = P(black_X)*H(Y|black_X) + P(white_X)*H(Y|white_X)

= (1/3)(0.86) + (2/3)(0.94) = 0.91 bit



Formal Derivation of Entropy

- Why do we have

$$H = -\sum_{i=1}^{n} p_i \log(p_i)?$$

Axiomatic Foundations

- Assuming that information measure should satisfy the three following requirements (Chaundy and McLeod (1960)):

  1. If all outcomes are split up into groups, then all the values of H for the various groups, multiplied by the statistical weights, should lead to the overall H.

  2. H should be continuous in $p_{i.}$

  3. If all p_i's are equal, i.e. for all i, ($p_i = 1/n$), then H will increase monotonically as a function of n. That means the uncertainty will increase for an increasing number of equal probabilities.

Derivation of Entropy

- Theorem: The only function that satisfy the three requirements above is

$$H = -K \sum_{i=1}^{n} p_i \log(p_i)$$

- Proof:

Summary

- History of information theory.
- Information theoretical entities
  - Information, self-information, entropy, conditional information, joint information, mutual information.
- Derivation of $H = -\sum p_i \log p_i$