Lecture 13+:
Nearest Neighbor Search

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VQ Encoding is Nearest Neighbor Search

- Given an input vector, find the closest codeword in the codebook and output its index.
- Closest is measured in squared Euclidean distance.
- For two vectors (w_1, x_1, y_1, z_1) and (w_2, x_2, y_2, z_2).

$$\text{Squared Distance} = (w_1 - w_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

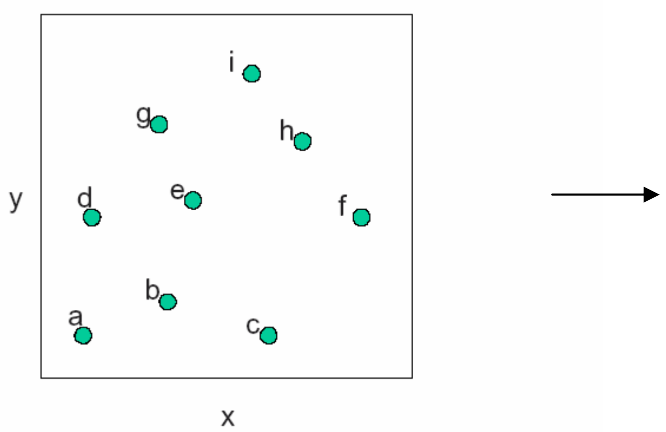
k-d Tree

- Jon Bentley, 1975
- Tree used to store spatial data.
  - Nearest neighbor search.
  - Range queries.
  - Fast look-up!
- k-d trees are guaranteed $\log_2 n$ depth where n is the number of points in the set.
  - Traditionally, k-d trees store points in *d*-dimensional space (equivalent to vectors in *d*dimensional space).

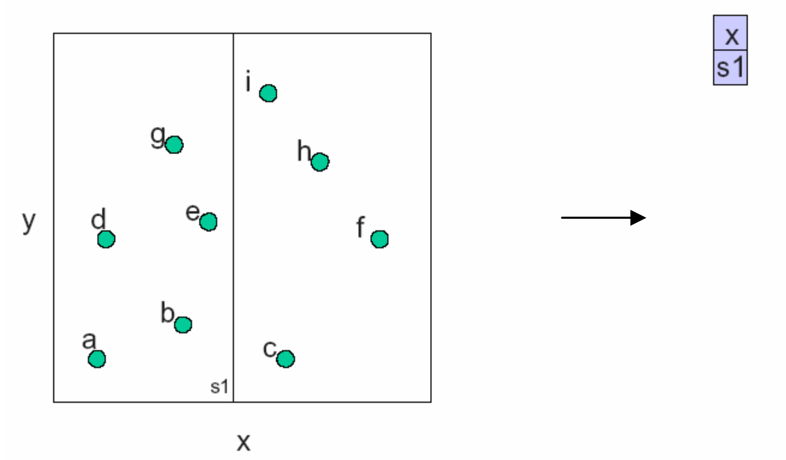
k-d tree construction

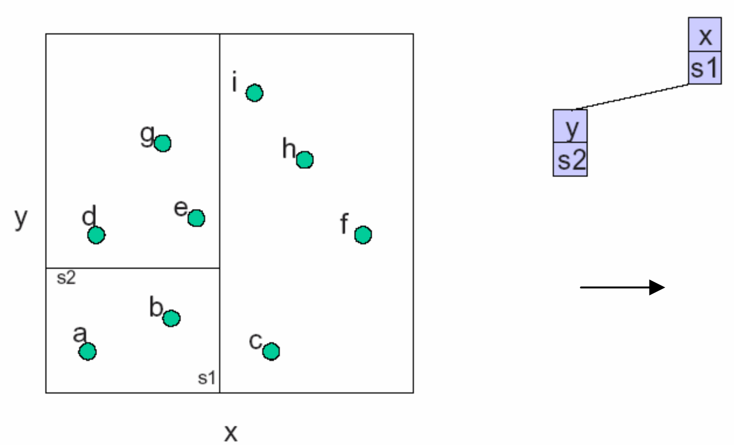
- If there is just one point, form a leaf with that point.
- Otherwise, divide the points in half by a line perpendicular to one of the axes.
- Recursively construct k-d trees for the two sets of points.
- Division strategies:
  - divide points perpendicular to the axis with widest spread.
  - divide in a round-robin fashion.

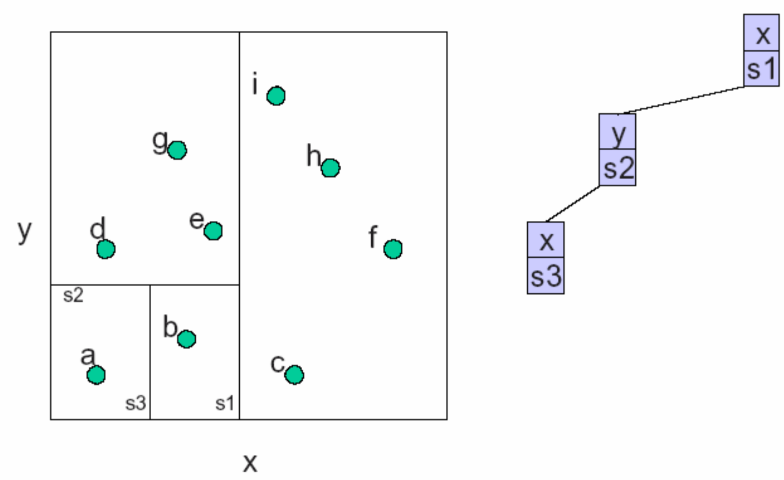
k-d tree construction example



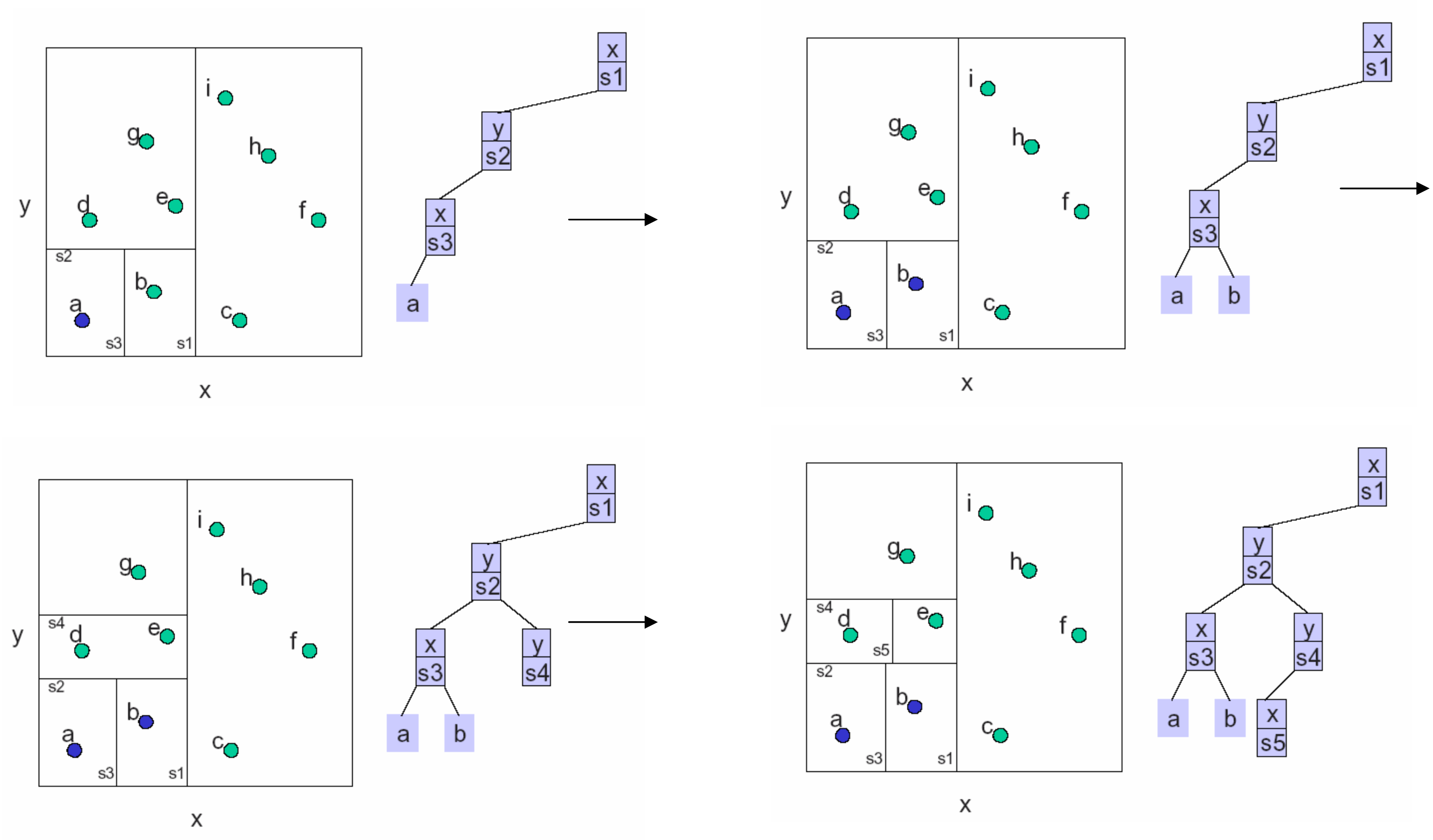
divide perpendicular to the widest spread.



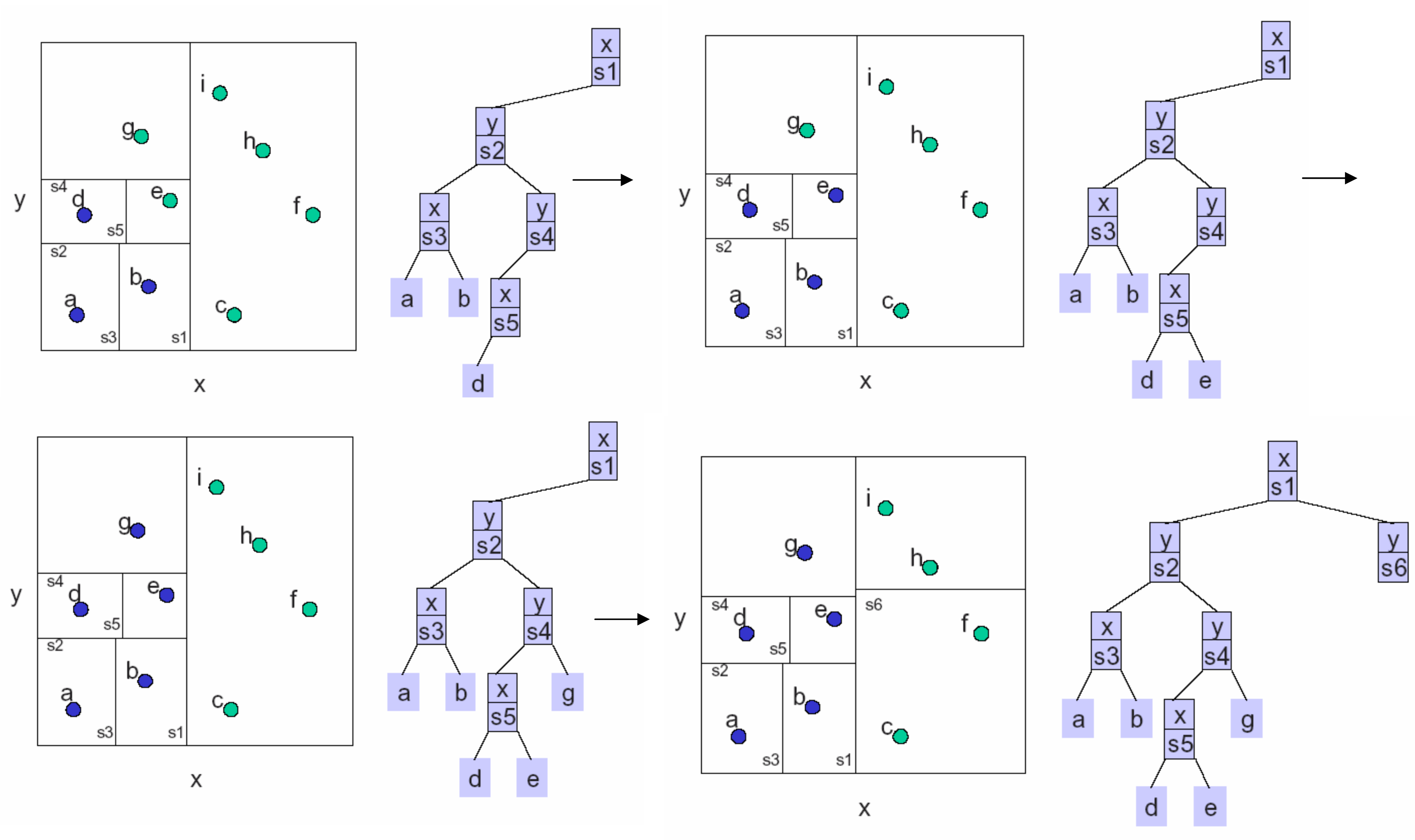




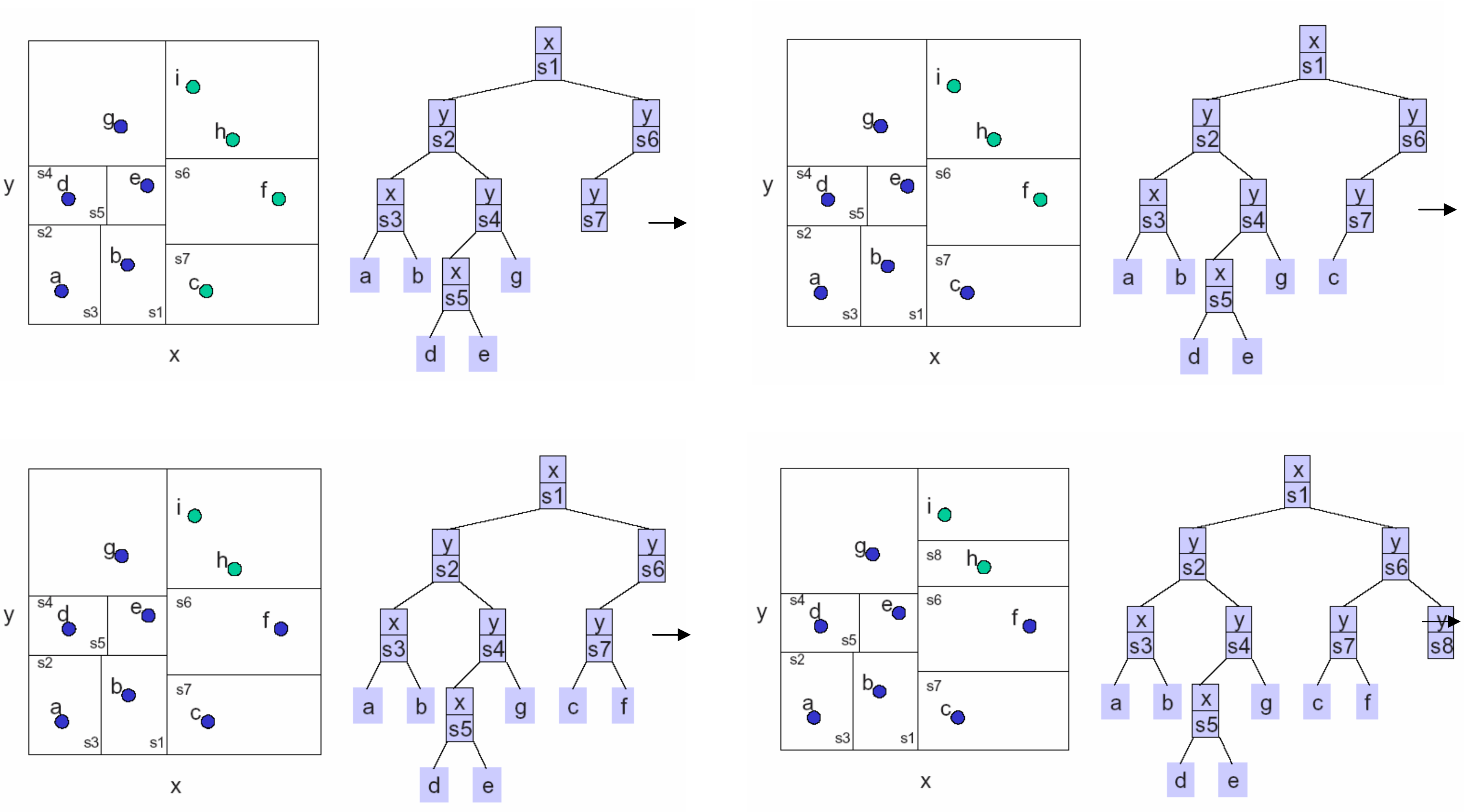
k-d tree construction example



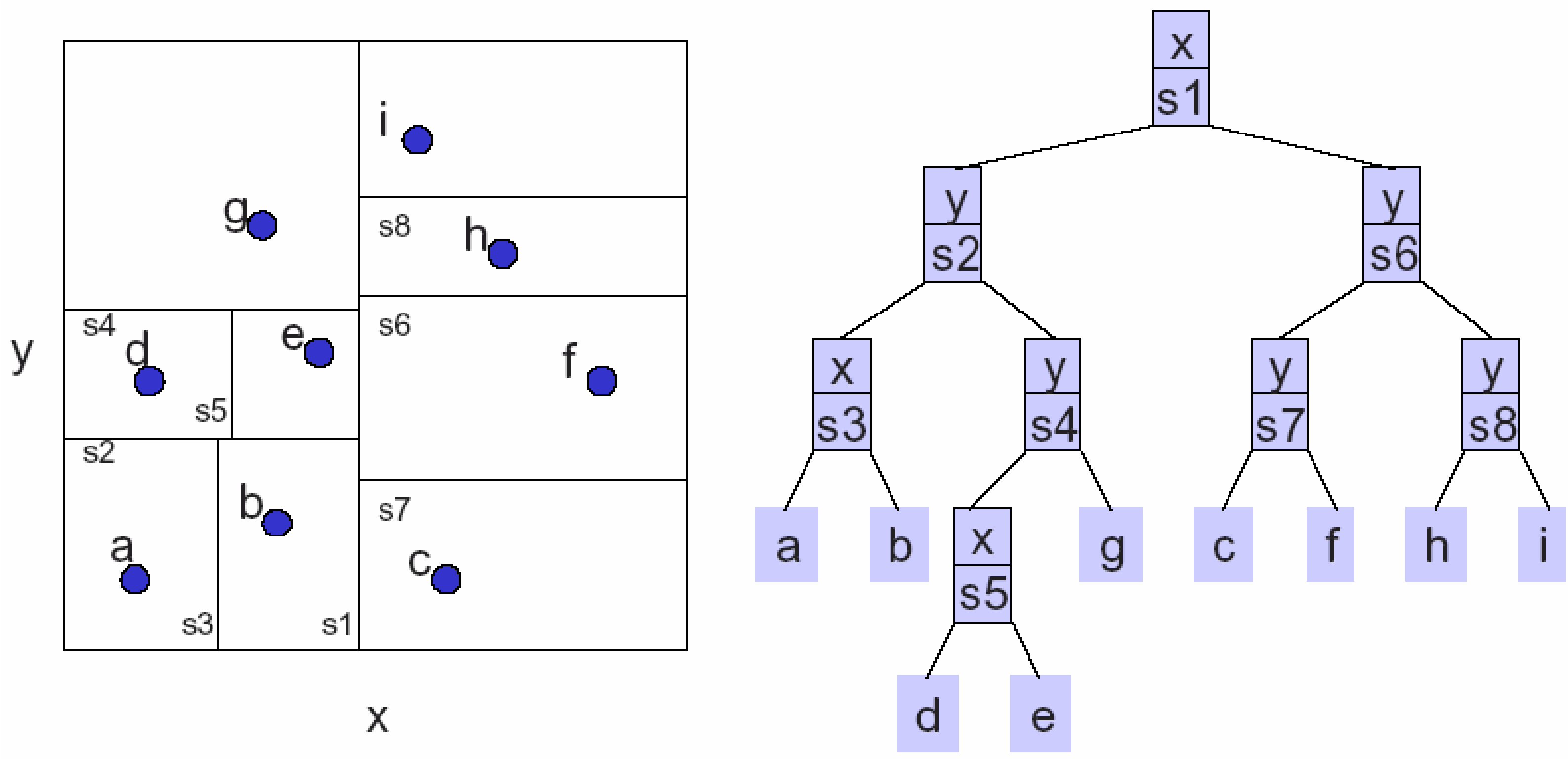
k-d tree construction example



k-d tree construction example



k-d tree construction example

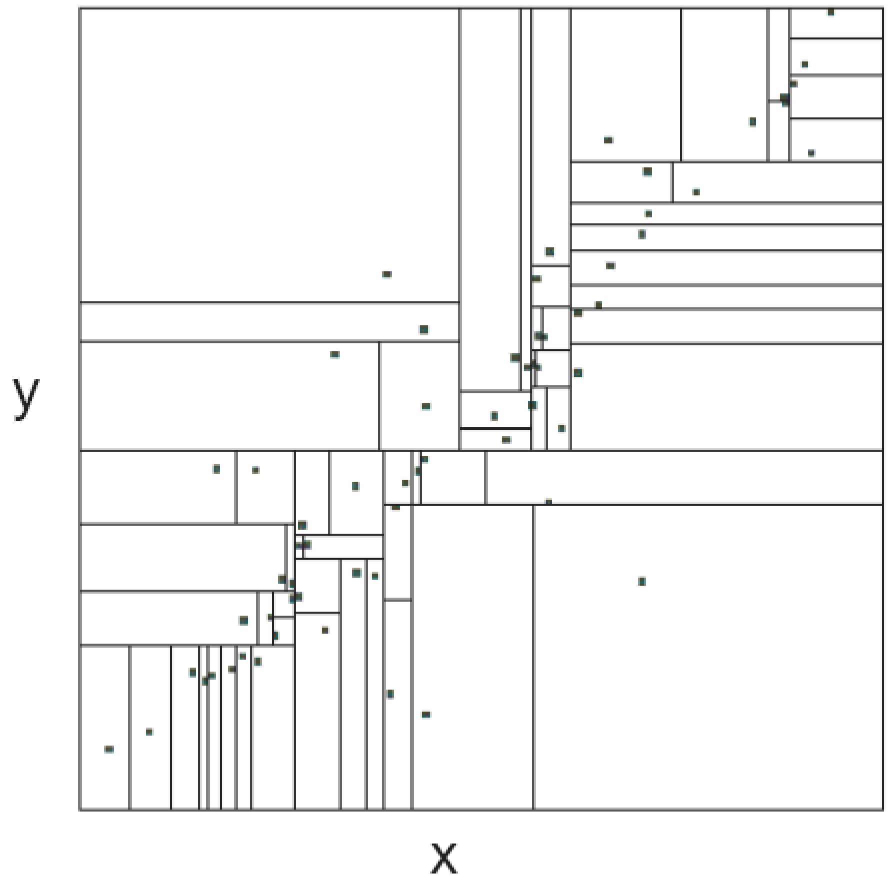


k-d tree Construction Complexity

- First sort the points in each dimension:
  - O(*dn* log *n*) time and *dn* storage.
  - These are stored in A[1..*d*,1..*n*]
- Finding the widest spread and equally dividing into two subsets can be done in O(*dn*) time.
- Constructing the k-d tree can be done in O(*dn* log *n*) and *dn* storage

Codebook for 2-d vector

2-d vectors
(x,y)



Node Structure for k-d Tree

- A node has 5 fields
  - axis (splitting axis)
  - value (splitting value)
  - left (left subtree)
  - right (right subtree)
  - point (holds a point if left and right children are null)

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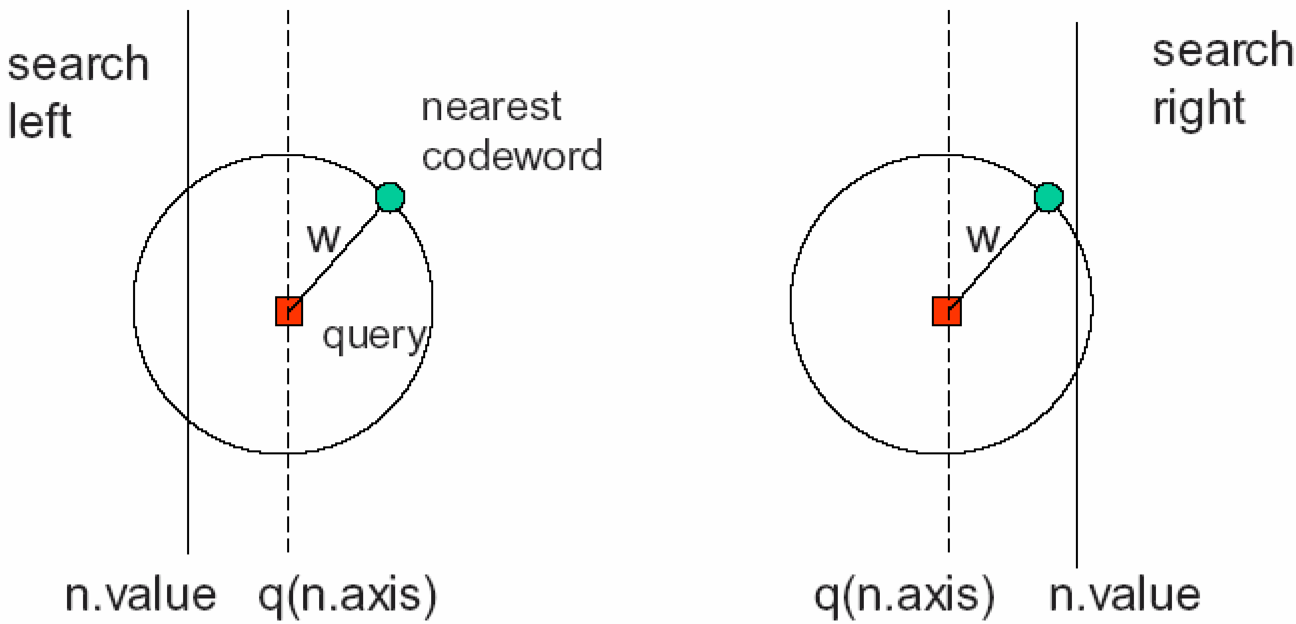
Why does k-d tree work?



$q(n.axis) - w \leq n.value$ means the circle overlaps the left subtree.

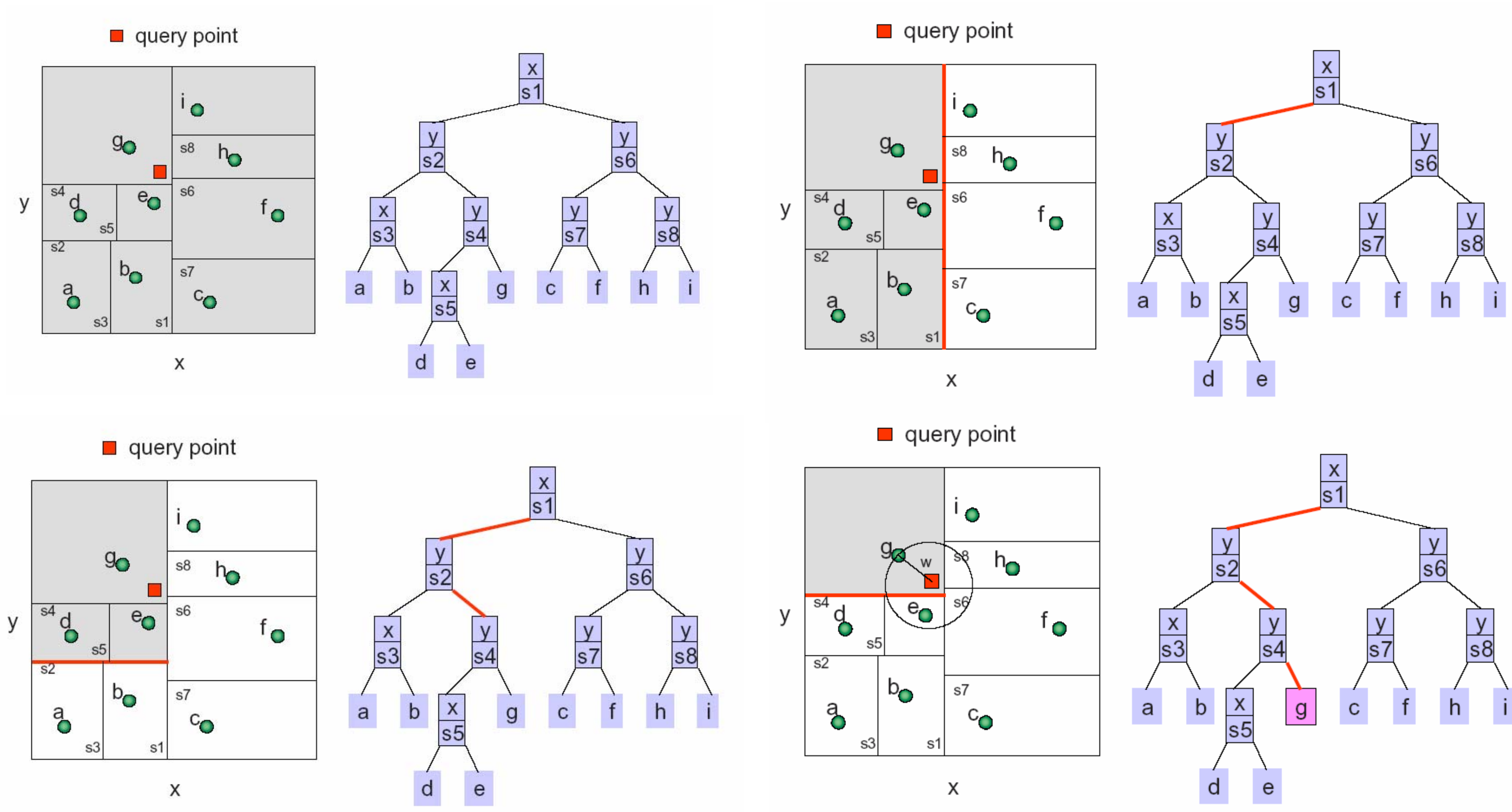
$q(n.axis) + w > n.value$ means the circle overlaps the right subtree.

k-d Tree Nearest Neighbor Search

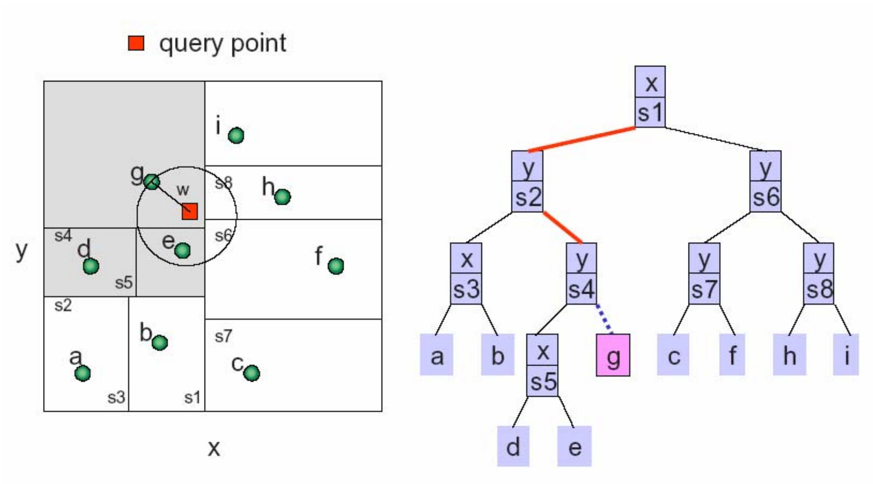
```
NNS(q: point, n: node, p: ref point w: ref distance)
if n.left = n.right = null then {leaf case}
    w' := ||q - n.point||;
    if w' < w then w := w'; p := n.point;
else
    if q(n.axis) ≤ n.value then
        search_first := left;
    else
        search_first := right;
    if (search_first == left)
        if q(n.axis) - w ≤ n.value then NNS(q, n.left, p, w);
        if q(n.axis) + w > n.value then NNS(q, n.right, p, w);
    else // search_first == right
        if q(n.axis) + w > n.value then NNS(q, n.right, p, w);
        if q(n.axis) - w ≤ n.value then NNS(q, n.left, p, w);
```

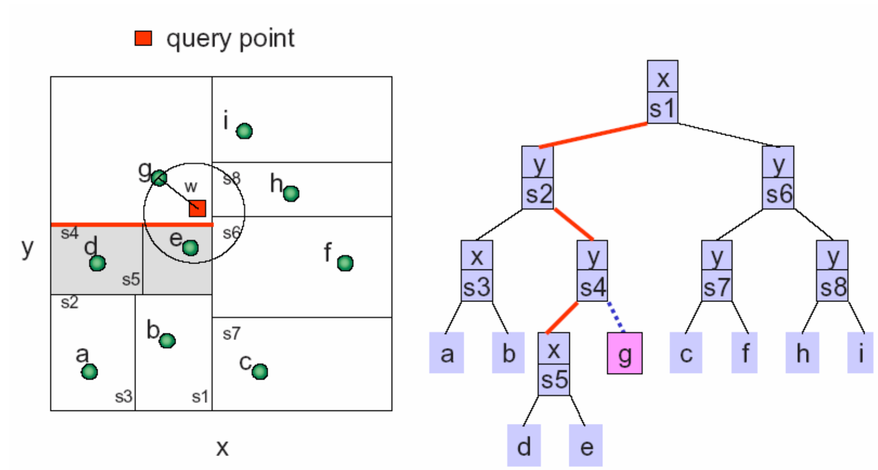
initial call `NNS(q, root, p, infinity)`

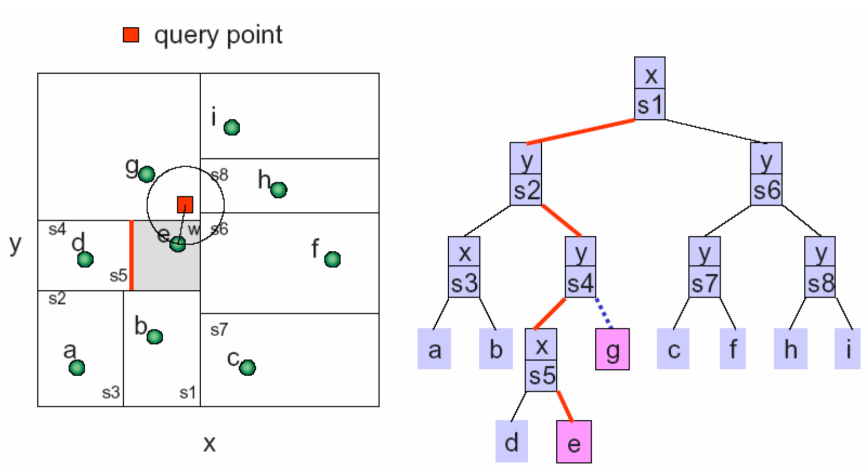
k-d Tree Nearest Neighbor Search

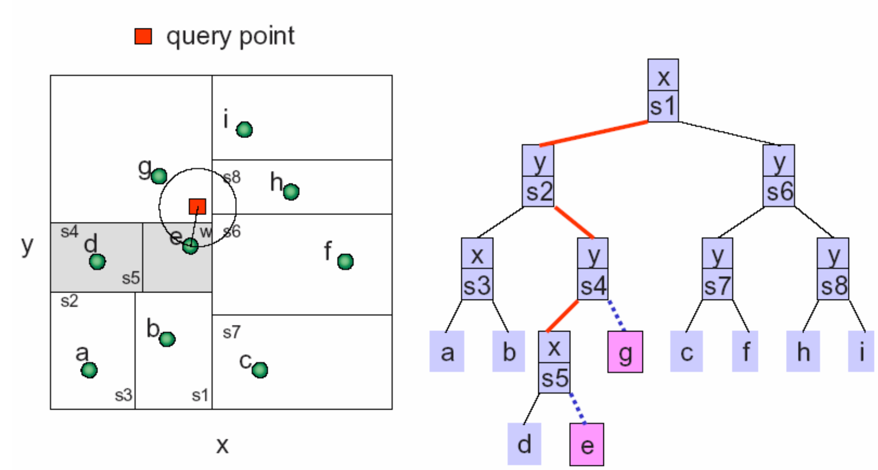


k-d Tree Nearest Neighbor Search

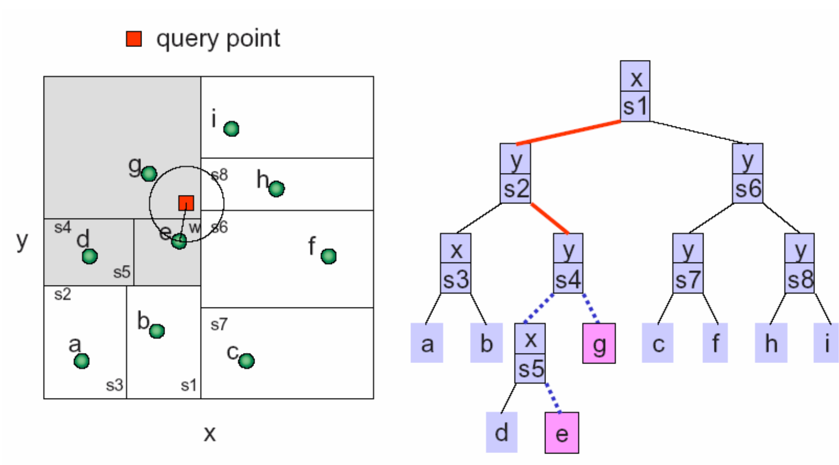


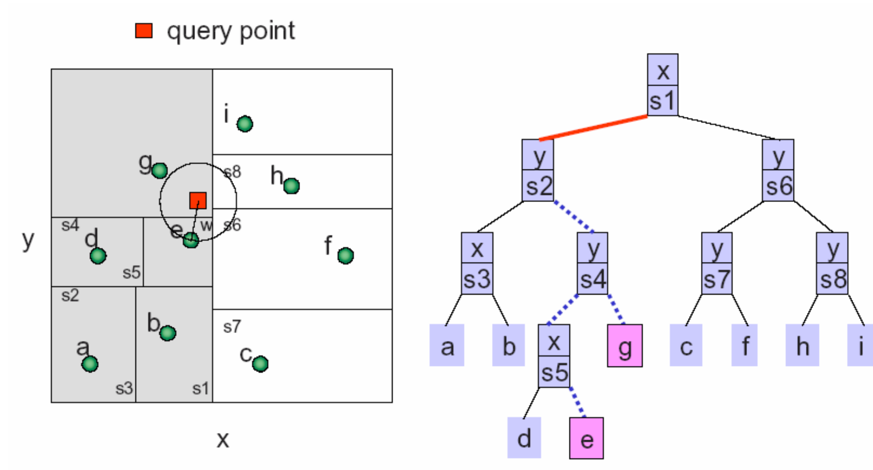


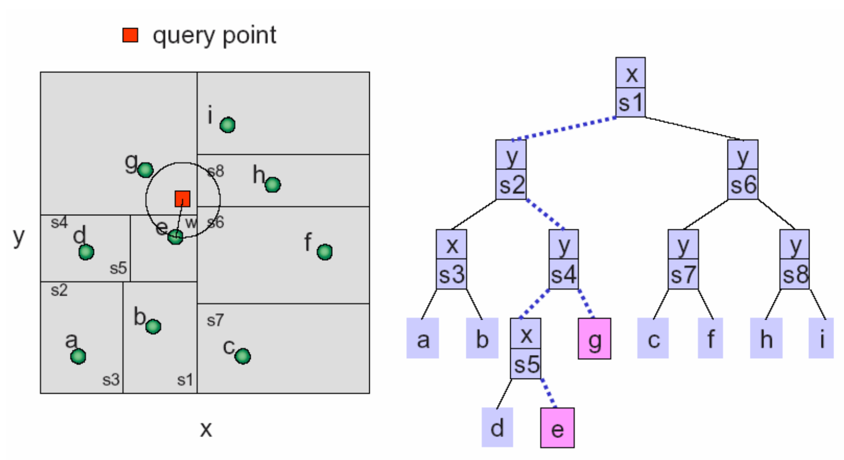


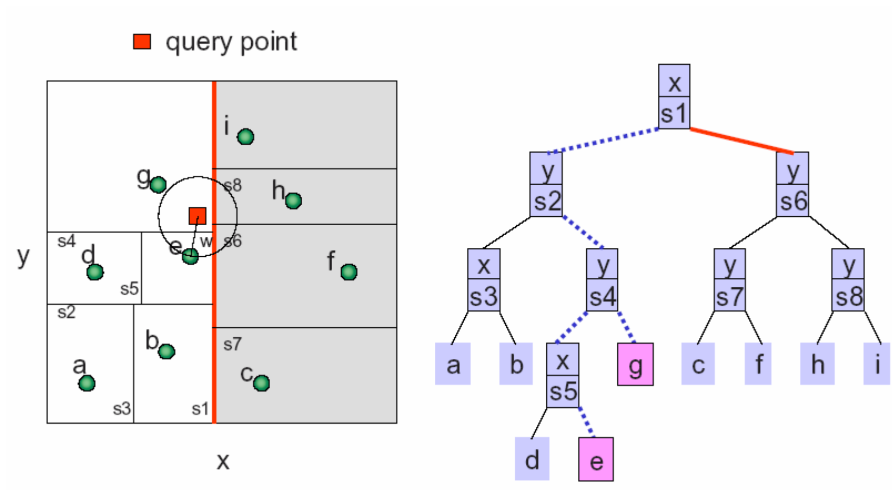


k-d Tree Nearest Neighbor Search

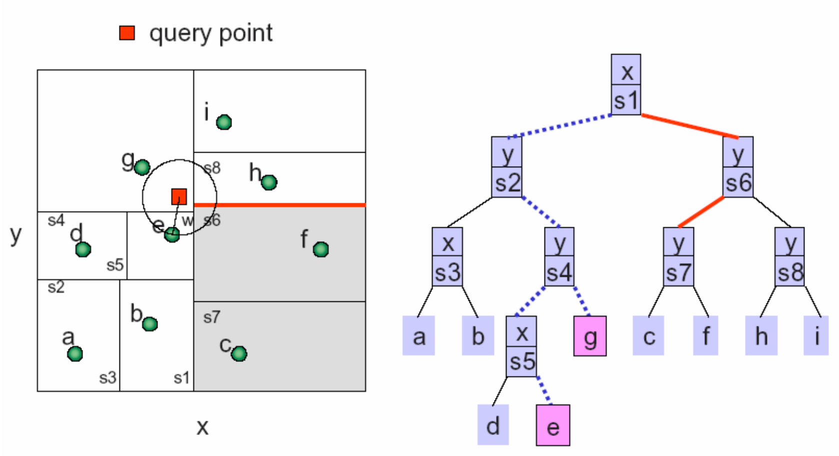


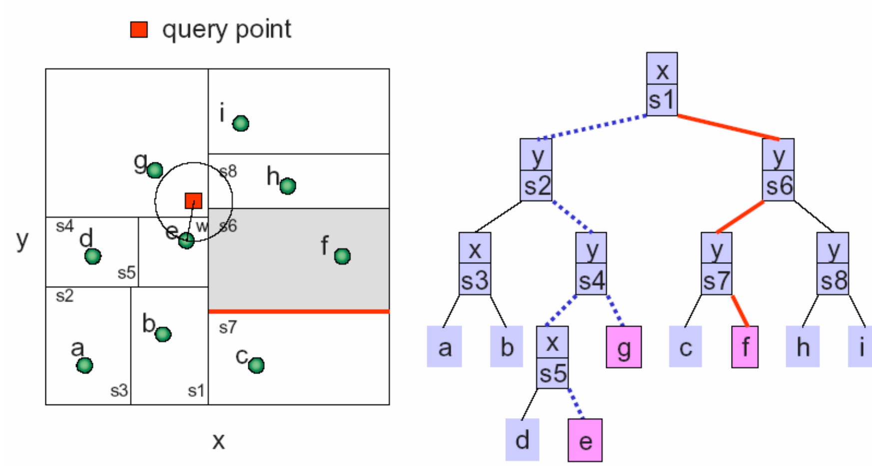


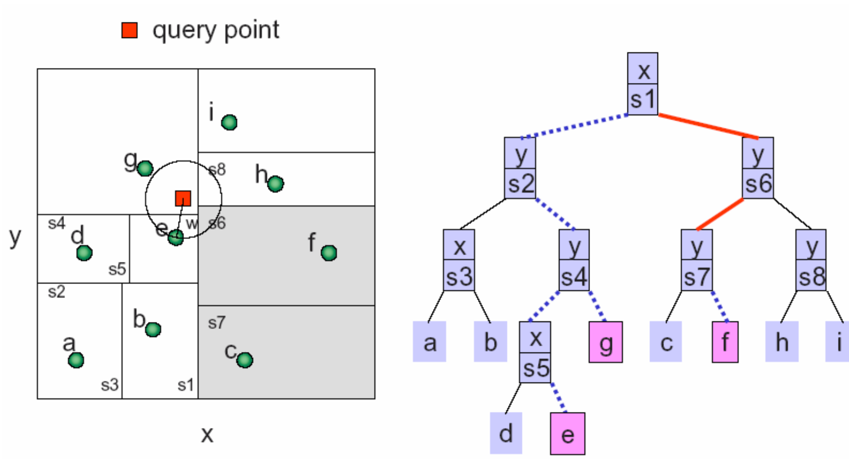


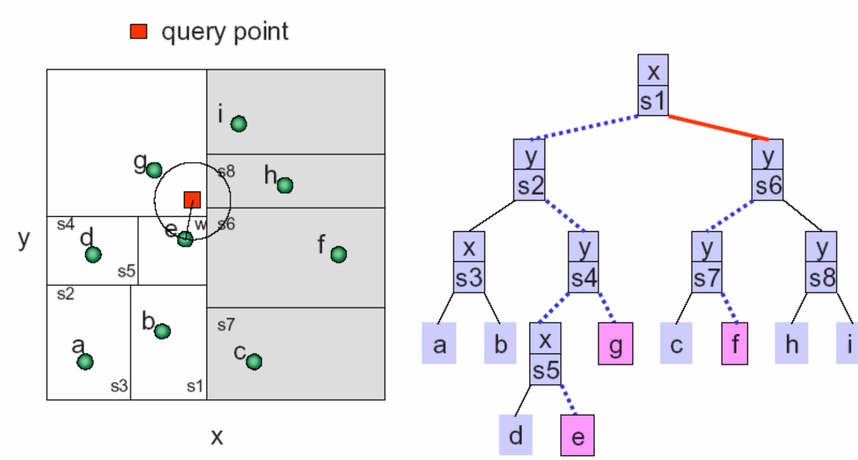


k-d Tree Nearest Neighbor Search

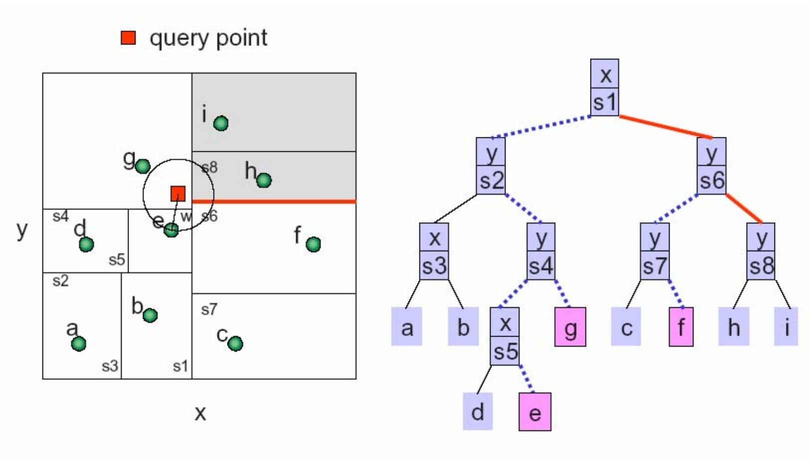


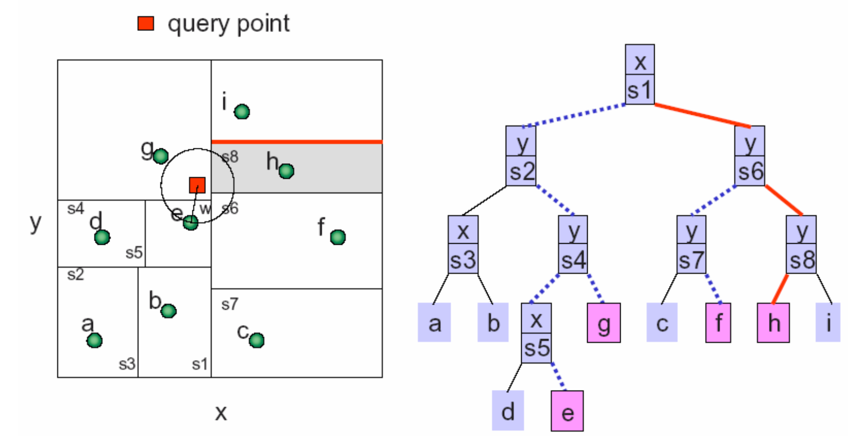


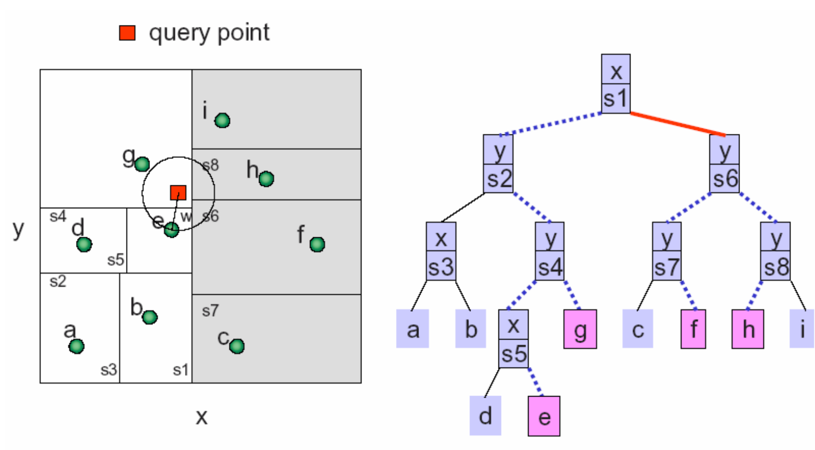


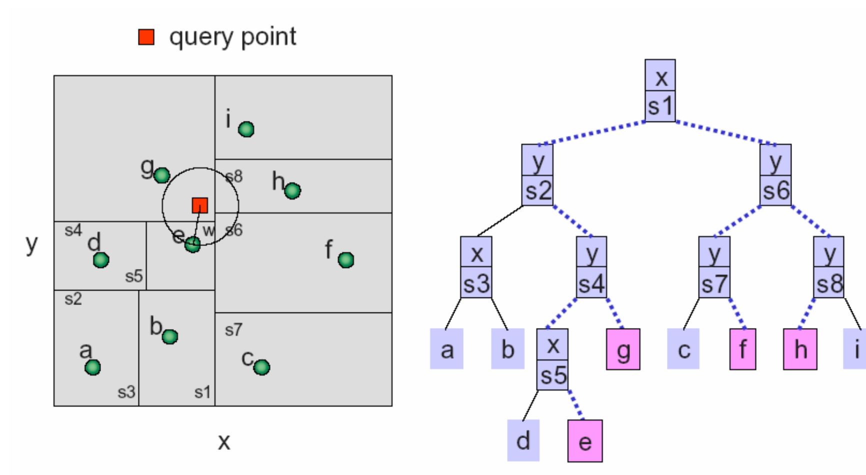


k-d Tree Nearest Neighbor Search









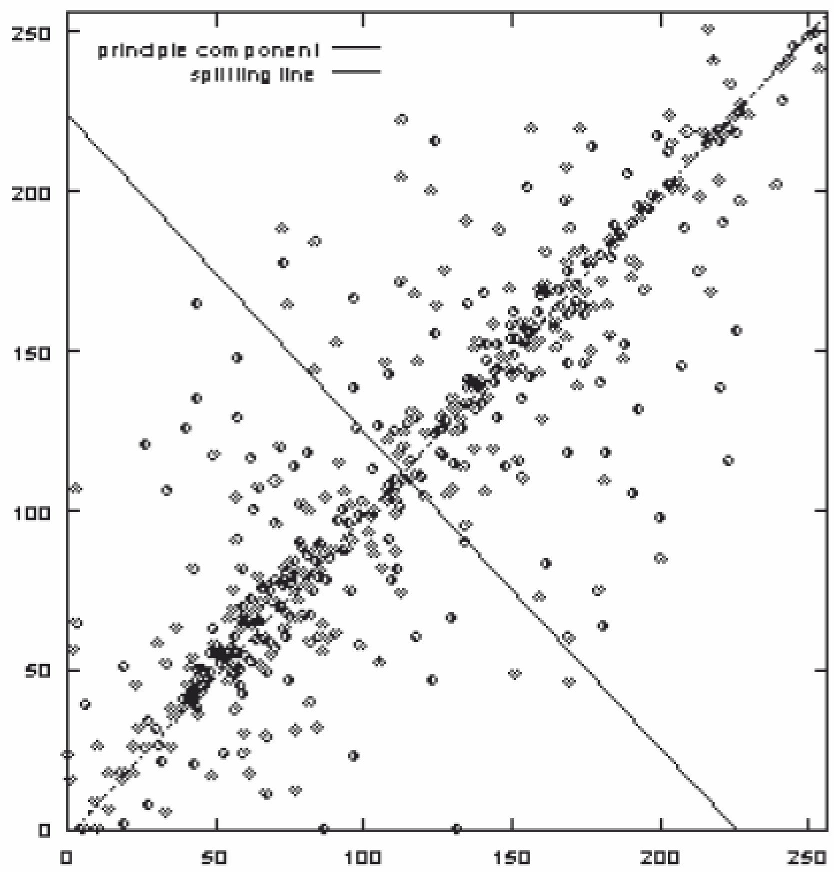
Notes on Nearest Neighbor Search

- Has been shown to run in O(log *n*) average time per search in a reasonable model. (Assuming *d* a constant)
- For VQ it appears that O(log *n*) is correct.
- Storage for the k-d tree is O(*n*).
- Preprocessing time is O(*n* log *n*) assuming *d* is a constant.

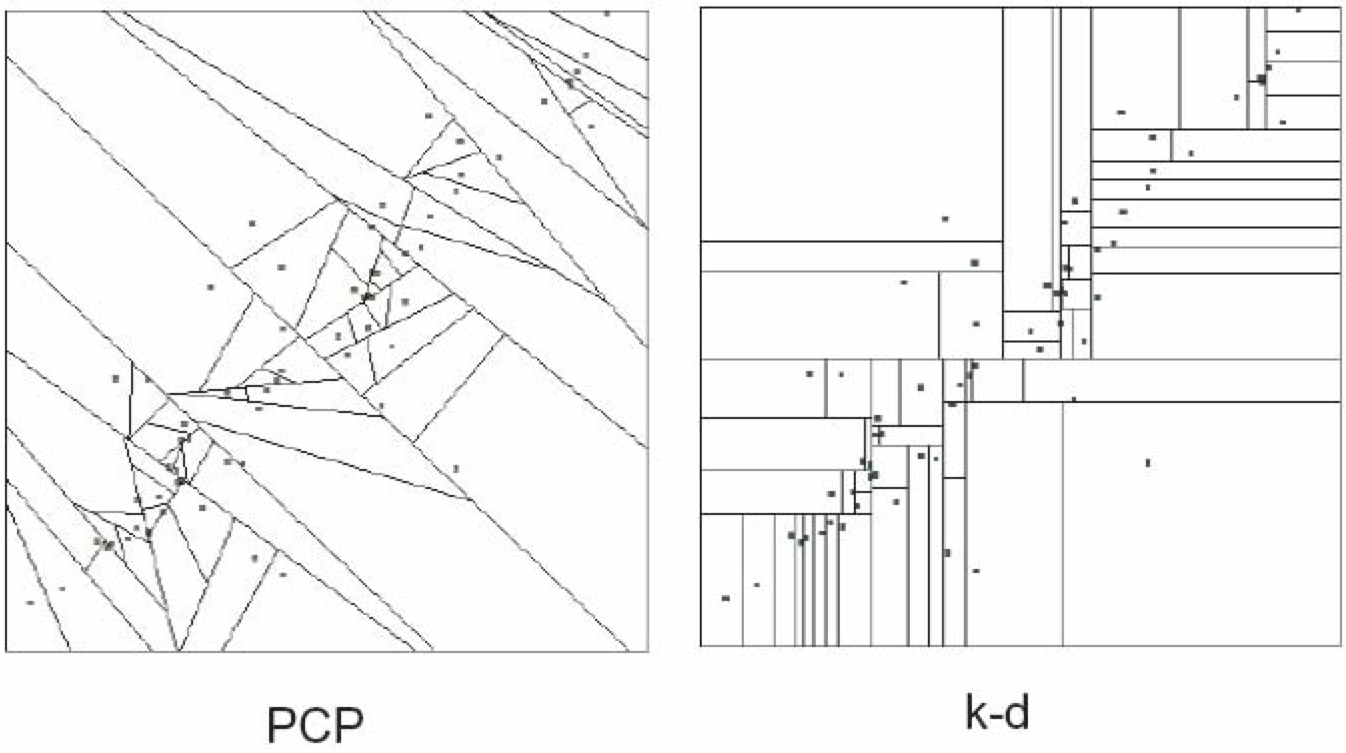
Notes on Nearest Neighbor Search

- Orchard's Algorithm (1991)
  - Uses O(n2) storage but is very fast
- Annulus Algorithm
  - Similar to Orchard but uses O(n) storage. Does many more distance calculations.
- Principal Component Partitioning (PCP)
  - Zatloukal, Johnson, Ladner (1999).
  - Similar to k-d trees.
  - Also very fast.

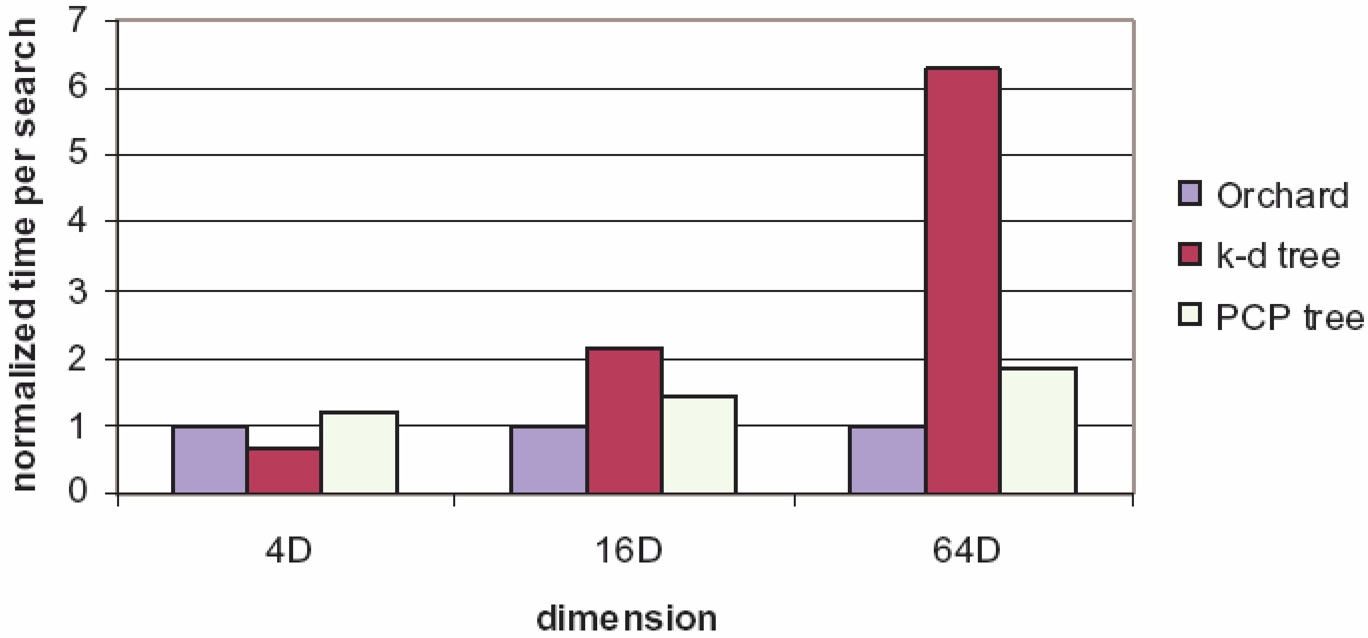
PCP Tree



PCP Tree vs. k-d Tree



Search Time



4,096 codewords

Notes on VQ

- Works well in some applications.
  - Requires training.
- Has some interesting algorithms:
  - Codebook design.
  - Nearest neighbor search.
- Variable length codes for VQ:
  - PTSVQ - pruned tree structured VQ (Chou, Lookabaugh and Gray, 1989)
  - ECVQ - entropy constrained VQ (Chou, Lookabaugh and Gray, 1989)